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On the strength of $\mathrm{PFA}(\aleph_2)$ in conjunction with a precipitous Ideal on ω_1 and Namba-like forcings on successors of regular cardinals

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On the strength of PFA($\aleph_2$) in conjunction with a precipitous Ideal on ω_1 and Namba-like forcings on successors of regular cardinals

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Preface

This PHD thesis divides rather neatly into two parts. This short introduction will give a brief summary of both of them and will also try to provide some mathematical context.

In the first part we will try to extract strength from PFA($\aleph_2$) - a strengthening of BPFA introduced by Todorčević in [Tod02] - and the existence of a precipitous Ideal on ω_1. It was shown by Claverie and Schindler in [CS] that this implies projective determinancy. When we first began working on this problem as a part of this thesis, the goal was to derive $AD^{L(\mathbb{R})}$ from these same hypotheses. Regrettably we are so far only able to prove this using additional assumptions.

The central method we use to extract consistency strength from these statements is the core model induction. Originally introduced by Woodin it was later streamlined by Steel. In this thesis we will use [SS] as our main reference.

Broadly speaking the core model induction is a method to propagate determinancy along the levels of the Jensen-hierarchy of $L(\mathbb{R})$. Though this is obviously a statement about sets of reals, we will be almost exclusively dealing with objects of inner model theory, so called mice. We can do this by using results by Martin,Neeman,Steel,Woodin and others, e.g. [MS89] or [Nee95], establishing a deep connection between the existence of certain mice and the determinancy of appropriate point classes.

The core model induction has been applied to both forcing axioms (cf. [Ste05]) and certain strong ideal properties (cf. [Ket00]), but the study of the conjunction of such properties is still relatively new. Closest comes perhaps [SZ] where the saturation of the non-stationary Ideal is used in conjunction with a stationary set reflection property. Unfortunately those hypotheses cannot be compared with ours, because while we only use proper forcing axioms instead of stationary set preserving forcing axioms like they do, their result is local while ours is global. It might be worthwile to try to localize the argument in this thesis.

We shall now proceed with a short summary of part 1. In chapter 1 we will introduce all the basic concepts used throughout this part of the thesis. We will discuss condensation properties of iteration strategies and mouse operators and introduce core models closed under certain operators. In chapter 2 we will prove reflection properties. Using PFA($\aleph_2$) we will show that both mouse operators and iteration strategies extend from H_{ω_2} over the whole set theoretic universe. Chapter 3 by far is the most important. Here we will give the argument that closure of the universe under one operator can be used to prove closure under the next. This chapter also introduces us to the problem that at certain points of the induction a precipitous ideal alone might not be enought to prove closure of H_{ω_2} under the appropriate operator. In chapter 3 we show how to get around this by assuming significantly stronger properties of the ideal. In chapter 4 we will then introduce a combinatorial principle that implies the same reflection, and which we then prove consistent with forcing axioms. In chapter 5 then we introduce an idea that might do away with the need for such additional principle altogether. By working in the universe after the collapse of ω_1 we can sidestep all issues with reflection. This of course introduces other problems, but we present an argument that at least a significant amount of determinancy can be reached that way.

The obvious next step would be to extend this result to all of $L(\mathbb{R})$. We are optimistic that this can be done.

Our goal in the second part will be to construct a Namba-like forcing on the succesor of a regular cardinal κ. The Namba forcing (cf. [Nam71]) is a partial order that will change the cofinality of ω_2 to ω without collapsing ω_1. Given a regular cardinal $\geq \omega_2$ our task is then to construct a forcing that will change the cofinality of κ^+ without collapsing or changing the cofinality of any cardinal $\leq \kappa$.

Core model theory tells us, that this can in fact not be done without involving measurable cardinals (cf. [Cox09b]). So the natural starting point is κ and an appropriate measurable cardinal $\mu > \kappa$. The first step must be to collapse μ to be κ^+, but the obvious next step of using Prikry forcing (cf. [Pri]) must lead to catastrophe.

It was at a small set theory workshop in Amsterdam that the question came up in a conversation between the author, Ralf Schindler and Peter Koepke. That is where the idea intertwining the Levy collapse and Prikry forcing to solve this problem first came up. Shortly after this conference the author received an e-mail from Peter Koepke, which contained a way to construct the Namba-like forcing under certain assumptions and in the case of countable cofinality.

From there we set out to construct the Namba-like forcing for uncountable cofinality also. For this we had to exchange the Prikry forcing for some partial order that would give certain cardinals uncountable cofinality. For this we used a forcing construction of Gitiks from [Git86].

Let us now give a short summary of part 2. In chapter 6 we mostly introduce the notation for part 2. In chapter 7 we have written up the construction in the case of countable cofinality. We introduce the Koepke forcing and show that we can factorize it in two ways. Basically a "collapse first" way and a "Prikry first" way. Using the first type of factorization we show that the Namba-like forcing exists in any Levy-generic extensions. And the second type of factorization is used to prove that the Koepke forcing is very well behaved. In chapter 8 we introduce the concepts that underlie Gitik's forcing construction, because we will need to refer to them in chapter 9, where we introduce a forcing construction that we then show factorizes in the same way the Koepke forcing did, but this time it contains Gitik's forcing instead of Prikry forcing, so the cofinality of μ will become uncountable. In chapter 10 we deal with the problems arising in iterating the Namba-like forcing. We also see that there can be no Namba-like forcing on two subsequent cardinals in universes without strong inner models. Chapter 11 stands out a bit, because we will actually work in those strong models there. We will see that in those models there are in fact plenty of Namba-like forcings, and we will explore the connection between the existence of strong Namba-like forcings and strong stationarity principles.

Contents

Part I.

On the strenth of $\mathrm{PFA}(\aleph_2)$ in conjunction with a precipitous Ideal on ω_1

1. Introduction

Theorem 1.1: *Assume $PFA(\aleph_2)$. Let I be a precipitous ideal on ω_1.*

(a) Then $J_{\omega_1}(\mathbb{R}) \models AD$;

(b) assume furthermore that I is presaturated and that $\mathcal{P}(\omega_1)/I$ is homogeneous, then $L(\mathbb{R}) \models AD$.

This extends earlier work by Claverie and Schindler (see [CS]).

The rest of this chapter will be devoted to an exposition of the concepts that lie at the heart of this part of the thesis. Chapter 2 will be devoted to reflection lemmata we will have to rely on in the proof of our main theorem. Chapter 3 will then contain the proof of our main theorem. In chapter 4 we shall discuss an alternate hypothesis from which the same result can be obtained with a minimally different proof. In chapter 5 we will sketch a more involved argument, which will give us inductive determinancy without invoking any extra hypotheses, and which we believe can be extended to eventually reach full determinancy.

Iteration trees

We shall exclusively use the Mitchell-Steel definition of premouse (see [MS94]). Premice will usually be designated $\mathcal{M}$ or $\mathcal{N}$.

By $r\Sigma_n$ we will refer to *fine structural* formulae, i.e. φ is $r\Sigma_{n+1}$, iff it is Σ_1 in a predicate coding the $r\Sigma_n$-theory of the premouse with parameters restricted to ordinals below an appropriate n-th projectum $\rho_n(\cdot)$ and an n-th standard parameter $p_n(\cdot)$.

Definition 1.2: Let $\mathcal{M}$ and $\mathcal{N}$. Let $\pi : \mathcal{N} \to \mathcal{M}$ be a function.

(a) π is called an n-embedding, iff it is $r\Sigma_{n+1}$-elementary. We shall write $\pi : \mathcal{N} \to_{n+1} \mathcal{M}$.

(b) π is called a weak n-embedding, iff it is $r\Sigma_n$-elementary everywhere and $r\Sigma_{n+1}$-elementary for a cofinal set of points in $\mathcal{N}$.

By $\mathrm{Hull}_n^{\mathcal{M}}(x)$ we will refer to the *collapse* of the $r\Sigma_n$-hull of x inside of $\mathcal{M}$.

All iteration trees appearing in this paper will be stacks of normal trees, i.e. we play the iteration in rounds such that during each round I has to play extenders of increasing length. We will allow truncations at the start of a new round, but will count any nontrivial truncation as a drop.

Iteration trees will usually be designated $\mathcal{T}$ or $\mathcal{U}$.

For an iteration tree $\mathcal{T}$ we write $D^{\mathcal{T}}$ for the set of drops (in model) and $\mathcal{M}_\alpha^{\mathcal{T}}$ for the α-th model of $\mathcal{T}$ and $E_\alpha^{\mathcal{T}}$ for the α-th extender used. If $\alpha \leq_{\mathcal{T}} \beta$ and $D^{\mathcal{T}} \cap (\alpha, \beta]_{\mathcal{T}} = \emptyset$, then we will write $i_{\alpha,\beta}^{\mathcal{T}}$ for the iteration embedding from $\mathcal{M}_\alpha^{\mathcal{T}}$ into $\mathcal{M}_\beta^{\mathcal{T}}$.

If $b \subseteq \mathcal{T}$ is a maximal branch, such that $D^{\mathcal{T}}$ is bounded in b, then we will write $\mathcal{M}_b^{\mathcal{T}}$ for the direct limit along b and $i_b^{\mathcal{T}} : \mathcal{M}_0^{\mathcal{T}} \to \mathcal{M}_b^{\mathcal{T}}$ for the direct limit embedding, if it exists, i.e $b \cap D^{\mathcal{T}} = \emptyset$.

An iteration strategy Σ on a premouse $\mathcal{M}$ is a partial function picking a cofinal well-founded branch for each $\mathcal{T} \in \mathrm{dom}(\Sigma)$. A tree $\mathcal{T} = \langle \leq_{\mathcal{T}}, \ldots \rangle$ is according to Σ, iff $\mathcal{T} \restriction \gamma \in \mathrm{dom}(\Sigma)$ and $[0, \gamma)_{\mathcal{T}} = \Sigma(\mathcal{T} \restriction \gamma)$ for all limit $\gamma < \mathrm{lh}(\mathcal{T})$.

Definition 1.3: Let $n \leq \omega$. An iteration tree $\mathcal{T}$ is n-maximal, iff for all $\alpha+1 < \mathrm{lh}(\mathcal{T})$ the following holds: let $\kappa := \mathrm{crit}(E_\alpha^{\mathcal{T}})$, let β be the predecessor of $\alpha+1$, let $\mathcal{M}^*$ be the longest initial segment of $\mathcal{M}_\beta^{\mathcal{T}}$ such that $\mathcal{P}(\kappa) \cap \mathcal{M}^* = \mathcal{P}(\kappa) \cap \mathcal{M}_\alpha^{\mathcal{T}}$ and let $k \leq \omega$ be maximal such that $\rho_k(\mathcal{M}^*) > \kappa$, then $\mathcal{M}_{\alpha+1}^{\mathcal{T}} = \mathrm{Ult}_m(\mathcal{M}^*; E_\alpha^{\mathcal{T}})$, where either $D^{\mathcal{T}} \cap [0, \beta]_{\mathcal{T}} = \emptyset$ and $m = \min\{n, k\}$ or $D^{\mathcal{T}} \cap [0, \beta]_{\mathcal{T}} \neq \emptyset$ and $m = k$.

Definition 1.4: Let $\mathcal{M}$ be a premouse and Σ an iteration strategy on $\mathcal{M}$.

(a) We say Σ is a (n, γ)-iteration strategy, iff Σ is total on all n-maximal trees of length less than γ, which are according to Σ.

(b) We say Σ is a no-drop-(n, γ)-iteration strategy, iff Σ is total on all n-maximal trees of length less than γ, which are according to Σ and do not drop, i.e. $D^{\mathcal{T}} = \emptyset$.

We will now introduce two condensation properties for iteration strategies.

