The small-N series in 0 dimensional $O(N)$ model

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Constructive QFT

[Glimm Jaffe '80, ... Rivasseau, ...]

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(Probably useless, but) find examples of QFTs that respect the OS axioms:

- regularity (slow growth with the number of external points)
- Euclidean covariance
- OS positivity
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$$S(\phi) = \int d^d x \left[\frac{1}{2} \phi(x)(-\Delta + m^2)\phi(g) + \frac{g}{4!}\phi(x)^4 \right]$$

$$Z = \int D\phi \; e^{-S(\phi)} \; , \qquad \langle \phi(x_1) \dots \phi(x_n) \rangle = \frac{1}{Z} \int D\phi \; e^{-S(\phi)} \phi(x_1) \dots \phi(x_n)$$

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Need regularization (cutoffs)!

Perturbation theory is not SO bad

$$Z = \int D\phi \; e^{-\frac{1}{2}\phi C^{-1}\phi - \frac{g}{4!}\int \phi^4} \,, \qquad W = \ln(Z)$$

Taylor expand in g (perturbed Gaussian measure):

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Resum the perturbation theory!

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Partial expansions testing links between blocks of interactions

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Convergent series representation for Z, W etc.

Typical result: $W = \ln Z$ is Borel summable in some domain in coupling $g \in D \subset \mathbb{C}$ uniformly in the cutoffs.

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$$Z = \int D\phi \; e^{-S(\phi)} \sim \sum_c e^{-S(\phi_c)^{\swarrow \sim \frac{1}{g}}} \sum_n c_n g^n \,, \qquad S'(\phi_c) = 0 \,.$$

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How are the non perturbative instanton effects and the resurgent transseries encoded in the convergent constructive expansions?

A toy model

$$Z(g,N) = \int_{-\infty}^{\infty} \left(\prod_{a=1}^{N} d\phi_a \right) \; e^{-S(\phi)} \,, \qquad S(\phi) = \frac{1}{2} \sum_{a=1}^{N} \phi_a \phi_a + \frac{g}{4!} \left(\sum_{a=1}^{N} \phi_a \phi_a \right)^2$$

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- finite dimensional integral: no cutoffs, no renormalization, no axioms
- hypergeometric function, known resurgence properties.
- $S'(\phi)=0$ has solutions $\phi=0$ and $(\phi_c)^2\sim-\frac{1}{g}$

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Ideal playground to find resurgence in a constructive expansion!

Study $Z(g,N)$, $W = \ln(Z(g,N))$ as functions of $g \in \mathbb{C}$.

Hubbard Stratonovich transformation

Intermediate field representation:

$$e^{-\frac{g}{4!}(\phi^2)^2} = \int_{-\infty}^{\infty} d\sigma \; e^{-\frac{1}{2}\sigma^2 + \imath\sqrt{\frac{g}{12}}\sigma\phi^2} \qquad Z(g,N) = \int d\phi \; e^{-\frac{1}{2}\phi^2 - \frac{g}{4!}(\phi^2)^2}$$

integrate out ϕ:

$$Z(g,N) = \int_{-\infty}^{\infty} d\sigma \; e^{-\frac{1}{2}\sigma^2 - \frac{N}{2}\ln(1-\imath\sqrt{\frac{g}{3}}\sigma)} = \sum_{n\geq 1} \frac{1}{n!}\left(-\frac{N}{2}\right)^n Z_n$$

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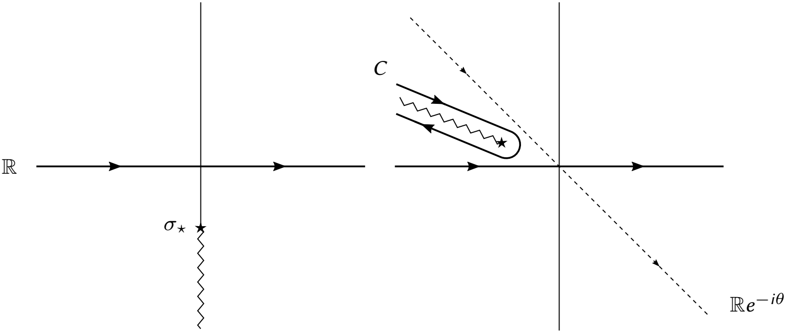
- traded ϕ^4 which dominates over the Gaussian at large field with $\ln(1-\imath\sqrt{\frac{g}{3}}\sigma)$ which does not!
- the perturbative expansion in N is convergent (infinite radius of convergence)!

