A class of singular SPDEs via convex integration

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Joint work with Martina Hofmanová and Rongchan Zhu

Naiver-Stokes equations driven by space-time white noise

Navier-Stokes equation

Consider the Navier-Stokes equation on $\mathbb{T}^3$:

$$\begin{aligned} \partial_t u + u \cdot \nabla u =& \nu \Delta u - \nabla p + \xi, \quad \mathrm{div} u = 0 \\ u(0) =& u_0 \end{aligned} \tag{1}$$

- $u(t, x) \in \mathbb{R}^3$: the velocity field at time t and position x,
- $p(t, x)$: the pressure,
- $\nu > 0$: the viscosity constant
- ξ: space-time white noise

Derivation of Navier-Stokes system: Newton's law

Suppose $u = u(t, x(t))$ and ρ: the density

$$\frac{\mathrm{d}}{\mathrm{d}t}u(t) = \underbrace{\partial_t u}_{\text{variation}} + \underbrace{u \cdot \nabla u}_{\text{convection}} = \underbrace{\nu \Delta u}_{\text{Diffusion}} - \underbrace{\nabla p}_{\text{Internal source}} + \underbrace{f}_{\text{External source}},$$

$$\underbrace{\partial_t \rho + \nabla \cdot (\rho u) = 0}_{\text{mass conservation}} \Rightarrow^{\text{if } \rho = \text{constant}} \mathrm{div} u = 0$$

$$u(0) = u_0.$$

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Motivation of space-time white noise

- Landau-Lifshitz-Navier-Stokes system: thermal fluctuations (critical or supercritical)
- scaling limit from vortex approximation/Euler perturbed by transport noise ([Flandoli, Luo20])
- regularization by noise

Deterministic: [Leray34], [Kato, Fujita62], [Temam84], [Constantin, Foias88] [Cafarelli,Kohn, Nirenberg84], [Fefferman 00], [Koch, Tataru01],...

- The global existence of **weak** solutions has been obtained in all dimensions.
- Existence and smoothness of solutions in the three dimensional case remains open (the Millennium Prize problem)./ Small initial data
- [Buckmaster, Vicol 19]: Non-uniqueness of analytic weak
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Stochastic : trace-class noise

- Martingale and Markov solutions have been constructed [Flandoli, Romito08]
- Nonuniqueness in law/ Nonuniqueness of Markov solutions/Global probabilistically strong solutions/ Nonuniqueness of stationary solution for NS and Euler [Hofmanová, Zhu, Z. 19, 21, 22]

Navier-Stokes equations: $d = 2$

$$\partial_t u + \mathrm{div}(u \otimes u) = \Delta u - \nabla p + \xi, \quad \mathrm{div} u = 0, \quad u(0) = \quad u_0$$

space-time white noise: a random Gaussian function with covariance
$\mathbf{E}\xi(s,x)\xi(t,y) = \delta(s-t)\delta(x-y)$

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$(f,g) \to fg$ is well-defined on $C^\alpha \times C^\beta$ to $C^{\alpha\wedge\beta}$ only if $\alpha + \beta > 0$.

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$$\partial_t v = \Delta v - \mathrm{div}(v \otimes v + v \otimes z + z \otimes v) - \mathrm{div}(\underbrace{z \otimes z}_{\text{Wick power}}) - \nabla p_2, \quad \mathrm{div} v = 0,$$

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- [Zhu, Z17]: Strong-Feller property and global existence and uniqueness from every starting point
- [Hairer, Rosati23]: Global well-posedness by PDE argument

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$$\mathcal{L}h + \mathrm{div}((h + z_1) \otimes (h + z_1) + z \otimes (h + z_1) + (z_1 + h) \otimes z) + \nabla p^h = 0,$$
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- Solution: regularity structures theory Hairer 14/ paracontrolled distribution method Gubinelli, Imkeller, Perkowski 15 $\Rightarrow$ local well-posedness in [Zhu, Z. 15]

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Singular SPDE:

- dynamical Φ^4 model: dissipation effect from $-\Phi^3$ [Mourrat, Weber 17, Albeverio, Kusuoka 18, Gubinelli, Hofmanová 19, Moinat, Weber 20, Chandra, Moinat, Weber 19]
- KPZ equation: Cole-Hopf transform [Hairer13, Perkowski, Rosati 19]/ Maximum principle+ Zvonkin transform in [Zhang, Zhu, Z. 20]

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3D NS equation driven by space time white noise

- no strong drift
- no maximum principle or Cole–Hopf's transform.
- existence of an invariant measure: open problem.
- No global energy (or other) estimates are available due to irregularity of solutions (L^2 estimate does not work here)

Main results

Theorem (Hofmanová, Zhu, Z. 21)

For any given divergence free initial condition $u_0 \in L^2 \cup B^{-1+\kappa}_{\infty,\infty}$ $\mathbf{P}$-a.s., $\kappa > 0$, there exist infinitely many global-in-time probabilistically strong solutions solving N-S driven by space-time white noise in a paracontrolled sense.

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1. Decomposition to regular and irregular parts by Bony's paraproduct and localizer
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Theorem (Lü, Z. 23)

Sharp nonuniqueness in 2D case.

- *Uniqueness of v in $C_T L^2 \cap L^2_T H^\zeta$ for some $\zeta > 0$*
- *Infinitely many solutions v in $C_T L^p \cap L^2_T H^\zeta$ for some $\zeta > 0$ and $1 < p < 2$.*

Convex integration

Iteration procedure a pair $(v_q^1, v_q^2, \mathring{R}_q)$ is constructed via

$$\mathcal{L}v_q^1 + \mathrm{div}(z_1 \otimes z_1 + V_q^1 + V_q^{1,*}) + \nabla p_q^1 = 0,$$
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- Key step: Let $w_{q+1} = v_{q+1}^2 - v_q^2$, then we have

$$\mathrm{div}\mathring{R}_{q+1} = \underbrace{-\Delta w_{q+1} + \partial_t w_{q+1} + \mathrm{div}(v_q \otimes w_{q+1} + w_{q+1} \otimes v_q)}_{\text{linear error}}$$
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  Choose $w_{q+1} \sim \sum_\xi a_\xi(\mathring{R}_q) W_\xi$ with W_ξ intermittent jets from [Buckmaster, Colombo, Vicol19]
- The space concentration ensure the linear error is small in L^1
- $\int W_\xi \otimes W_\xi \simeq 1$ and $a_\xi(\mathring{R}_q) \approx \sqrt{-\mathring{R}_q}$ oscillates slowly

Iteration scheme

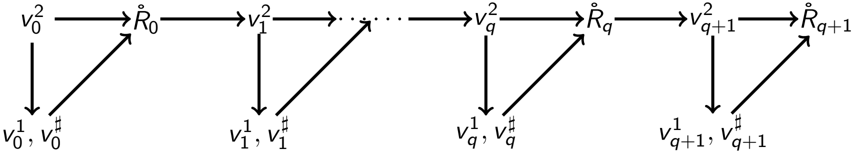

Figure: Iteration scheme.

- $v_q^2 \to^{\text{Schauder estimates+paracontrolled calculus}} v_q^1, v_q^\sharp$
  $\mathring{R}_q \to^{\text{convex integration}} v_{q+1}^2$

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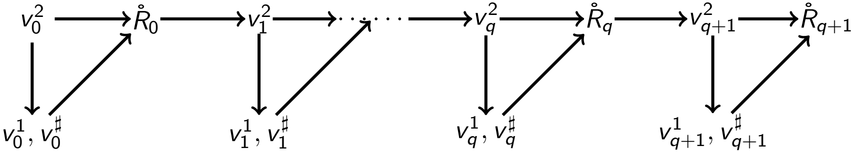

Figure: Iteration scheme.