Definition 1.5: Let $\mathcal{N}, \mathcal{M}$ be premice and let $\mathcal{U}, \mathcal{T}$ be iteration trees on $\mathcal{N}$ and $\mathcal{M}$ respectively. We say $(\mathcal{N}, \mathcal{U})$ is a hull of $(\mathcal{M}, \mathcal{T})$ iff there is some $\sigma : \mathrm{lh}(\mathcal{U}) \to \mathrm{lh}(\mathcal{T})$ and $\langle \pi_\xi : \xi < \mathrm{lh}(\mathcal{U}) \rangle$ such that

- $\forall n \leq \omega$ ($\mathcal{U}$ is n-maximal $\Leftrightarrow \mathcal{T}$ is n-maximal),
- $\alpha \leq_{\mathcal{U}} \beta \Leftrightarrow \sigma(\alpha) \leq_{\mathcal{T}} \sigma(\beta)$ and $\sigma(0) = 0$,
- $D^{\mathcal{U}} \cap (\alpha, \beta]_{\mathcal{U}} = \emptyset \Leftrightarrow D^{\mathcal{T}} \cap (\sigma(\alpha), \sigma(\beta)]_{\mathcal{T}} = \emptyset$
- $\pi_\xi : \mathcal{M}_\xi^{\mathcal{U}} \to \mathcal{M}_{\sigma(\xi)}^{\mathcal{T}}$ is a $\deg^{\mathcal{U}}(\xi)$-embedding,
- $\pi_\alpha \restriction \mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1 = \pi_\beta \restriction \mathrm{lh}(E_\alpha^{\mathcal{T}}) + 1$ for all $\alpha < \beta$ in the same normal component,
- if $\alpha \leq_{\mathcal{U}} \beta$ and $(\alpha, \beta]_{\mathcal{U}} \cap D^{\mathcal{U}} = \emptyset$, then $\pi_\beta \circ i_{\alpha,\beta}^{\mathcal{U}} = i_{\sigma(\alpha),\sigma(\beta)}^{\mathcal{T}} \circ \pi_\alpha$,
- if $\alpha = \mathrm{pred}_{\mathcal{U}}(\beta+1)$, then $\sigma(\alpha) = \mathrm{pred}_{\mathcal{T}}(\sigma(\beta+1))$ and $\pi_{\beta+1}([a, f]_{E_\beta^{\mathcal{U}}}) = [\pi_\beta(a), \pi_\alpha(f)]_{E_{\sigma(\beta)}^{\mathcal{T}}}$,
- if $\mathcal{U}' \subseteq \mathcal{U}$ is a normal component, then σ" $[\mathcal{U}']$ is contained in a normal component of $\mathcal{T}$.

Examples 1.6:

(a) Let $\mathcal{T}$ be an iteration tree on the premouse $\mathcal{M}$. Let θ be sufficiently big, and let $\pi : H \to H_\theta$ - H transitive - be fully elementary with $\mathcal{M}, \mathcal{T}$ in the range of π, say $\pi(\bar{\mathcal{M}}) = \mathcal{M}$ and $\pi(\bar{\mathcal{T}}) = \mathcal{T}$. Then both $(\bar{\mathcal{M}}, \bar{\mathcal{T}})$ and $(\mathcal{M}, \pi\bar{\mathcal{T}})$ are hulls of $(\mathcal{M}, \mathcal{T})$.

(b) Let $j : V \to M$ be elementary. Let $\mathcal{T}$ be an iteration tree on the premouse $\mathcal{M}$. Assume that $j \upharpoonright \mathcal{T} \in M$, then both $(\mathcal{M}, \mathcal{T})$ and $(j(\mathcal{M}), j\mathcal{T})$ are hulls of $(j(\mathcal{M}), j(\mathcal{T}))$ inside of M.

When there is no confusion about the base model of iteration trees $\mathcal{T}, \mathcal{U}$, we will often just say "$\mathcal{U}$ is a hull of $\mathcal{T}$" instead of "$(\mathcal{M}, \mathcal{U})$ is a hull of $(\mathcal{M}, \mathcal{T})$".

Definition 1.7: An iteration strategy Σ on the premouse $\mathcal{M}$ has hull condensation, iff the following holds: let $\mathcal{U}, \mathcal{T}$ be iteration trees on $\mathcal{M}$ such that $\mathcal{T}$ is according to Σ and $(\mathcal{M}, \mathcal{U})$ is a hull of $(\mathcal{M}, \mathcal{T})$, then $\mathcal{U}$ is according to Σ.

Definition 1.8: An iteration strategy Σ on the premouse $\mathcal{M}$ has branch condensation, iff the following holds: let $\mathcal{T}$ be an iteration tree on $\mathcal{M}$ according to Σ with last model $\mathcal{M}_\theta^\mathcal{T}$ such that the tree embedding $i^\mathcal{T} : \mathcal{M} \to \mathcal{M}_\theta^\mathcal{T}$ exists, let $\mathcal{U}$ be an iteration tree on $\mathcal{M}$ of limit length according to Σ, let b be a cofinal wellfounded branch through $\mathcal{U}$ such that $b \cap D^\mathcal{U} = \emptyset$. If there is some Σ_1-elementary embedding $\pi : \mathcal{M}_b^\mathcal{U} \to \mathcal{M}_\theta^\mathcal{T}$ such that the diagram

$$\begin{tikzcd} \mathcal{M}_\theta^\mathcal{T} & & \mathcal{M}_b^\mathcal{U} \arrow[ll, "\sigma"'] \\ & \mathcal{M} \arrow[ul, squiggly, "i^\mathcal{T}"] \arrow[ur, squiggly, "i_b^\mathcal{U}"'] & \end{tikzcd}$$

commutes, then $b = \Sigma(\mathcal{U})$.

It is easy to see, that branch condensation implies hull condensation for non-dropping trees. The author is not aware of any strategies with branch condensation but not hull condensation, the converse though seems possible. It appears reasonable to consider branch condensation to be a strengthening of hull condensation.

We shall now prove two technical but easy lemmas.

Lemma 1.9: *Let Σ be an (n, ω_1)-iteration strategy with hull condensation on the countable premouse $\mathcal{M}$. Then an iteration tree $\mathcal{T}$ is according to Σ, iff all its countable hulls are according to Σ.*

PROOF: The "$\Rightarrow$" direction is trivial. For the "$\Leftarrow$" direction it suffices to prove the following: Let $\mathcal{T}$ be according to Σ, let b be a cofinal wellfounded branch through $\mathcal{T}$, such that all hulls of $\mathcal{T}^\frown\mathcal{M}_b^\mathcal{T}$ are according to Σ, then $b = \Sigma(\mathcal{T})$.

Assume not: Fix a tree $\mathcal{T}$ and b as above such that $b \neq \Sigma(\mathcal{T})$. Let $X \prec H_\theta$ be countable, with $b, \Sigma(\mathcal{T}), \mathcal{T}$ all in X. Let $\pi : X \to H$ be the transitive collapse.

$\pi(\mathcal{T}^\frown\mathcal{M}_b^\mathcal{T})$ is then a hull of $\mathcal{T}^\frown\mathcal{M}_b^\mathcal{T}$ and thus by assumption is according to Σ, i.e. $\pi(b) = \Sigma(\pi(\mathcal{T}))$.

By a similar argument using hull condensation we have $\pi(\Sigma(\mathcal{T})) = \Sigma(\pi(\mathcal{T}))$. But then $\pi(b) = \pi(\Sigma(\mathcal{T}))$, which implies $b = \Sigma(\mathcal{T})$. Contradiction! $\dashv$

Lemma 1.10: *Let Σ be an (n,ω_1)-iteration strategy with hull condensation on the countable premouse $\mathcal{M}$. Assume that the restriction of Σ to countable trees has branch condensation, then Σ has branch condensation.*

Proof: Assume not: Let $\mathcal{U}$ be a tree according to Σ and let b be some cofinal wellfounded branch such that there is $\mathcal{M}^*$ some iterate of $\mathcal{M}$ by Σ - say $\mathcal{M}^*$ is the last model of the iteration tree $\mathcal{T}$ - and some σ, which embeds $\mathcal{M}_b^{\mathcal{U}}$ into $\mathcal{M}^*$ such that $\sigma \circ i_b^{\mathcal{U}} = i^{\mathcal{T}}$, but $b \neq \Sigma(\mathcal{U})$.

Let $X \prec H_\theta$ be countable, with $\mathcal{T}, \mathcal{U}, b, \sigma, \Sigma(\mathcal{U})$ all in X. Let $\pi : X \to H$ be the transitive collapse. By hull condensation, we then have that $\pi(\mathcal{M}^*)$ is an Σ-iterate of $\mathcal{M}$ and that $\pi(\mathcal{U})$ is according to Σ. Also $\pi(\Sigma(\mathcal{U})) = \Sigma(\pi(\mathcal{U}))$.

By elementarity $\pi(\mathcal{M}_b^{\mathcal{U}}) = \mathcal{M}_{\pi(b)}^{\pi(\mathcal{U})}$ is embeddable back into $\pi(\mathcal{M}^*)$ by some $\bar{\sigma}$ such that $\bar{\sigma} \circ i_{\pi(b)}^{\pi(\mathcal{U})} = i^{\pi(\mathcal{T})}$, but then by hypothesis $\pi(b) = \Sigma(\pi(\mathcal{U}))$, which implies $b = \Sigma(\mathcal{U})$. Contradiction! ⊣

Mouse operators

We will now give the definition of mouse we will use throughout this thesis.

Definition 1.11: Let $\mathcal{M}$ be a premouse.

(a) $\mathcal{M}$ is (n,γ)-iterable, iff there is some (n,γ)-iteration strategy on $\mathcal{M}$.

(b) $\mathcal{M}$ is a mouse, iff $\mathcal{N}$ is $(n,\omega+1)$-iterable for all countable weak n-embeddings $\pi : \mathcal{N} \to \mathcal{M}$.

Remark: For the rest of the thesis we shall ignore the "n" and just refer to γ-iteration strategys and γ-iterable premice, as this much attention to detail will frankly never matter for what is to follow. If there is a drop (in model or degree) at the beginning of an iteration tree, there isn't even any functional difference between say 1-maximal or 2-maximal and the majority of iteration trees in this thesis will be of this kind.

The same definition also applies to premice build above a self-wellordered set A.

Let x be a set. Write:

$$\mathcal{C}_x := \{y | x \in L_1(y) \text{ and } y \text{ is self-wellordered } \}$$

We call $\mathcal{C}_x$ the cone above x.

We say a relativized premouse is above x, iff it is an A-premouse, where $A \in \mathcal{C}_x$.

Definition 1.12: A mouse operator (defined on a cone) above x is a partial function $M : \mathcal{C}_x \to V$ such that for all $A \in \mathrm{dom}(M)$ $M(A)$ is an A-mouse, that is sound above A, i.e. $\mathrm{Hull}_1^{M(A)}(A) = M(A)$.

The canonical example for a mouse operator would be $M_n^{\#}$ for some natural number. In fact all the mouse operators appearing in this thesis will be $M_n^{\#}$-like. We will now make precise, what we mean by that.

Definition 1.13: Let M be a mouse operator above x.

(a) M condenses well, iff for all $A \in \mathrm{dom}(M)$ and all $\pi : \mathcal{M} \to_1 M(A)$ with $x \in \mathcal{M}$ and $\pi \upharpoonright \mathrm{tc}(\{x\}) = \mathrm{id}$ $\pi^{-1}(A) \in \mathrm{dom}(M)$ and $\mathcal{M} = M(\pi^{-1}(A))$.

(b) M relativizes well, iff there is a formula $\varphi(a, b, c, d)$ such that for all $A, B \in \mathrm{dom}(M)$ with $B \in L_1(A)$ and all ZFC$^-$-models N with $M(A) \in N$ $M(B) \in N$ and $M(B)$ is the unique a with $N \models \varphi(a, x, B, M(A))$.

(c) M determines itself on generic extensions, iff for all ν such that M is total on $\mathcal{C}_x \cap H_\nu$ and all partial orders $\mathbb{P} \in H_\nu$, M extends to a mouse operator defined on a cone above x, that is total on $\mathcal{C}_x \cap H_\nu^{V^{\mathbb{P}}}$.

(d) M is called nice, iff it condenses well, relativizes well and determines itself on generic extensions.

Let $\varphi(a, b)$ be a Σ_1-formula. We call an A-premouse $\mathcal{M}(A)$ (φ, x)-small, iff $x \in \mathcal{M}(A)$ and $\mathcal{M}(A) \not\models \varphi(A, x)$.

Definition 1.14: A mouse operator M defined on a cone above x is extraordinarily nice, iff M is nice and there is some Σ_1-formula $\varphi(a, b)$ such that for all A $M(A)$ is the least sound A-mouse, which is not (φ, x)-small.

Let $\mathcal{M}$ be a premouse and $\mathcal{T}$ a normal iteration tree on $\mathcal{M}$ of limit length. Write $\delta(\mathcal{T}) := \sup\limits_{\alpha < \mathrm{lh}(\mathcal{T})} \mathrm{lh}(E_\alpha^\mathcal{T})$ and $\mathcal{M}(\mathcal{T}) := \bigcup\limits_{\alpha < \mathrm{lh}(\mathcal{T})} \mathcal{M}_\alpha^\mathcal{T} | \mathrm{lh}(E_\alpha^\mathcal{T})$. Let b be a cofinal wellfounded branch through $\mathcal{T}$.

We say b has a Q-structure, iff $\rho_\omega(\mathcal{M}_b^\mathcal{T}) < \delta(\mathcal{T})$ or there is some initial segment $\mathcal{P}$ of $\mathcal{M}_b^\mathcal{T}$ with

$$\mathcal{P} \models \delta(\mathcal{T}) \text{ is not Woodin .}$$

In this case we define $\mathcal{Q}_b^\mathcal{T}$ as the largest initial segment of $\mathcal{M}_b^\mathcal{T}$ extending $\mathcal{M}(\mathcal{T})$ in which $\delta(\mathcal{T})$ is Woodin. $\mathcal{Q}_b^\mathcal{T}$ is then called a Q-structure for b.