Where did the instanton go?

$$Z^{\mathbb{R}}(g,N) = \int_{\mathbb{R}} d\sigma \; e^{-\frac{1}{2}\sigma^2} \frac{1}{(1 - \imath\sqrt{\frac{g}{3}}\sigma)^{N/2}}$$

$\arg(g) = 0$ $\quad\quad$ $\arg(g) > \pi$



Properties of $Z(g)$

Theorem

$Z(g)$ is analytic and Borel summable along all the directions in $\mathbb{C} \setminus \mathbb{R}_-$; has a cut singularity at $\mathbb{R}_-$; a second Stokes line is found at $\mathbb{R}_+$ on the second Riemann sheet:

$2k\pi < |\varphi| < (2k+1)\pi$:

$$Z(g,N) = \omega_{2k}\, Z^{\mathbb{R}}(g,N) + \eta_{2k}\, \frac{\sqrt{2\pi}}{\Gamma(N/2)}\, e^{\imath\tau\frac{\pi}{2}}\, e^{\frac{3}{2g}} \left(e^{\imath(2k+1)\tau\pi}\, \frac{g}{3} \right)^{\frac{1-N}{2}} Z^{\mathbb{R}}(-g, 2-N) \,,$$

$(2k+1)\pi < |\varphi| < (2k+2)\pi$:

$$Z(g,N) = \omega_{2k+1}\, Z^{\mathbb{R}}(g,N) + \eta_{2k+1}\, \frac{\sqrt{2\pi}}{\Gamma(N/2)}\, e^{\imath\tau\frac{\pi}{2}}\, e^{\frac{3}{2g}} \left(e^{\imath(2k+1)\tau\pi}\, \frac{g}{3} \right)^{\frac{1-N}{2}} Z^{\mathbb{R}}(-g, 2-N) \,,$$

where $\tau = -\text{sgn}(\varphi)$ and the Stokes parameters (ω, η) are

$$(\omega_{2k}, \eta_{2k}) = \begin{cases} e^{\imath\tau\pi N\frac{k}{2}}\,(1,0) & , k \text{ even} \\ e^{\imath\tau\pi N\frac{k+1}{2}}\,(1,-1) & , k \text{ odd} \end{cases} .$$

For g in the sector $k\pi < |\varphi| < (k+1)\pi$ we have:

$$Z(g,N) \simeq \omega_k \sum_{n=0}^{\infty} \frac{\Gamma(2n+N/2)}{2^{2n} n!\, \Gamma(N/2)} \left(-\frac{2g}{3} \right)^n$$

$$+ \eta_k\, e^{\imath\tau\pi(1-\frac{N}{2})}\, \sqrt{2\pi} \left(\frac{g}{3} \right)^{\frac{1-N}{2}} e^{\frac{3}{2g}} \sum_{q\geq 0} \frac{1}{2^{2q} q!\, \Gamma(\frac{N}{2} - 2q)} \left(\frac{2g}{3} \right)^q \,,$$

$W(g, N)$

$$Z(g,N) = \sum_{n\geq 0} \frac{1}{n!} \left(-\frac{N}{2}\right)^n Z_n(g) \ , \qquad Z_n(g) = \int d\sigma \ e^{-\frac{\sigma^2}{2}} \ \left[\ln(1 - \imath\sqrt{\tfrac{g}{3}}\sigma)\right]^n$$

Z_n has n "loop vertices"

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The free energy also has a small N expansion:

$$W(g,N) = \ln(Z(g,N)) = \sum_{n\ge 1} \frac{1}{n!} \left(-\frac{N}{2} \right)^n W_n(g) \; ,$$

Möebius inversion in the sense of formal power series:

$$W_n(g) = \sum_{k=1}^{n} (-1)^{k-1} (k-1)! \sum_{\substack{n_1,\dots,n_{n-k+1}\ge 0 \\ \sum i n_i = n, \; \sum n_i = k}} \frac{n!}{\prod_i n_i! (i!)^{n_i}} \prod_{i=1}^{n-k+1} Z_i(g)^{n_i} \; .$$

To make Möebius inversion rigorous

Copies of the field with degenerate covariance:

$$Z_n(g) = \left[e^{\frac{1}{2}\frac{\partial}{\partial\sigma}C\frac{\partial}{\partial\sigma}} [V(\sigma)]^n \right]_{\sigma=0} = \left[e^{\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial}{\partial\sigma^{(i)}}C\frac{\partial}{\partial\sigma^{(j)}}} \prod_{i=1}^{n} V(\sigma^{(i)}) \right]_{\sigma^{(i)}=0}$$

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Introduce weakening parameters x^{ij} between the copies:

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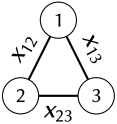
Interpolation on x^{ij} leads to forests

$$e^{\frac{1}{2} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(1)}} + \frac{1}{2} \frac{\partial}{\partial \sigma^{(2)}} C \frac{\partial}{\partial \sigma^{(2)}} + x^{12} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(2)}}} \Big|_{x^{12}=1} = e^{\frac{1}{2} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(1)}} + \frac{1}{2} \frac{\partial}{\partial \sigma^{(2)}} C \frac{\partial}{\partial \sigma^{(2)}}}$$
$$+ \int_0^1 du^{12} \; e^{\frac{1}{2} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(1)}} + \frac{1}{2} \frac{\partial}{\partial \sigma^{(2)}} C \frac{\partial}{\partial \sigma^{(2)}} + u^{12} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(2)}}} \frac{\partial}{\partial \sigma^{(1)}} C \frac{\partial}{\partial \sigma^{(2)}}$$