- $v_q^2 \rightarrow^{\text{Schauder estimates+paracontrolled calculus}} v_q^1, v_q^\sharp$
  $\mathring{R}_q \rightarrow^{\text{convex integration}} v_{q+1}^2$
- energy of $v^1 + v^2$ is different $\Rightarrow$ nonuniqueness of solution

Surface quasi-geostrophic equation in the critical and supercrtical regime

Surface quasi-geostrophic equation

- Surface quasi-geostrophic equation with irregular spatial perturbation

$$\partial_t\theta + u\cdot\nabla\theta + (-\Delta)^{\gamma/2}\theta = \zeta, \quad u = \nabla^{\perp}(-\Delta)^{-1/2}\theta$$

on $[0,\infty)\times\mathbb{T}^2$, $\gamma\in[0,3/2)$ and $\zeta\in C^{-2+\kappa}, \kappa>0$.

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- Example: $\zeta = (-\Delta)^{\alpha/2}\xi, \alpha<1$, ξ a space white noise in two dimensions. Let

$$\tilde{\xi}(x) := \lambda\xi(\lambda x), \quad \tilde{\theta}(t,x) := \lambda^{1+\alpha-\gamma}\theta(\lambda^{\gamma}t,\lambda x),$$
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- Equation is subcritical if $2\gamma-2-\alpha>0$, critical if $2\gamma-2-\alpha=0$ and supercritical if $2\gamma-2-\alpha<0$.

Surface quasi-geostrophic equation

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- Write for ψ smooth

$$\langle u\cdot\nabla\theta,\psi\rangle = \frac{1}{2}\langle\theta,[\mathcal{R}^\perp\cdot,\nabla\psi]\theta\rangle,$$

- The commutator

$$[\mathcal{R}^\perp\cdot,\nabla\psi] = -[\mathcal{R}_2\cdot,\partial_1\psi] + [\mathcal{R}_1\cdot,\partial_2\psi]$$

  maps $\dot{H}^{-1/2}$ to $\dot{H}^{1/2}$

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$$\langle u\cdot\nabla\theta,\psi\rangle = \frac{1}{2}\langle\theta,[\mathcal{R}^\perp\cdot,\nabla\psi]\theta\rangle,$$

- The commutator

$$[\mathcal{R}^\perp\cdot,\nabla\psi] = -[\mathcal{R}_2\cdot,\partial_1\psi] + [\mathcal{R}_1\cdot,\partial_2\psi]$$

  maps $\dot{H}^{-1/2}$ to $\dot{H}^{1/2}$
  $\Rightarrow$ For $\theta\in\dot{H}^{-1/2}$, the nonlinear term is well-defined in the analytical weak sense

$$\partial_t\theta + u\cdot\nabla\theta + (-\Delta)^{\gamma/2}\theta = \zeta, \quad u = \nabla^{\perp}(-\Delta)^{-1/2}\theta$$
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- energy method breaks down due to singularity of ζ

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There exist infinitely many

1. *weak solutions on* $[0, \infty)$ *for any prescribed initial condition* $\theta_0 \in C^\eta$ $\mathbf{P}$*-a.s.,* $\eta > 1/2$,
2. *weak solutions on* $[0, T]$ *for any prescribed initial and terminal condition* $\theta_0, \theta_T \in C^\eta$ $\mathbf{P}$*-a.s.,* $\eta > 1/2$, $T \geq 4$.

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Coming down from infinity: for any $\varepsilon > 0$ there exists a solution θ

$$\|\theta\|_{C_b([T,\infty),B^{-1/2-\delta}_{\infty,1})} \leq \varepsilon.$$

independent of the size of initial value and the noise

Idea of proof

- Iteration scheme: at each step n, a pair $(\theta_{\leq n}, q_n) \in C_0^\infty \times C_0^\infty$ is constructed solving the following system

$$\partial_t \theta_{\leqslant n} + \nabla \cdot (u_{\leqslant n} \theta_{\leqslant n}) - P_{\leq \lambda_n} \zeta = (-\Delta)^{\gamma/2} \theta_{\leqslant n} + \Delta q_n.$$

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- Error from $(P_{\le\lambda_{n+1}} - P_{\le\lambda_n})\zeta \Rightarrow q_{n+1}$

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There exist infinitely many non-Gaussian

1. *stationary solutions,*
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Moreover, the ergodic stationary solutions are time dependent. The point (3) additionally implies existence and non-uniqueness of solutions to the corresponding elliptic and wave equation.

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We could also use this method for fractional NS equation with rough spatial perturbation.

Thank you !