If there is no Q-structure for b, then $Q_b^\mathcal{T}$ is not defined. Thus if we are refering to $Q_b^\mathcal{T}$ in any context, this is meant to imply that there is some Q-structure.

Note that there is at most one cofinal welfounded branch with a sufficiently iterable Q-structure, as long as the iteration doesn't reach a non-tame mouse. We can and will silently assume that there are no non-tame mice throughout this thesis.

Definition 1.15: Let M be a mouse operator above x. Let $\mathcal{M}$ be a premouse above x and Σ an iteration strategy on $\mathcal{M}$. Σ is guided by M iff, for all iterates $\mathcal{N}$ of $\mathcal{M}$ by Σ and all normal trees $\mathcal{T}$ on $\mathcal{N}$ according to Σ such that $\Sigma(\mathcal{T})$ exists, $\mathcal{M}(\mathcal{T}) \in \mathrm{dom}(M)$ and $\mathcal{Q}_{\Sigma(\mathcal{T})}^\mathcal{T} \trianglelefteq M(\mathcal{M}(\mathcal{T}))$.

Remark 1.16: Let Σ be an iteration strategy on a premouse above x. If Σ is guided by the nice mouse operator above x M, then it has hull condensation.

The basis for the core model induction lies in the fact, that $M_{n+1}^{\#}$'s unique iteration strategy is guided by $M_n^{\#}$. The importance of having guided strategies will be demonstrated in the next lemma. It or more precisely its proof will be our most useful tool in proving iterability hypotheses.

Lemma 1.17: *Let M be a nice mouse operator above some real x, which is total on $\mathcal{C}_x \cap H_\kappa$. Assume that the ω_1-iteration strategy Σ on the countable premouse $\mathcal{M}$ above x is guided by M. Then Σ extends to a κ-iteration strategy guided by M.*

Proof: Let $\mathcal{T}$ be an iteration tree according to Σ on $\mathcal{M}$ of length $<\kappa$. W.l.o.g assume that $\mathcal{T}$ is normal; note that $M(\mathcal{M}(\mathcal{T}))$ exists. Take some countable $X \prec H_\theta$ with $\mathcal{M}, \mathcal{T}, M(\mathcal{M}(\mathcal{T})) \in X$. Let $\pi : X \to H$ be the transitive collapse.

Because M condenses well $\pi(M(\mathcal{M}(\mathcal{T}))) = M(\mathcal{M}(\pi(\mathcal{T})))$. By assumption $b := \Sigma(\pi(\mathcal{T}))$ exists and is the unique cofinal wellfounded branch through $\pi(\mathcal{T})$ such that $\mathcal{Q}_b^{\pi(\mathcal{T})} \trianglelefteq M(\mathcal{M}(\pi(\mathcal{T})))$. Consider now the formula

$$\exists b\, \mathcal{Q}_b^{\pi(\mathcal{T})} \trianglelefteq M(\mathcal{M}(\pi(\mathcal{T}))).$$

This is Σ_1^1 in codes for $\pi(\mathcal{T})$ and $M(\mathcal{M}(\pi(\mathcal{T})))$, so by absoluteness there is such a branch in $H[h]$, where $h \subset \mathrm{Col}(\omega, \delta(\pi(\mathcal{T})))$ is generic over H.

But the branch is unique and thus $b \in H[h]$. Because this does not depend on the choice of the generic, we in fact have $b \in H$. So then H believes, that there is a cofinal wellfounded branch with a Q-structure below $M(\mathcal{M}(\pi(\mathcal{T})))$ through $\pi(\mathcal{T})$. By elementarity we are done. $\dashv$

F-mice

Definition 1.18: Let A be a self-wellordered set. A model operator (defined on a cone above) A is a partial function $F : \mathcal{C}_A \to V$ such that each $\mathcal{M} \in \mathrm{dom}(F)$ is a transitive amenable rudimentarily closed model of the form $(M; \in, A, E, B, S)$ and $F(\mathcal{M})$ is a transitive amenable rudimentarily closed model of the form $(N; \in, A, E, B')$ with $F(\mathcal{M}) = \mathrm{Hull}_1^{F(\mathcal{M})}(M \cup \{M\})$.

Note that the models appearing in this part of the thesis will admit a fine structure, so the use of "Hull" makes sense in this context.

Definition 1.19: Let F be a model operator over A. A potential F-premouse over x - where x is over A - is a transitive amenable rudimentarily closed structure

$$\mathcal{M} = (M; \in, A, \vec{E}, B, \langle \mathcal{M}_i : i < \theta \rangle)$$

such that

(i) for all $i < \theta$ $\mathcal{M}_i$ is a transitive amenable rudimentarily closed structure of the form $(M_i; \in, A, \vec{E}_i, B_i, \langle \mathcal{M}_j : j < i \rangle)$; (From now on also write $\mathcal{M}_\theta := \mathcal{M}$ and $M_\theta := M$.)

(ii) $\vec{E} = (E_\alpha : \alpha \in \mathrm{dom}(\vec{E}))$ codes a fine extender sequence (see [MS94]). Furthermore for all $i < \theta$ $\vec{E}_i = (E_\alpha : \alpha \in \mathrm{dom}(\vec{E} \cap (i+1))$;

(iii) $\mathcal{M}_0 = (\mathrm{tc}(x \cup \{x\}); \in, A, \emptyset, \emptyset, \emptyset)$;

(iv) if $i+1 \leq \theta$, then $F(\mathcal{M}_i) = (M_{i+1}; \in, A, \vec{E}_{i+1}, B_{i+1})$;

(v) if $\lambda \leq \theta$ is a limit, then $\mathcal{M}_\lambda = (\bigcup_{i<\lambda} M_i, \vec{E}_\lambda, \emptyset, \langle \mathcal{M}_i : i < \lambda \rangle)$.

As it stands the hull of a potential F-premouse need not be a potential F-premouse, which is why we will restrict ourselves to operators that condense well.

Definition 1.20: A model operator F over A condenses well, iff for all $\mathcal{M} \in \mathrm{dom}(F)$ and all $\pi : \mathcal{N} \to_1 F(\mathcal{M})$ such that $\pi^{-1}(\mathcal{M})$ is over A and $\pi \restriction A \cup \{A\} = \mathrm{id}$ $\mathcal{N} = F(\pi^{-1}(\mathcal{M}))$.

Proposition 1.21: *Let F be a model operator over A that condenses well. Let $\mathcal{M}$ be a potential F-premouse and let $\pi : \mathcal{N} \to_1 \mathcal{M}$ with $\mathcal{N}$ over A and $\pi \restriction A \cup \{A\} = \mathrm{id}$. Then $\mathcal{N}$ is a potential F-premouse.*

With this result it is possible to set up a fine structure for potential F-premice and define solidity,soundness and so on. As usual a F-premouse will be a potential F-premouse all of whose initial segments are sound.

We will also want to talk about F-mice, but there is an important change here.

Definition 1.22: A F-premouse $\mathcal{M}$ is κ-iterable, iff there is an iteration strategy Σ, which is total on trees of length $<\kappa$, which are according to Σ, and whenever $\mathcal{T}$ is an iteration tree on $\mathcal{M}$, which is according to Σ and has a last model $\mathcal{N}$, $\mathcal{N}$ is a F-premouse.

From here on everything we have defined for premice generalizes in a straightforward fashion.

Now that we have given all the basic definitions our reader can feel confident in ignoring them. All the F-mice appearing in this thesis will be sufficiently close to standard mice so that our reader will miss nothing by just considering F-mice as normal mice with additional closure properties, which go down under elementary embeddings and are preserved by iterations.

Notation: Let F be a model operator over A, which condenses well. Let B be a set over A.

(a) If F is total on $\mathcal{C}_A$, then $L^F(B)$ refers to the unique class-size F-mouse over B without any extenders on its sequence; (if F is not total only an initial segment of $L^F(B)$ might exist;)

(b) $F^\#(B)$ refers to the smallest sound active F-mouse over B;

(c) $M_1^F(B)$ refers to the smallest sound F-mouse over B such that $M_1^F(B) \models \exists\delta > \mathrm{rank}(B) : \delta$ is Woodin .

The model operators appearing in this thesis can be sorted into two types. The first type are the model operators from the previous section (modulo perhaps some coding). If M is some mouse operator and $\mathcal{M}$ is a M-mouse, then it can be reorganized as a standard mouse, which happens to be closed under M. It is important though to make a distinction between the extenders sitting on the M-mouse list and the standard mouse list.

A M-mouse might be highly nontrivial as a standard mouse, but be an initial segment of L^M as a M-mouse. Also iterations using extenders not on the F-mouse list won't preserve closure under M. This is desirable though as it will reduce the complexities of our proofs immensely.

The second type of model operator we will refer to as F_Σ defined over $\mathcal{N}$ a (standard) mouse, on which Σ is an iteration strategy. F_Σ codes Σ in such a way, that the universe of any sufficiently strong F_Σ-mouse will be closed under Σ. If Σ has hull condensation, then F_Σ condenses well.

This class of F-mice are collectively known also as *hybrid-mice.* To save ink we will refer to F_Σ-mice just as Σ-mice.

The main use of F-mice is that they allow us to formulate the theory of a F-core model analogous to [Ste96]. In the end we will have a K^F-dichotomy, which will allow us to extend the core model induction to the $L(\mathbb{R})$-hierarchy of pointclasses.

Definition 1.23: Let F be a model operator over the real x, which condenses well and is total on $\mathcal{C}_x$. Let y be a real over x. A $K^{c,F}$-construction over y is a sequence $\langle \mathcal{N}_\xi : \xi \leq \theta \rangle$ of F-premice such that

(a) $\mathcal{N}_0 = (V_\omega \cup \{y\}; \in, x, \emptyset, \emptyset, \emptyset)$;

(b) if $\xi < \theta$ then $\mathcal{N}_\xi$ is solid and we let $\mathcal{C}_\omega(\mathcal{N}_\xi) =: \mathcal{M} = (M; \in, x, \vec{E}, B, \vec{\mathcal{M}})$, then:

  (i) either $\mathcal{M}$ is passive and $\mathcal{N}_{\xi+1} := (M; \in, x, \vec{E}^\frown E, B, \vec{\mathcal{M}})$, where E is some extender cohering with $\mathcal{M}$, which is certified in the sense of [Ste96],

  (ii) or $\mathcal{N}_{\xi+1} := (M'; \in, x, \vec{E}, B', \vec{\mathcal{M}}^\frown \mathcal{M})$, where $F(\mathcal{M}) = (M', \in, , \vec{E}, B')$;

(c) if $\lambda \leq \theta$ is a limit, then $\mathcal{N}_\lambda = (N_\lambda; x, (E^\alpha_\lambda : \alpha \in \mathrm{dom}(\vec{E}_\lambda)), \emptyset, (\mathcal{M}^\beta_\lambda : \beta \in \mathrm{dom}(\vec{M}_\lambda)))$, where $\alpha \in \mathrm{dom}(\vec{E}_\lambda)$ iff $\alpha \in \mathrm{dom}(\vec{E}_\eta)$ for all but boundably many $\eta < \lambda$ and the sequence of the E^α_η is eventually constant, E^α_λ is then this eventual value, analogously for $\vec{\mathcal{M}}_\lambda$.

Lemma 1.24: *Let F be a model operator over the real x, which condenses well and is total on $\mathcal{C}_x$. Let y be a real over x. Let $\langle \mathcal{N}_\xi : \xi \leq \theta \rangle$ be a $K^{c,F}$-construction over y. Let us assume, that $M^F_1(y)$ does not exist. $\mathcal{N}_\xi$ then is a F-mouse for all $\xi \leq \theta$.*

PROOF: The proof of [Ste96] works here as well. The only thing left to check is that following the realizable branch strategy produces F-premice. But any iterate by this strategy can be embedded into one of the $\mathcal{N}_\xi$, which are F-premice, so by condensation the iterate is a F-premice as well. ⊣

By standard arguments ([MS94]) we can then show, that there is a unique maximal $K^{c,F}$-construction over y. The last (class size) model of this construction we will refer to as $K^{c,F}(y)$.

Lemma 1.25 ($\boldsymbol{K^F}$-dichotomy): *Let F be a model operator over the real x, which condenses well and is total on $\mathcal{C}_x$. Let y be a real over x. Then exactly one of the following holds true:*

(a) $M_1^F(y)$ exists;

(b) $K^{c,F}$ exists and is fully iterable, thus the F-closed core model $K^{c,F}$ exists.

The theory of the F-closed core model can be stated in greater generality, but this will be enough for our purposes. See [SS] and [Ste96] for greater detail on core model theory and how it relates to the core model induction.

Bounded proper forcing axioms

We shall make use of the axiom $PFA(\aleph_2)$ in our core model induction, which is a fragment of the full PFA.