The Brydges-Kennedy-Abdesselam-Rivasseau formula



$$f(x_{12}, x_{13}, x_{23})$$

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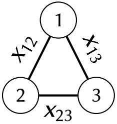
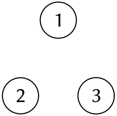
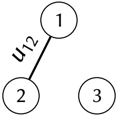


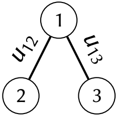
$$f(x_{12}, x_{13}, x_{23})$$

$$f(1,1,1) = f(0,0,0) + \int_0^1 du_{12}\ \frac{\partial f}{\partial x_{12}}(u_{12},0,0) + \ldots$$

$$+ \int_0^1 du_{12} du_{13}\ \frac{\partial^2 f}{\partial x_{12}\partial x_{13}}(u_{12}, u_{13}, \inf(u_{12}, u_{13})) + \ldots$$







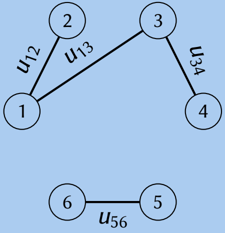
The BKAR formula (2)

Consider the complete graph over n vertices labelled $\{1, \ldots n\}$ and let $f(x_{ij})$ be a function of the $\binom{n}{2}$ link variables x_{ij}. Then

$$f(1, \ldots 1) = \sum_F \int_0^1 \left(\prod_{(k,l)\in F} du_{kl} \right) \left(\frac{\partial^{|F|} f}{\prod_{(k,l)\in F} \partial x_{kl}} \right) (w_{ij}^F) \ ,$$

- F runs over the forests (acyclic subgraphs) of the complete graph
- to each edge (k, l) in the forest we associate a variable u_{kl} which is integrated from 0 to 1
- we take the derivative of f with respect to the variables associated to the edges in the forest
- we evaluate this derivative at $x_{ij} = w_{ij}^F$, the infimum of u along the path in F connecting the vertices i and j

The w_{ij}^F matrix



$$w^F = \begin{pmatrix} 1 & u_{12} & u_{13} & \inf(u_{13}, u_{34}) & 0 & 0 \\ \dots & 1 & \inf(u_{12}, u_{13}) & \inf(u_{12}, u_{13}, u_{34}) & 0 & 0 \\ \dots & \dots & 1 & u_{34} & 0 & 0 \\ \dots & \dots & \dots & 1 & 0 & 0 \\ \dots & \dots & \dots & \dots & 1 & u_{56} \\ \dots & \dots & \dots & \dots & \dots & 1 \end{pmatrix} \geq 0!$$

Loop vertex expansion

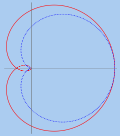
$$W_1(g) = Z_1(g) = \int_{-\infty}^{+\infty} [d\sigma] \; e^{-\frac{1}{2}\sigma^2} \; \ln\left[1 - \imath\sqrt{\frac{g}{3}}\,\sigma\right] \; ,$$

$$W_n(g) = -\left(\frac{g}{3}\right)^{n-1} \sum_{\mathcal{T}\in T_n} \int_0^1 \prod_{(i,j)\in\mathcal{T}} du_{ij}$$

$$\int_{-\infty}^{+\infty} \frac{\prod_i [d\sigma_i]}{\sqrt{\det \underbrace{w^{\mathcal{T}}}_{\text{positive matrix}}}} \; e^{-\frac{1}{2}\sum_{i,j}\sigma_i (w^{\mathcal{T}})^{-1}_{ij}\sigma_j} \; \prod_i \frac{(d_i - 1)!}{\left(1 - \imath\sqrt{\frac{g}{3}}\,\sigma_i\right)^{d_i}} \; ,$$

$\sum_n \frac{1}{n!}\left(-\frac{N}{2}\right)^n W_n(g)$ convergent in some domain in g.

just enough for Borel summability in $\mathbb{C}\setminus\mathbb{R}_-$



Resurgent transseries for $W(n,N)$, $W_n(g)$: Möebius inversion + $Z_n(g)$

Lessons for constructive Quantum Field theory

In the intermediate field / loop vertex expansion:

- the instantons are replaced by singularities crossing integration contours
- for the transseries of $W_n(g)$ and $W(g, N)$ we had to resort to the explicit Möebius inversion – try to find the instantons directly from the LVE expression.
- the logarithmic interaction has good large field properties

However:

- counterterms + subtraction of divergences in intermediate field are non trivial
- multi-series?
- decay of correlations?