Definition 1.26: Let λ be a cardinal. Then $PFA(\lambda)$ holds iff for all proper complete boolean algebras $\mathbb{P}$ and all sequences $\langle A_\xi : \xi < \omega_1 \rangle$ of maximal antichains of $\mathbb{P}$ such that $\mathrm{Card}(A_\xi) \leq \lambda$ for all $\xi < \omega_1$, there is some Filter $G \subset \mathbb{P}$ such that $G \cap A_\xi \neq \emptyset$ for all $\xi < \omega_1$.

Remark: $BPFA \Leftrightarrow PFA(\aleph_1)$.

Instead of this definition, we will use a different formulation, which is analogous to $BPFA$. For that we shall need to quote two lemmas.

Lemma 1.27 (Moore): $BPFA \rightarrow 2^{\aleph_0} = \aleph_2$.

Lemma 1.28 (Todorčević): *$PFA(\lambda)$ holds, iff for every $A \subseteq \lambda$ and every Σ_0-formula φ if some proper poset forces that $\exists x \varphi(x, A)$ holds in some transitive model, then there exist stationarily many $N \prec H_{\omega_3}$ of size $\aleph_1$ such that $A \in N$ and H_{ω_2} satisifies $\exists x \varphi(x, \pi_N(A))$, where π_N is the transitive collapse of N.*

See [Moo05] and [Tod02] respectively.

Lemma 1.29: *Assume $PFA(\aleph_2)$. Let $A \subseteq \mathbb{R}$. Let $\mathcal{L}_A$ be the language of set theory with an added unary predicate symbol $\dot{A}$. Let $\varphi(x, y)$ be Δ_0 in $\mathcal{L}_A$ such that A occurs positively in φ. We then have, that for all proper posets $\mathbb{P}$ and all $\vec{p} \in H_{\omega_2}$:*

$$\langle (H_{\omega_2})^{V^{\mathbb{P}}}; \in, A \rangle \models \exists x : \varphi(x, \vec{p}) \Rightarrow \langle H_{\omega_2}; \in, A \rangle \models \exists x : \varphi(x, \vec{p})$$

Proof: Assume that the left side holds. Because $2^{\aleph_0} = \aleph_2$ we can turn both A and $\mathrm{tc}(\{\vec{p}\})$ into some $A^* \subseteq \omega_2$ using some easy coding.

Reformulating we then get $\langle (H_{\omega_2})^{V^{\mathbb{P}}}; \in, A^* \rangle \models \exists x : \varphi^*(x, A^*)$. By the preceding lemma we can then find some $N \prec H_{\omega_3}$ of size $\aleph_1$ such that $A^* \in N$ and H_{ω_2} satisifies $\exists x \varphi^*(x, \pi_N(A^*))$, where π_N is the transitive collapse of N.

We then have $\pi_N(\vec{p}) = \vec{p}$ and $\pi_N(A) = A \cap N$. Thus $\langle H_{\omega_2}; \in, A \cap N \rangle \models \exists x : \varphi(x, \vec{p})$. Because A occured positively in φ we are done. ⊣

At this point we have assembled all the tools we will need in the course of this part of the thesis. At this point we would like to mention some simple results of Todorčević's, which should provide some context on the strength of $PFA(\aleph_2)$.

The main point here is, that the consistency strength of $PFA(\aleph_2)$ in itself is pretty much trivial, justifying our use of additional hypotheses.

Definition 1.30: An uncountable regular cardinal κ is H_{κ^+}-reflecting, iff for any $a \in H_{\kappa^+}$ and all formula φ if $H_\theta \models \varphi(a)$ for some regular θ then there exist stationarily many $N \prec H_\theta$ of size $<\kappa$ such that $a \in N$ and there is some $\theta' < \kappa$ with $H_{\theta'} \models \varphi(\pi_N(a))$.

It is easy to see, that this is a strengthening of being reflecting. Furthermore H_{κ^+}-reflecting is to reflecting as $PFA(\aleph_2)$ is to $BPFA$.

Lemma 1.31 (Todorčević): *The following theories are equiconsistent:*

(a) ZFC $+PFA(\aleph_2)$*,*

(b) ZFC $+\exists\kappa$ κ *is* H_{κ^+}*-reflecting .*

Lemma 1.32 (Todorčević): *Assume* $0^\#$ *exists. Let* κ *be an indiscernible, then* $L \models$ κ *is* H_{κ^+}*-reflecting .*

Again see [Tod02] for details.

2. Reflection

Lemma 2.1: *Let F be a model operator over a real x. Let $\mathcal{M}$ be an uncountable F-mouse. Assume that there is a club C on $\mathcal{P}_{\omega_2}(\mathcal{M})$ such that for all $X \in C$, there is some F-mouse $\mathcal{N}$ and natural number n with $\mathcal{M}_X \trianglelefteq \mathcal{N}$ and $\rho_{n+1}(\mathcal{N}) < \mathcal{M}_X \cap \mathrm{On}$, where we let $\pi_X : \mathcal{M}_X \to X$ refer to the reversal of the transitive collapse. For every $X \in C$ let $\mathcal{N}_X$ and n_X be minimal with the above properties.*

Then there is a set S stationary in $[\mathcal{M}]^{\omega_1}$ such that $\mathrm{Ult}_{n_X}(\mathcal{N}_X; \pi_X)$ is a F-mouse for all $X \in S$.

Proof (Sketch): We will borrow some ideas from the proof of the covering lemma (see [MS95]). Let C^* be some club on $\mathcal{P}_{\omega_2}(\mathcal{M})$. Let $\langle X_\xi : \xi \leq \omega_1 \rangle$ be an increasing continuous sequence of countable - except for X_{ω_1} naturally - subsets of $\mathcal{M}$ in $C^* \cap C$ such that for all $\xi < \omega_1$ if $\mathrm{Ult}_{n_{X_\xi}}(\mathcal{N}_{X_\xi}; \pi_{X_\xi})$ is not a F-mouse, i.e. there is some hull - indexed as $\langle [a_i, f_i] : i < \omega \rangle$- of it, that does not collapse to a F-mouse, then $\{a_i | i < \omega\} \subseteq X_i$.

Then X_{ω_1} is as wanted. Assume not. Let $[a_i, f_i]$ witness this. We can then find some countable $Z \prec H_\theta$ - θ big enough - containing the sequence $\langle X_i : i \leq \omega_1 \rangle$ such that $Z \cap \mathcal{M} = X_\xi$ for $\xi = Z \cap \omega_1$ and $f_i \in Z$ for all $i < \omega$. Let $\bar{f}_i$ be the image of f_i under the transitive collapse of Z.

We then have, that $\langle [a_i, \bar{f}_i] : i < \omega \rangle$ witnesses that $\mathrm{Ult}_{n_{X_\xi}}(\mathcal{N}_{X_\xi}; \pi_{X_\xi})$ is not a F-mouse, so there are $\langle [b_i, g_i] : i < \omega \rangle$ witnessing this such that the b_i are in X_{ω_1}, but using the map

$$[b_i, g_i] \mapsto \sigma(g_i)((\pi_{X_{\omega_1}})^{-1}(b_i))$$

where $\sigma : H \to Z$ is the reverse of the transitive collapse we can embed the collapse of that hull into $\mathcal{M}$. Contradiction! $\dashv$

Lemma 2.2: *Let M be an extraordinarily nice (hybrid)-mouse operator above the real a, that is total on $H_{\omega_2} \cap \mathcal{C}_a$. Assume $PFA(\aleph_2)$ holds, then M is total on $\mathcal{C}_a$.*

Proof: Let $X \subseteq \kappa$ be arbitrary above a, where $\kappa \geq \aleph_2$. Assume that $M(X)$ does not exist. Then

$$S(X) := \bigcup \{\mathcal{M} | \mathcal{M} \text{ is a sound above } X \text{ } (\Sigma\text{-})\text{mouse over } X\}$$

is (φ, a)-small. Write $\lambda := S(X) \cap \mathrm{On}$.

Claim 1: $\mathrm{cof}(\lambda) \geq \aleph_2$.

Proof of Claim: Assume not. Let $Y \prec S(X)$ be of size at most $\aleph_1$, then $\mathcal{M}_Y$ is a (φ, a)-small (Σ)-mouse over some $\bar{X} \in H_{\omega_2}$. So $M(\bar{X})$ exists and $\mathcal{M}_Y \trianglelefteq M(\bar{X})$ and thus $\mathcal{N}_Y$ and n_Y exist.

By the above lemma we can then find some $Y \prec S(X)$ cofinal in $S(X)$ such that $\mathcal{N} := \mathrm{Ult}_{n_Y}(\mathcal{N}_Y; \pi_Y)$ is a (Σ)-mouse. But then $S(X) \trianglelefteq \mathcal{N}$ and $\mathcal{N}$ is sound above X. Contradiction! □

Define a tree $T_{S(X)}$: the members of $T_{S(X)}$ are (Σ)-mice $\mathcal{Q} \trianglelefteq S(X)$ over X with biggest cardinal $\lambda_{\mathcal{Q}}$, $Q|\lambda \prec S(X)$ and there is some $n < \omega$ such that $\rho_{n+1}(\mathcal{Q}) \leq \kappa$ - the smallest such we shall call $n_{\mathcal{Q}}$.

For $\mathcal{Q}_0, \mathcal{Q}_1 \in T_{S(X)}$ $\mathcal{Q}_0 \leq \mathcal{Q}_1$, iff $n_{\mathcal{Q}_0} = n_{\mathcal{Q}_1} =: n$ and there is some weak n-embedding $\sigma : \mathcal{Q}_0 \to \mathcal{Q}_1$ with $\sigma \restriction \kappa = \mathrm{id}$ and $\sigma(\lambda_{\mathcal{Q}_0}) = \lambda_{\mathcal{Q}_1}$ - the unique such embedding we shall call $\sigma_{\mathcal{Q}_0, \mathcal{Q}_1}$.

Fix now some $G \subset \mathrm{Col}(\omega_1, \lambda)$ generic over V. In $V[G]$ take some club $C \subseteq \lambda$ of ordertype ω_1.

$$T := T_{S(X)} \restriction C := \{\mathcal{Q} \in T_{S(X)} | \lambda_{\mathcal{Q}} \in C\}$$

Note that T is a tree of height ω_1.

Claim 2: *There is no cofinal branch through T (or equivalently $T_{S(X)}$)) in $V[G]$.*

Proof of Claim: Assume not. So there is some cofinal branch say indexed as $\langle \mathcal{Q}_i : i < \omega_1 \rangle$. Let $\langle \mathcal{Q}^*, \sigma_i : i < \omega_1 \rangle := \mathrm{dirlim} \langle \mathcal{Q}_i, \sigma_{\mathcal{Q}_i, \mathcal{Q}_j} : i \leq j < \omega_1 \rangle$.

Note that for any countable $\sigma : \mathcal{Q} \to \mathcal{Q}^*$, there is some $i < \omega_1$ such that $\mathrm{ran}(\sigma) \subseteq \mathrm{ran}(\sigma_i)$ and thus $\sigma_i^{-1} \circ \sigma : \mathcal{Q} \to \mathcal{Q}_i$. So $\mathcal{Q}^*$ is wellfounded and we shall identify it with a transitive structure.

We will now show, that $\mathcal{Q}^*$ only depends on the height of this structure and not necessarily on the choice of the branch. If there were two such structures $\mathcal{Q}^*_0, \mathcal{Q}^*_1$ of equal height, then take countable $\sigma^*_k : \mathcal{Q}_k \to \mathcal{Q}^*_k$ for $k < 2$ such that the $\mathcal{Q}_k$ are two distinct strucutres over some countable $\bar{X}$.

By the above there we can find elements $\mathcal{P}_k$ of $T_{S(X)}$ such that $\mathcal{Q}_k$ is embeddable into $\mathcal{P}_k$. By countable closure both the $\mathcal{Q}_k$ and the embeddings are in V. But then they are in fact (Σ)-mice there, and thus $\mathcal{Q}_0 \trianglelefteq \mathcal{Q}_1$ or vice versa. Contradiction!

By homogeneity we can thus assume $\mathcal{Q}^* \in V$. A similar argument as above shows that Q^* is a (Σ)-mouse in V. But then $S(X) \trianglelefteq \mathcal{Q}^*$ and $\mathcal{Q}^*$ is sound above X contradicting the choice of $S(X)$. □

Now take $\vec{\mathcal{M}} = \langle \mathcal{M}_i, \pi_{i,j} : i \leq j < \omega_1 \rangle$ a directed system of countable models whose direct limit is $S(X)$ inside of $V[G]$. By countable closure the individual models are all in V and are in fact (Σ)-mice. By A we will refer to the set of all countable (Σ)-mice in V.

Let $\mathbb{P}$ be the specializing forcing for T. Let $H \subset \mathbb{P}$ be generic over $V[G]$. $V[G][H]$ is an extensions by a proper poset. Then in $H_{\omega_2}^{V[G][H]}$ the following holds true:

There is some set of Ordinals X, an ordinal λ of uncountable cofinality, a club $C \subset \lambda$ of ordertype ω_1 and a (φ, a)-small premouse S over X such that there is $\vec{\mathcal{M}} = \langle \mathcal{M}_i, \pi_{i,j} :$

$i \leq j < \omega_1\rangle$ a directed system of countable models whose direct limit is S and all of whose individual models are in A and the tree $T_S \restriction C$ of height λ is special.

This is a Σ_1-statement using A as a predicate and it appears positively in that statement. So by Lemma 1.29 the same statement is true in H_{ω_2}. So let us take some $\bar{S}, \bar{\lambda}, \bar{X}, \bar{C}$ as in the statement.

First note that $\bar{S}$ is a (Σ)-mouse, for if $\pi : \mathcal{M} \to \bar{S}$ is countable, then by the argument from the proof of claim 2, $\mathcal{M}$ is already embeddable in one of the $\mathcal{M}_i$ and thus a (Σ)-mouse.

We then must have $\bar{S} \trianglelefteq M(\bar{X})$, so take $\mathcal{Q}^* \trianglelefteq M(\bar{X})$ and n minimal such that $\bar{S} \trianglelefteq \mathcal{Q}^*$ and $\rho_{n+1}(\mathcal{Q}^*) \leq \sup \bar{X}$. Using an argument used in the construction of $\square$-sequences (cf [SZ10]), we can show that there must exist a club of $\mathcal{Q}$'s, which are weakly n-embeddable into $\mathcal{Q}^*$. But this induces a branch through $T_{\bar{S}}$. Contradiction! $\dashv$

Remark: In the proof of the lemma we have used the notation $S(X)$. This was meant to evoke the notion of the stack over some mouse. What we actually worked with may have been closer to what is usually called the lower part closure, but it will be important later that the argument works equally well with the stack as it is usually understood.

Lemma 2.3: *Assume $PFA(\aleph_2)$. Let $\mathcal{N}$ be a countable premouse. Let Σ be an ω_2-iteration strategy with hull condensation. Then Σ extends to an* On*-iteration strategy Γ with hull condensation.*

Proof: We shall define Γ as the unique iteration strategy such that a tree is according to Γ iff all its countable hulls are according to Σ. By Lemma 1.9 this extends Σ.

Claim 1: Γ *has hull condensation.*

Proof of Claim: Assume not. So there is some $\mathcal{T}$ according to Γ and some hull $\mathcal{U}^\frown c$ of $\mathcal{T}^\frown\Gamma(T)$, such that $c \neq \Gamma(U)$. Let then $X \prec H_\theta$, where $\mathrm{lh}(\mathcal{T}) < \theta$, with $\mathcal{U}^\frown c, \mathcal{U}^\frown\Gamma(\mathcal{U}) \in X$. Let $\pi : X \to H$ be the transitive collapse. On the one hand $\pi(\mathcal{U}^\frown c)$ is a hull of $\mathcal{U}^\frown c$ and thus of $\mathcal{T}^\frown\Gamma(\mathcal{T})$ and is therefore according to Σ, on the other hand $\pi(\mathcal{U}^\frown\Gamma(\mathcal{U}))$ is a hull of $\mathcal{U}^\frown\Gamma(\mathcal{U})$ and thus by Σ, but $\pi(c) \neq \pi(\Gamma(\mathcal{U}))$. Contradiction! $\square$

It remains to show, that Γ is total. Let $\mathcal{T}$ be some arbitrary tree according to Γ. Let $\lambda := \mathrm{lh}(\mathcal{T})$. There are two cases:

1st case:
Let $\mathrm{cof}(\lambda) > \omega$. Assume for a contradiction, that $\mathcal{T}$ has no cofinal branch. Let $\varphi_0(\mathcal{T}, \langle \mathcal{T}_i, \sigma_{ij}, \vec{\pi}_{ij} : i \leq j < \omega_1\rangle)$ be the conjunction of the following statements:

- $\forall i \leq j < \omega_1 : \mathcal{T}_i$ is a hull of $\mathcal{T}_j$ as witnessed by $\sigma_{ij}, \vec{\pi}_{ij}$,
- $\forall i \leq j \leq k < \omega_1 : \sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$,
- $\forall i \leq j \leq k < \omega \forall l < \mathrm{lh}(\mathcal{T}_i) : \pi_{jk}^{\sigma_{ij}(l)} \circ \pi_{ij}^l = \pi_{ik}^l$,

- $\forall i < \omega_1 : \mathcal{T}_i$ is according to Σ,
- $\operatorname{dirlim}\langle \mathcal{T}_i, \sigma_{ij}, \vec{\pi}_{ij} : i \leq j < \omega_1 \rangle = \mathcal{T}$.

Working in $V^{\mathrm{Col}(\omega_1,\lambda)}$ fix some club $C \subseteq \lambda$ of ordertype ω_1. Let $\dot{\mathbb{P}}$ refer to the specializing forcing in $V^{\mathrm{Col}(\omega_1,\lambda)}$ for $\mathcal{T} \restriction C$. Let $\varphi_1(\mathcal{T}, C)$ refer to the conjunction of the following statements:

- $\mathcal{T}$ is an iteration tree on $\mathcal{N}$,
- $C \subseteq \mathrm{lh}(\mathcal{T})$ is a club of ordertype ω_1,
- $\mathcal{T} \restriction C$ is special

Then

$$(H_{\omega_2})^{\mathrm{Col}(\omega_1,\lambda)*\dot{\mathbb{P}}} \models \exists \mathcal{T} \exists \vec{\mathcal{T}} \exists C \varphi_0(\mathcal{T}, \vec{\mathcal{T}}) \wedge \varphi_1(\mathcal{T}, C)$$

This is Σ_1 in the language with Σ as a predicate. So by $PFA(\aleph_2)$ this is true in $H_{\omega_2}^V$. So let us fix some $\mathcal{T}^*, C^*$ and $\langle \mathcal{T}_i^*, \sigma_{ij}^*, \vec{\pi}_{ij}^* : i \leq j < \omega_1 \rangle$ that witness this. Let $\sigma_{i\omega_1}$ and $\vec{\pi}_{i\omega_1}$ refer to the direct limit embeddings.

Claim 2: *$\mathcal{T}^*$ is by Γ.*

Proof of Claim: It is enough to show, that all countable hulls of $\mathcal{T}^*$ are by Σ. So let $\bar{\mathcal{T}}$ be some countable hull as witnessed by $\bar{\sigma}, \langle \bar{\pi}^i : i < \mathrm{dom}(\bar{\sigma}) \rangle$. There is then some $j < \omega_1$, such that $\mathrm{ran}(\bar{\sigma}) \subseteq \mathrm{ran}(\sigma_{j\omega_1}^*)$ and $\mathrm{ran}(\bar{\pi}^i) \subseteq \mathrm{ran}((\pi_{j\omega_1}^*)^{(\sigma_{j\omega_1}^*)^{-1}(\bar{\sigma}(i))})$ for all $i < \mathrm{dom}(\bar{\sigma})$. Then $(\sigma_{j\omega_1}^*)^{-1} \circ \bar{\sigma}, \langle (\pi_{j\omega_1}^*)^{(\sigma_{j\omega_1}^*)^{-1}(\bar{\sigma}(i))} \circ \bar{\pi}^i : i < \mathrm{dom}(\bar{\sigma}) \rangle$ witnesses, that $\bar{\mathcal{T}}$ is a hull of $\mathcal{T}_j^*$. But $\mathcal{T}_j^*$ is by Σ and thus so is $\bar{\mathcal{T}}$. $\square$

$\mathcal{T}^*$ is in H_{ω_2} so $\Gamma(\mathcal{T}^*)$ exists and is a cofinal branch. Then $\Gamma(\mathcal{T}^*) \cap C^*$ is cofinal in $\mathcal{T}^* \restriction C^*$, but $\mathcal{T}^* \restriction C^*$ is special. Contradiction!

2nd case:
Let $\mathrm{cof}(\lambda) = \omega$. We will call a countable hull $\bar{\mathcal{T}}$ of $\mathcal{T}$ stable, iff for all countable hulls of $\mathcal{T}$ $\mathcal{U}$, of which $\bar{\mathcal{T}}$ is a hull as witnessed by σ, σ" $\left[\Sigma(\bar{\mathcal{T}})\right] \subset \Sigma(\mathcal{U})$.

Claim 3: *There is a stable hull of $\mathcal{T}$.*

Proof of Claim: Assume not. Then there is a system of hulls $\langle \mathcal{T}_i, \sigma_{ij}, \vec{\pi}_{ij} : i \leq j < \omega_1 \rangle$ such that

- $\forall i \leq j < \omega_1 : \mathcal{T}_i$ is a hull of $\mathcal{T}_j$ as witnessed by $\sigma_{ij}, \vec{\pi}_{ij}$,
- $\forall i \leq j \leq k < \omega_1 : \sigma_{ij} \circ \sigma_{jk} = \sigma_{ik}$,
- $\forall i \leq j \leq k < \omega \forall l < \mathrm{lh}(\mathcal{T}_i) : \pi_{jk}^{\sigma_{ij}(l)} \circ \pi_{ij}^l = \pi_{ik}^l$,
- $\forall i < \omega_1 : \mathcal{T}_i$ is according to Σ,

- $\forall i < \omega_1 : \sigma_{ii+1}" [\Sigma(\mathcal{T}_i)] \not\subset \Sigma(\mathcal{T}_{i+1})$.

Let $\mathcal{T}^* := \mathrm{dirlim}\langle \mathcal{T}_i, \sigma_{ij}, \vec{\pi}_{ij} : i < j < \omega_1 \rangle$ and let $\sigma_{i\omega_1}$ and $\vec{\pi}_{i\omega_1}$ refer to the direct limit embeddings. $\mathcal{T}^*$ is an iteration tree of size $\aleph_1$. An argument like above, shows that $\mathcal{T}^*$ is according to Γ.

Let $b := \Gamma(\mathcal{T}^*)$. There is some $i < \omega_1$, such that for all $i \leq j < \omega_1$ $\mathrm{ran}(\sigma^*_{j\omega_1}) \cap b$ is cofinal in b. Let $b_j := (\sigma^*_{j\omega_1})^{-1}" [b]$. Then $\mathcal{T}_j{}^\frown b_j$ is a hull of $\mathcal{T}^{*\frown} b$ for all $j \geq i$. Thus $b_j = \Sigma(\mathcal{T}_j)$ for all $j > i$. But then b_j is moved correctly by the σ_{ij}. Contradiction! □

Let us fix a stable Hull $\bar{\mathcal{T}}$ of $\mathcal{T}$ as witnessed by $\bar{\sigma}, \vec{\bar{\pi}}$. Let b be the downward closure of $\bar{\sigma}" \left[\Sigma(\bar{\mathcal{T}})\right]$.

Claim 4: *All countable hulls of $\mathcal{T}^\frown b$ are according to Σ.*

Proof of Claim: Let $\mathcal{U}^\frown c$ be some countable hull of $\mathcal{T}$ witnessed by σ. We have to show, that $c = \Sigma(\mathcal{U})$. We can certainly find some countable hull $\mathcal{T}^*$ of $\mathcal{T}$ (witnessed by σ^*), such that both $\bar{\mathcal{T}}$ and $\mathcal{U}$ are hulls of $\mathcal{T}^*$, e.g. by taking the collapse of some substructure of some transitive set containing $\bar{\mathcal{T}}$,$\mathcal{U}$ and $\mathcal{T}$, which knows that the former are hulls of the latter. Because $\bar{\mathcal{T}}$ was stable, σ^* maps $\Sigma(\mathcal{T}^*)$ into b. Thus $\mathcal{U}^\frown c$ is a hull of $\mathcal{T}^{*\frown}\Sigma(\mathcal{T}^*)$ as witnessed by $(\sigma^*)^{-1} \circ \sigma$. But then $c = \Sigma(\mathcal{U})$ by hull condensation of Σ. □

Thus we get that $b = \Gamma(\mathcal{T})$. ⊣

3. The induction

Now assume $PFA(\aleph_2)$ and let I be a precipitous ideal on ω_1. At certain points we will need I to be presaturated and $\mathcal{P}(\omega_1)/I$ to be homogeneous. We shall call out all these points explicitely. Otherwise we will assume no more than precipitousness.

The proof is by core model induction. We will follow the general outline of [SS]. We will write $M_{n,\alpha}$ for the (hybrid-)mouse operators witnessing $W^*_{\alpha+1}$. (Note: $M_{n+1,\alpha} = M_1^{M_{n,\alpha}}$). Write a_α for some real, above which $M_{0,\alpha}$ is defined.

All these operators are extraordinarily nice. It is shown in [SS] that the operators condense and relativize well. We will later give a proof, that they determine themselves on generic extensions.

At this point we should also mention, that in the core model induction we usually only deal with ω_1-iterability, while we have usually demanded $\omega_1 + 1$-iterability. This is not really problematic though, because the way the core model induction works makes sure that we can always transport iterations to inner models, where ω_1^V is inaccessible. And anyway the first thing we shall do once introducing new models is to show, that they are in fact fully iterable.

Our "cycling" argument will have the following outline:

$(0)_\alpha$ W^*_α,

$(1)_{n,\alpha}$ H_{ω_1} is closed under $M^\#_{n,\alpha}$,

$(2)_{n,\alpha}$ H_{ω_2} is closed under $M^\#_{n,\alpha}$,

$(3)_{n,\alpha}$ V is closed under $M^\#_{n,\alpha}$

As an additional hypotheses we will inductively verify the following property of the (hybrid)-mouse operators, which we shall refer to as $(3)'_{n,\alpha}$:

"For all regular uncountable cardinals θ, there are trees $T^\theta_{n,\alpha}, U^\theta_{n,\alpha}$ such that for all partial orders $\mathbb{P} \in H_\theta$

$$\begin{aligned} p\left[T^\theta_{n,\alpha}\right]^{V^{\mathbb{P}}} &= \{(x,y) \in \mathbb{R}^2 | y = M^\#_{n,\alpha}(x)\}, \\ p\left[U^\theta_{n,\alpha}\right]^{V^{\mathbb{P}}} &= \{(x,y) \in \mathbb{R}^2 | y \neq M^\#_{n,\alpha}(x)\}." \end{aligned}$$

We will prove the following:

- $(0)_\alpha \to (1)_{\alpha,0}$,
- $\forall n\ [(1)_{n,\alpha} \to (2)_{n,\alpha} \to (3)_{n,\alpha} \to (1)_{n+1,\alpha}]$,

- $\forall n\ (1)_{n,\alpha} \to (0)_{\alpha+1}$.

Let us fix an I-generic G and let $j : V \to N$ be the ultrapower-embedding.

The steps of our larger overarching induction along the Jensen hierarchy of $L(\mathbb{R})$ divide into two cases. In the first case we will deal exclusively with fine-structural mice. The second case features hybrid-mice.

Standard mice

Let $M_{0,\beta}$ be one of the following three operators:

(a) $\bigoplus_{n<\omega} M_{n,\beta'}$ defined above $a_{\beta'}$, where $\beta' + 1 = \beta$,

(b) $\bigoplus_{n<\omega} M_{0,\beta_n}$ defined above $\bigoplus_{n<\omega} a_{\beta_n}$, where $\langle \beta_n : n < \omega \rangle$ is a sequence of critical ordinals cofinal in β,

(c) the smallest initial segment M of $\mathrm{lp}(\cdot)$, such that for all $\mathrm{Col}(\omega, \cdot)$-generics H and all $n < \omega$, there is some $\gamma_n < M \cap \mathrm{On}$, such that $M||\gamma_n\,[H]$ is a $\langle \varphi_n^*, \sigma_A^H \rangle$-prewitness defined above z. (See [SS] for the exact nature of φ_n^* and z.)

By induction hypothesis, we can assume that $M_{0,\beta}$ is total on H_{ω_1} above a_β.

Lemma 3.1: *$M_{0,\beta}$ is total on V above a_β. This (partially) uses homogeneity and pre-saturatedness.*

Proof: Because of Lemma 2.2 it will be enough to show that $M_{0,\beta}$ is total on H_{ω_2} above a_β. The cases (a) and (b) are extremely similar so we will deal with them at one fell swoop.

1st case:
So let $A \in H_{\omega_2}$ be arbitrary. We then have $A \in N$ and by elementarity $\mathcal{M} := j(M_{0,\beta})(A)$ exists in N. We will show, that $\mathcal{M}$ is a mouse in $V[G]$ and will remain a mouse in all further forcing extensions. Standard arguments will then show that $\mathcal{M}$ is in fact in V and is a mouse there, so in fact $\mathcal{M} = M_{0,\beta}(A)$.

$M_{0,\beta}$ is the amalgamation of countably many mouse operators, which we shall call $\langle N_m : m < \omega \rangle$ for the moment. Using inductive hypothesis $(3)'$ we can assume that these mouse operators have universally baire representations, say the trees $\langle T_m : m < \omega \rangle$ project to codesets for these operators in all possible forcing extensions.

It is easy to see that $j(T_m)$ will project to the same set as T_m (inside of $V[G]$ and any further extension), so N actually knows all of the $\langle N_m : m < \omega \rangle$. Now assume that in some forcing extensions of $V[G]$ there is some (w.l.o.g. countable) bad iteration tree on $\mathcal{M}$. $\mathcal{M}$ will have to drop at the first stage of the game and the iteration must then be guided by N_m for some $m < \omega$.

We can then define a tree searching for such a bad iteration tree with Q-structures below N_m using the tree $j(T_m)$. This tree must then be illfounded in some forcing extension, but by the above the tree will be in N, where there are no bad iteration trees.

Contradiction!

2nd Case:
Let $A \in H_{\omega_2}$ be arbitrary. We then have $A \in N$ and by elementarity $j(M_{0,\beta})(A)$ exists in N. It is furthermore definable in $j(J_\beta(\mathbb{R}))$ as the unique ω_1-iterable mouse of a certain height, which is sound above A. As $j(J_\beta(\mathbb{R})) = (J_{j(\beta)}(\mathbb{R}))^N = (J_{j(\beta)}(\mathbb{R}))^{V[G]}$, we get that $j(M_{0,\beta})(A)$ is OD in $V[G]$ from A, and thus by homogeneity in V. By a similar argument it is futhermore ω_2-iterable in V. $\dashv$

Remark 3.2: In the cases (a) and (b) we didn't need homogeneity or presaturation. And this is also the only time these properties will appear in this section. As all the cases below ω_1 are of this form this will give us the first part of Theorem 1.1.

Lemma 3.3: *$M_{0,\beta}$ is nice.*

Proof: As mentioned at the beginning of this chapter, we will only need to show that $M_{0,\beta}$ extends itself on generic extensions. So let us fix some partial order $\mathbb{P}$ and some θ, such that $\mathbb{P} \in H_\theta$. We want to show, that for all generics $H \subseteq \mathbb{P}$, and all $x \in (H_{\omega_1})^{V[H]}$, $M_{0,\beta}(x)$ exists and is $\omega_1^{V[H]}$-iterable. This will suffice, because it is true for all forcing relations.

We will use a reflection argument. So let us fix some big enough λ, and some countable substructure $X \prec H_\lambda$ with $\mathbb{P}, M_{0,\beta}(H_\theta) \in X$. Let $\pi : X \to H$ be it's transitive collapse. Write $\bar{\mathbb{P}} = \pi(\mathbb{P}), \bar{H} = \pi(H_\theta)$. We then have $\pi(M_{0,\beta}(H_\theta)) = M_{0,\beta}(\bar{H}) \in H$. Let us take some $h \subseteq \bar{\mathbb{P}}$ generic over H.

Claim 1: *Let $x \in \mathbb{R} \cap H\,[h]$. Then $M_{0,\beta}(x) \in H\,[h]$.*

Proof of Claim: As before the cases (a) and (b) are extremely similar:

1st case:
$M_{0,\beta}$ is the amalgamation of countably many mouse operators, which we shall call $\langle N_m : m < \omega \rangle$ for the moment. Using induction assume that the background construction inside of $N_m(\bar{H})\,[h]$ over x reaches $N_m(x)$. But $N_m(\bar{H})\,[h] \in M_{0,\beta}(\bar{H})\,[h]$ for all $m < \omega$ and so the background construction over x in $M_{0,\beta}(\bar{H})\,[h]$ reaches all of them and thus reaches $M_{0,\beta}(x)$.

We should mention here that $M_{0,\beta}(\bar{H})\,[h]$ is iterable in V and thus the background construction will succeed.

2nd case:
Consider $M_{0,\beta}(\bar{H})\,[h]$. First note that $\mathbb{R} \cap H\,[h] \subseteq M_{0,\beta}(\bar{H})\,[h]$.

Furthermore for any $g \subseteq \mathrm{Col}(\omega, \bar{H})$ generic over $M_{0,\beta}(\bar{H})\,[h]$ we can find some $g^* \subseteq \mathrm{Col}(\omega, \bar{H})$ generic over $M_{0,\beta}(\bar{H})$, such that $M_{0,\beta}\,[h]\,[g] = M_{0,\beta}\,[g^*]$.

So for all $n < \omega$ there is some γ_n, such that $M_{0,\beta}\,[h]\,[g]\,||\gamma_n$ is a $\langle \varphi_n^*, \sigma_{\bar{H}}^{g^*} \rangle$-prewitness.

Of course $M_{0,\beta}(\bar{H})\,[h]$ is still iterable (in V), so we do indeed correctly compute $M_{0,\beta}(x)$ inside of it. Thus it is an element of $H\,[h]$. $\square$

So let us fix some $x \in \mathbb{R} \cap H[h]$. By the claim we have $\mathcal{M} := M_{0,\beta}(x) \in H[h]$, but a priori $H[h]$ might not believe it to be ω_1-iterable. Let Σ refer to $\mathcal{M}$'s unique iteration strategy. Let $\mathcal{T} \in H[h]$ be some countable (in $H[h]$) tree according to Σ. We want to show, that $\Sigma(\mathcal{T}) \in H[h]$.

De facto $\mathcal{T}$ has a Q-structure, call it $\mathcal{Q}$. It is easy to see, that $\mathcal{Q} \in M_{0,\beta}(M(\mathcal{T}))$. But the latter is in $H[h]$ by the claim, so $\mathcal{Q} \trianglelefteq H[h]$.

Consider now the statement:

$$\exists b \text{ cofinal branch through } \mathcal{T} : \mathcal{Q} \trianglelefteq M_b^{\mathcal{T}}$$

This is Σ^1_1 in codes for $\mathcal{Q}$ and $\mathcal{T}$. In V there exists such a branch and it is also unique. By absoluteness then, this b is in $H[h]$ and it must be equal to $\Sigma(\mathcal{T})$. ⊣

The proof shows more. Write T^θ_β and U^θ_β for trees searching for

- a countable transitve $\mathcal{P}$,
- an elementary embedding $\sigma : \mathcal{P} \to M_{0,\beta}(H_\theta)$,
- some h, which is generic for some $\mathbb{P} \in \mathcal{P}$ over $\mathcal{P}$ with $\sigma(\mathbb{P}) \in H_\theta$,

such that $y = M_{0,\beta}(x)$ or respectively $y \neq M_{0,\beta}(x)$ as computed from $\mathcal{P}[h]$.

Corollary 3.4: *Let θ be some regular uncountable cardinal. For all $\mathbb{P} \in H_\theta$ and all $H \subseteq \mathbb{P}$ generic over V*

$$\begin{aligned} p\left[T^\theta_\beta\right]^{V[H]} &= \{(x,y) \in \mathbb{R}^2 | y = M_{0,\beta}(x)\}, \\ p\left[U^\theta_\beta\right]^{V[H]} &= \{(x,y) \in \mathbb{R}^2 | y \neq M_{0,\beta}(x)\}. \end{aligned}$$

We can now show $(1)_{0,\beta}$. For that we need:

Lemma 3.5: *$M_{0,\beta} \upharpoonright N = j(M_{0,\beta})$.*

Proof: First let $x \in \mathbb{R}^N$ and let y code $j(M_{0,\beta})(x)$. By elementarity then $(x,y) \in p\left[j(T^\theta_\beta)\right]$. A standard argument shows, that $p^{V[G]}\left[T^\theta_\beta\right] \subset p^{V[G]}\left[j(T^\theta_\beta)\right]$ and $p^{V[G]}\left[U^\theta_\beta\right] \subset p^{V[G]}\left[j(U^\theta_\beta)\right]$.

But $T^\theta_\beta, U^\theta_\beta$ project to complements and the projections of $j(T^\theta_\beta), j(U^\theta_\beta)$ are disjunct, so $p^{V[G]}\left[T^\theta_\beta\right] = p^{V[G]}\left[j(T^\theta_\beta)\right]$ and $p^{V[G]}\left[U^\theta_\beta\right] = p^{V[G]}\left[j(U^\theta_\beta)\right]$.

So $(x,y) \in p^{V[G]}\left[T^\theta_\beta\right]$ and thus y codes $M_{0,\beta}(x)$.

So let $x \in N$ be arbitrary, such that $M_{0,\beta}(x) \neq j(M_{0,\beta})(x)$. The internal definition is certainly satisfied, so it must be a failure of iterability. So in $V[G]$ there is some countable hull of $j(M_{0,\beta})(x)$, which is not iterable.

Formulated differently this means by the above argument, that the tree R, searching for a countable hull of $j(M_{0,\beta})(x)$, which is in $p\left[j(U^\theta_\beta)\right]$ is illfounded. But $R \in N$. Contradiction! ⊣

Looking at some arbitrary real $x \in V$ now, we see that $j : L^{M_{0,\beta}}(x) \to L^{M_{0,\beta}}(x)$ is a non-trivial elementary embedding into itself by the lemma. So in $V[G]$ there is a club class of indiscernibles for $L^{M_{0,\beta}}(x)$. Then there is also a club class of indiscernibles in V, and thus $M^\#_{0,\beta}(x)$ exists.

This gives $(1)_{0,\beta}$. $(2)_{0,\beta}$ follows immediately by the same argument plus elementarity. We then get $(3)_{0,\beta}$ by Lemma 2.2.

So let us now assume $(3)_{n,\beta}$ for some n. Write $T^\theta_{n,\beta}$ and $U^\theta_{n,\beta}$ for trees searching for

- a countable transitve $\mathcal{P}$,
- an elementary embedding $\sigma : \mathcal{P} \to M^\#_{n,\beta}(H_\theta)$,
- some h, which is generic for some $\mathbb{P} \in \mathcal{P}$ over $\mathcal{P}$ with $\sigma(\mathbb{P}) \in H_\theta$,

such that $y = M^\#_{n,\beta}(x)$ or respectively $y \neq M^\#_{n,\beta}(x)$ as computed from $\mathcal{P}[h]$.

As before we get that for all regular uncountable cardinals θ, for all $\mathbb{P} \in H_\theta$ and all $H \subseteq \mathbb{P}$ generic over V

$$\begin{aligned} p\left[T^\theta_{n,\beta}\right]^{V[H]} =& \{(x,y) \in \mathbb{R}^2 | y = M^\#_{n,\beta}(x)\}, \\ p\left[U^\theta_{n,\beta}\right]^{V[H]} =& \{(x,y) \in \mathbb{R}^2 | y \neq M^\#_{n,\beta}(x)\}. \end{aligned}$$

By a previous argument we get:

Lemma 3.6: $M^\#_{n,\beta} \upharpoonright N = j(M^\#_{n,\beta})$.

Let us fix some real $x \in V$ now. We want to show, that $K_\beta(x) := K^{M_{0,\beta}}(x)$ does not exist. Because the universe is closed under $M^\#_{n,\beta}$, this would show that $M^\#_{n+1,\beta}(x)$ exists. Nothing will be lost if we ignore the real parameter x from now on, so we will do so.

For this we need to be able to compare K_β with K^N_β.

They are in the same hierarchy, so the only problem might be a failure of K^N_β to be iterable. So assume for a contradiction, that there is some countable hull $\bar{K}$ of K^N_β and some countable iteration tree $\mathcal{T}$ on $\bar{K}$, such that

- either the last model of $\mathcal{T}$ is illfounded,
- or $\mathcal{T}$ has limit length and there is no cofinal wellfounded branch through $\mathcal{T}$ with a Q-structure below $M^\#_{n,\beta}(M(\mathcal{T}))$.

We can reformulate this as saying, that the tree R searching for some countable hull $\bar{K}$ of K^N_β and some countable tree $\mathcal{T}$ on $\bar{K}$, such that

- either the last model of $\mathcal{T}$ is illfounded,

- or $\mathcal{T}$ has limit length and there is some y, such that $(M(\mathcal{T}), y) \in p\left[j(T^{\theta}_{n,\beta})\right]$ and there is some transitive structure containing y and $\mathcal{T}$ but not a cofinal wellfounded branch through $\mathcal{T}$ with a Q-structure below y.

But $R \in N$. Contradiction!

The same argument can be repeated in all further forcing extensions of $V[G]$, so we get full iterability for K^N_β.

Lemma 3.7: $(\omega_1^+)^{K_\beta} = \omega_2$.

Proof: Assume not. Let $\lambda := (\omega_1^+)^{K_\beta} < \omega_2$. By standard arguments we then have $j \restriction \mathcal{P}(\omega_1) \cap K_\beta \in N$. Let F be the $j(\omega_1)$-extender derived from that.

By [JS] the theory of the core model doesn't need this, but it might be more comfortable in the following argument to pretend, that there is some sufficiently large cardinal Ω up to which the core model is defined. We will not need to make this explicit though.

We would like to show that $F \in K^N_\beta$ and would like to invoke Theorem 8.6. from [Ste96] to that end. Strictly speaking we would then need to work with a fitting very soundness witness. For presentations sake we shall ignore this though and work with K_β instead, which we will assume has the definability property at all ordinals. It should be easy to see that the following argument can be made to fit an very soundness witness instead.

Let $\mathcal{T}$ on K_β and $\mathcal{U}$ on K^N_β be the iteration trees arising in the coiteration of K_β and K^N_β.

By the Dodd-Jensen lemma both must iterate to a common model, call it $\mathcal{Q}$. The Dodd-Jensen lemma is applicable here because K_β has a strategy, which is guided by $M^{\#}_{n,\beta}$.

Claim 1: $\mathrm{crit}(i^{\mathcal{T}}), \mathrm{crit}(i^{\mathcal{U}}) \geq \omega_1$.

Proof of Claim: Let $\Gamma = \{\gamma \in \mathrm{On} \,|\, i^{\mathcal{T}}(\gamma) = i^{\mathcal{U}} \circ j(\gamma)\}$. Γ is thick. We will now show $i^{\mathcal{T}} = i^{\mathcal{U}} \circ j$.

Let us assume there was some minimal ordinal witness to the contrary say α. Using the definabilty property of K_β at α there must be some term τ such that $\alpha = \tau^W(\gamma_0, \ldots, \gamma_k)$, where $\gamma_i \in \Gamma$. But then

$$i^{\mathcal{T}}(\alpha) = \tau^{\mathcal{Q}}(i^{\mathcal{T}}(\gamma_0), \ldots, i^{\mathcal{T}}(\gamma_k)) = \tau^{\mathcal{Q}}(i^{\mathcal{U}}(j(\gamma_0)), \ldots, i^{\mathcal{U}}(j(\gamma_k))) = i^{\mathcal{U}}(j(\alpha))$$

Contradiction!

So if we now assume that one of $i^{\mathcal{T}}, i^{\mathcal{U}}$'s critical points is less than ω_1, then the critical points are equal. Say the value is $\alpha < \omega_1$. For any $a \in [\mathrm{On}]^{<\omega}$ and any $X \subset \alpha$ in K_β we then have

$$a \in i^{\mathcal{T}}(X) \Leftrightarrow a \in i^{\mathcal{U}}(j(X)) \Leftrightarrow a \in i^{\mathcal{U}}(X)$$

implying that the first extenders used along the main branch of $\mathcal{T}$ and $\mathcal{U}$ were compatible. Contradiction! □

By the claim we then have $\lambda = (\omega_1^+)^{K_\beta^N}$ and $J_\lambda^{K_\beta} = J_\lambda^{K_\beta^N}$. So $\mathrm{Ult}(K_\beta^N; F)$ makes sense but more is true.

Claim 2: *The phalanx $(K_\beta^N, \mathrm{Ult}(K_\beta^N; F); j(\omega_1))$ is fully iterable in $V[G]$ and N.*

Proof of Claim: Because both K_β and K_β^N strategies are guided by $M_{n,\beta}^{\#}$ we can copy the tree $\mathcal{T}$ onto K_β^N via j. Let $j_\alpha : \mathcal{M}_\alpha^{\mathcal{T}} \to \mathcal{M}_\alpha^{j\mathcal{T}}$ be the copy maps. Note that

$$j_\alpha \restriction \mathrm{lh}(E_\alpha^{\mathcal{T}}) = j_\gamma \restriction \mathrm{lh}(E_\gamma^{\mathcal{T}})$$

for $\alpha \leq \gamma$ and that $\mathrm{lh}(E_\alpha^{\mathcal{T}}) > \omega_1$, where α is the minimal element of the main branch of $\mathcal{T}$. This implies that the $j_\alpha(\omega_1) = j(\omega_1)$ extender derived from j_α is in fact independent of α for every α on the main branch and thus equal to F.

Now let us write $\mathcal{Q}^*$ for the last model of $j\mathcal{T}$ and $j^* : \mathcal{Q} \to \mathcal{Q}^*$ for the copy map. By the above we can apply F to $\mathcal{Q}$ and if $k^* : \mathrm{Ult}(\mathcal{Q}; F) \to \mathcal{Q}^*$ is the map

$$[a, f]_F \longmapsto j^*(f)(a)$$

then $\mathrm{crit}(k^*) \geq j^*(\omega_1) = j(\omega_1)$ and $j^* = k^* \circ i_{\mathcal{Q}}^F$, where $i_{\mathcal{Q}}^F$ of course refers to the ultrapower embedding.

Now define the map $k : \mathrm{Ult}(K_\beta^N; F) \to \mathrm{Ult}(\mathcal{Q}; F)$ by

$$[a, f] \longmapsto \left[a, i^{\mathcal{U}}(f)\right]$$

This is elementary because for any natural number m and formula φ

$$\{b \in [\omega_1]^m \,|K_\beta^N \models \varphi(f(b))\} = \{b \in [\omega_1]^m \,|\mathcal{Q} \models \varphi(i^{\mathcal{U}}(f)(b))\}$$

Remember $\mathrm{crit}(i^{\mathcal{U}}) \geq \omega_1$. Additionally

$$k(a) = k(i_{K_\beta^N}^F(\mathrm{id})(a)) = i_{\mathcal{Q}}^F(i^{\mathcal{U}}(\mathrm{id}))(a) = \mathrm{id}(a) = a$$

for any $a \in [j(\omega_1)]^{<\omega}$.

So $\mathrm{crit}(k^* \circ k) \geq j(\omega_1)$ and so the phalanx can be embedded into $\mathcal{Q}^*$ via $i^{\mathcal{U}}, k^* \circ k$. Thus she is iterable in $V[G]$. The proof for the iterability of K_β^N in $V[G]$ works in reverse as well, and therefore the phalanx is also iterable in N. □

By a result from core model theory (see [Ste96] Theorem 8.6) then $F \in K_\beta^N$. But F is superstrong inside of K_β^N. Contradiction!

Lemma 3.8: $(\omega_1^+)^{K_\beta} < \omega_2$.

Proof: Assume not. Then $K_\beta||\aleph_2$ is equal to the stack of $M_{0,\beta}$-mice above $K_\beta||\omega_1$. By applying the main argument from the proof of Lemma 2.2, we must have some $M_{0,\beta}$-mouse $\mathcal{M}$ of size and cofinality $\aleph_1$ with largest cardinal ω_1 end-extending $K_\beta||\omega_1$ such that the tree $T_{\mathcal{M}}$ is special (restricted to some appropriate club, but let's ignore that).

Claim 1: $\mathcal{M} \trianglelefteq K_\beta$.

Proof of Claim: Let ξ, η be ordinals such that $\omega_1 < \xi, \eta < \omega_2$ and $\rho_\omega(K_\beta||\xi) = \rho_\omega(\mathcal{M}||\eta) = \omega_1$. It will be enough to show, that $K_\beta||\xi \trianglelefteq \mathcal{M}||\eta$ or $\mathcal{M}||\eta \trianglelefteq K_\beta||\xi$.

Assume not. Then take countable $\pi : \bar{\mathcal{M}} \to \mathcal{M}||\eta$ and $\sigma : \bar{K} \to K_\beta||\xi$ such that $\bar{\mathcal{M}}, \bar{K}$ are incomparable.

But by the condensation lemma then $\bar{\mathcal{M}} \trianglelefteq \mathcal{M}||\eta$ and $\bar{K} \trianglelefteq K_\beta||\xi$ and thus $\bar{\mathcal{M}} \trianglelefteq K_\beta||\omega_1$ and $\bar{K} \trianglelefteq K_\beta||\omega_1$. Contradiction! □

So there is some mouse $\mathcal{Q}$ with $\mathcal{M} \trianglelefteq \mathcal{Q} \trianglelefteq K_\beta$ such that $\rho_\omega(\mathcal{Q}) = \omega_1$. We can then derive a contradiction using the same argument from the proof of Lemma 2.2. ⊣

So we have shown $(1)_{n+1,\beta}$. Let us fix some $X \in H_{\omega_2}$ and a $H \subseteq \mathrm{Col}(\omega, I)$ generic over $V[G]$.

N certainly contains some X-premouse $\mathcal{M}$, which it believes to be $M^\#_{n+1,\beta}(X)$ by elementarity. Using the argument we applied to K^N_β, we see that $\mathcal{M}$ is ω_1-iterable in $V[G][H]$ (using $p\left[T^\theta_{n,\beta}\right] = p\left[j(T^\theta_{n,\beta})\right]$). But $V[G][H]$ is equivalent to a homogeneous forcing extension of the ground model. So $\mathcal{M} \in V$ and it is ω_2-iterable there, so it is $M^\#_{n+1,\beta}(X)$.

By Lemma 2.2 we then get $(3)_{n+1,\beta}$. So we can now conclude, that our core model induction will reach $\beta + 1$.

Hybrid mice

Let β be such, that

- it either ends a weak gap beginning at α,
- or $\beta - 1$ exists and it ends a strong gap beginning at α.

In both cases we want W_α to hold. Let $\langle A_i : i < \omega \rangle$ be a sjs "cofinal" in β, such that each A_i is $\mathrm{OD}_{<\beta}$.

Let $\mathcal{N}$ be an $\langle A_i : i < \omega \rangle$-iterable mouse and Σ it's unique $\langle A_i : i < \omega \rangle$-guided ω_1-iteration strategy, which has hull condensation (and branch condensation).

Lemma 3.9: *Σ extends to a total iteration strategy Γ with hull condensation. This uses homogeneity and presaturatedness.*

Proof: By Lemma 2.3 it is enough to show, that Σ extends to a ω_2-iteration strategy. Let $\mathcal{T}$ be some iteration tree on $\mathcal{N}$ of size at most $\aleph_1$. Then $\mathcal{T} \in N$, and in N it is a countable iteration tree on $\mathcal{N}$. In M there is then $b := j(\Sigma)(\mathcal{T})$ a cofinal wellfounded branch through $\mathcal{T}$. b is

either the unique wellfounded branch through $\mathcal{T}$ with a Q-structure in $j(J_\beta(\mathbb{R}))$, or the unique wellfounded branch moving the terms for $\langle j(A_i) : i < \omega\rangle$ correctly, which is OD in $j(J_{\beta+1}(\mathbb{R}))$.

As $j(J_\beta(\mathbb{R})) = J_{j(\beta)}(\mathbb{R}^N) = J_{j(\beta)}(\mathbb{R}^{V[G]})$ by presaturation, we then have that b is $OD_V^{V[G]}$ and thus in V by homogeneity. In fact $j(\Sigma) \restriction H_{\omega_2} \in V$.

Claim 1: *$j(\Sigma) \restriction H_{\omega_2} \in V$ has hull condensation and extends Σ.*

Proof of Claim: Let $\bar{\mathcal{T}}$ be a countable hull of $\mathcal{T}$ witnessed by $\sigma, \vec{\pi}$. All of these are in N, so N knows that $\bar{\mathcal{T}}$ is a hull of $\mathcal{T}$. As $j(\Sigma)$ satisfies hull condensation in N, we have that $\bar{\mathcal{T}}$ is by $j(\Sigma)$.

Furthermore if $\mathcal{T}$ is a countable tree on $\mathcal{N}$ by Σ, then $\mathcal{T} = j(\mathcal{T})$ is by $j(\Sigma)$ by elementarity. □

Note here that by Lemma 1.10 Γ has branch condensation.

We will now show, that in every forcing extension $V[H]$ there is an ω_1-no-drop-iteration strategy on $\mathcal{N}$, which we shall call Σ^H, that condenses well and agrees with Γ on the intersection of their domains.

For that let us assume without loss of generality, that there are class-many regular countably closed cardinals κ, such that $\kappa^{<\kappa} = \kappa$. Let $\mathbb{P}$ be some partial order and fix some regular θ, such that $\mathbb{P} \in H_\theta$.

Lemma 3.10: *There is some non-dropping iteration $\mathcal{N} \to \mathcal{N}^*$ according to Γ, such that for all total extender E on the sequence of $\mathcal{N}^*$ the following holds:*

- $\mathrm{crit}(E) \geq \theta$,
- *E is certified, i.e. it is derived from some elementary embedding $\pi : H \to H_{\kappa^+}$, where $\kappa^{<\kappa} = \kappa$ and H is transitive and countably closed.*

Proof: The idea is simple: first hit the smallest measurable of $\mathcal{N}$ θ-many times, then begin removing non-certified extenders. We will have to show, that this terminates.

Fix a regular countably closed $\kappa > \theta$, such that $\kappa^{<\kappa} = \kappa$. Assume the iteration reaches an iteration tree $\mathcal{T}$ of length κ. Let $X \prec H_{\kappa^+}$ be countably closed, such that $\mathcal{T}, \Gamma(\mathcal{T}) \in X$, and $\gamma := X \cap \kappa \in \kappa$. Let $\pi : H \to H_{\kappa^+}$ be the inverse of the transitive collapse.

By the proof of the comparison lemma, we have that $i^{\mathcal{T}}_{\gamma,\kappa} = \pi \restriction \mathcal{M}^{\mathcal{T}}_\gamma$. But then the first extender used along this branch above γ is certified by π. Contradiction! ⊣

Lemma 3.11: *Let $\mathcal{T}$ be a countable non-dropping iteration tree on $\mathcal{N}$ in $V[H]$, such that for all limit $\alpha < \mathrm{lh}(\mathcal{T})$ the branch $[0,\alpha)_{\mathcal{T}}$ is realizable into $\mathcal{N}^*$ as above. Then there is exactly one maximal branch through $\mathcal{T}$, which is realizable into $\mathcal{N}^*$.*

Proof: By the choice of $\mathcal{N}^*$ all of it's total extenders are certified. Thus there is indeed at least one such branch, by the proof of iterability for K^c. We will show, that there is at most one such branch by a reflection argument.

Let us assume for a contradiction that there is some tree with two realizable branches. Let λ be big enough and let $X \prec H_\lambda$ be countable, with $\mathcal{N}, \mathcal{N}^*, p, \mathbb{P} \in X$, where p is some conditon forcing the existence of a tree with two realizable branches. Let $\mathcal{H}$ be the transitive collapse of X, and write $\bar{\mathcal{N}}^*, \bar{p}, \bar{\mathbb{P}}$ for the images of $\mathcal{N}^*, p$ and $\mathbb{P}$ respectively. Let h be some $\bar{\mathbb{P}}$-generic over $\mathcal{H}$, such that $\bar{p} \in h$.

Thus in $\mathcal{H}[h]$ there is some $\mathcal{T}$ picking realizing branches, such that there are two distinct realizing branches b, c. First note that, because Γ condenses well $\bar{\mathcal{N}}^*$ is a Σ-iterate of $\mathcal{N}$. But then by branch condensation $b = \Sigma(\mathcal{T}) = c$. Contradiction! $\dashv$

We then define Σ^H as the strategy picking the unique branch realizing back into $\mathcal{N}^*$. Using the homogeneity of the $\mathrm{Col}(\omega, \lambda)$ line of forcings, it is then very simple to show, that Σ^H extends to a full iteration strategy (for non-dropping trees) Γ^H on $V[H]$, that condenses well.

The proof shows more. Write T^θ_β and U^θ_β for trees searching for

- a countable transitve $\mathcal{P}$,
- an elementary embedding $\sigma : \mathcal{P} \to (H_{(2^\theta)^+}; \Gamma)$,
- some h, which is generic for some $\mathbb{P} \in \mathcal{P}$ over $\mathcal{P}$ with $\sigma(\mathbb{P}) \in H_\theta$,

such that $(x, y) \in \Sigma^h$ or respectively $(x, y) \notin \Sigma^h$ as computed from $\mathcal{P}[h]$.

Corollary 3.12: *Let θ be some uncountable regular cardinal. For all $\mathbb{P} \in H_\theta$ and all $H \subseteq \mathbb{P}$ generic over V*

$$p\left[T^\theta_\beta\right]^{V[H]} = \Sigma^H,$$
$$p\left[U^\theta_\beta\right]^{V[H]} = \mathbb{R}^{V[H]} \backslash \Sigma^H.$$

This implies $L^{j(\Sigma)} = L^\Sigma$, which then implies the existence of $\Sigma^\#$'s for all reals. From there the induction proceeds in exactly the same way it did on p.21 just relativized to Σ-mice.

This finishes the analysis of all the cases of the core model induction and thus the proof of Theorem 1.1.

4. Alternate hypothesis

In the preceding chapter we showed, that $AD^{L(\mathbb{R})}$ followed from the existence of a homeogeneous presaturated ideal on ω_1 under $PFA(\aleph_2)$. Unfortunately the following result from [MF88] puts in doubt the consistency of this hypothesis.

Lemma 4.1 (Foreman-Magidor-Shelah): *Assume MA. Let I be a presaturated ideal on I. Then forcing with I doesn't add a cohen real.*

Fortunately we have used the homogenity and presaturation properties of the ideal in only minor ways, and thus it is an easy task to replace it with another property, that we will subsequently show to be consistent relative to forcing axioms.

Going back we see that we used these properties exactly twice, namely in Lemma 3.1 and Lemma 3.9. Both of these results have to do with reflection, and in both cases this is achieved by showing that the image of an operator under the generic ultrapower embedding is indepedent of the generic.

So what we need is an ideal property, that implies independence of the generic for certian functions on countable sets inside of $L(\mathbb{R})$.

Definition 4.2: Let I be a normal ideal on ω_1. By $(+)_I$ we refer to the following property:

For all $A \subseteq \omega_1$, for all $F : H_{\omega_1} \to H_{\omega_1}$ in $L(\mathbb{R})$, for all $\xi < \omega_1$ either

(1) $\{\eta < \omega_1 | \xi \in F(A \cap \eta)\} \in I^*$, or

(2) $\{\eta < \omega_1 | \xi \notin F(A \cap \eta)\} \in I^*$.

If the ideal in question is NS_{ω_1} we shall omit the subscript and just write $(+)$.

It is easy to see, that if I is precipitous and $(+)_I$ holds, then $j(F)(A) \cap \omega_1$ (where j is the generic embedding) does not depend on the choice of the generic.

On the other hand if there were some countable ordinal ξ and I-positive sets S, T, such that $S \Vdash \check{\xi} \in j(\check{F})(\check{A})$ and $T \Vdash \check{\xi} \notin j(\check{F})(\check{A})$, then by normality

- $S \cap R \subseteq \{\eta < \omega_1 | \xi \in F(A \cap \eta)\}$, and
- $T \cap R \subseteq \{\eta < \omega_1 | \xi \notin F(A \cap \eta)\}$,

where $R \in I^*$. This clearly contradicts $(+)_I$. It is in fact not necessary to invoke the precipitousness of I to plug the gaps in our core model induction.

Lemma 4.3: *Let I be some normal Ideal on ω_1. Assume $(+)_I$ holds.*

(a) *Let $M \in L(\mathbb{R})$ be a mouse operator, that condenses well. Then mouse reflection holds for M at $(\aleph_1, \aleph_2)$, i.e. if M was total on H_{ω_1} (above some real) then it is total on H_{ω_2} (above the same real).*

(b) *Let $\Sigma \in L(\mathbb{R})$ an iteration strategy on some countable $\mathcal{N}$ that has hull condensation. Assume that Σ is total on H_{ω_1}, then Σ extends to an iteration strategy on H_{ω_2} that has hull condensation.*

Proof: (a): Fix some $A \subseteq \omega_1$. Let $F : H_{\omega_1} \to H_{\omega_1}$ code M in some nice fashion, say

$$F(a) := \{\ulcorner \varphi, n, \xi_1, \ldots, \xi_n \urcorner | \xi_i \in a, \varphi\ \Sigma_1, M(a) \models \varphi(\tau_n(\xi_1, \ldots, \xi_n))\} \qquad \dashv$$

for some countable $a \subset \omega_1$. $F \in L(\mathbb{R})$, and thus for all $\xi < \omega_1$, there is some I-measure one set C_ξ, such that either $\xi \in F(A \cap \eta)$ for all $\eta \in C_\xi$ or $\xi \notin F(A \cap \eta)$ for all $\eta \in C_\xi$. Let C be the diagonal intersection of the C_ξ.

For $\eta_0 < \eta_1$ both in C there exists then a Σ_1 elementary embedding from $M(A \cap \eta_0)$ into $M(A \cap \eta_1)$. Let $M(A)$ be the direct limit of the $\langle M(A \cap \eta) : \eta \in C\rangle$ under these embeddings. By standard arguments this limit is both wellfounded and sufficiently iterable.

(b): Fix some tree $\mathcal{T}$ on $\mathcal{N}$ of size $\aleph_1$, which we shall assume to be according to some strategy extending Σ, such that all countable hulls are according to Σ. Let $A \subseteq \omega_1$ code $\mathcal{T}$ in some simple fashion. Let $F \in L(\mathbb{R})$ code Σ. Fix I-measure one sets C_ξ, such that either $\xi \in F(A \cap \eta)$ for all $\eta \in C_\xi$ or $\xi \notin F(A \cap \eta)$ for all $\eta \in C_\xi$. Let D be the (club-)set of $\eta < \omega_1$, such that $A \cap \eta$ codes a tree on $\mathcal{N}$ according to Σ and let C be the intersection of D with the diagonal intersection of the C_ξ.

We can then define a branch through $\mathcal{T}$ by putting the node corresponding to some $\xi < \omega_1$ into the branch iff $\xi \in F(A \cap \eta)$ I-measure one often. We will show, that this branch is cofinal.

Assume not, then there is some γ, such that for all $\xi < \omega_1$ lying on a level above γ, there are measure one many η, such that that $\xi \notin F(A \cap \eta)$. Take some $\eta \in C$, such that $\gamma < \eta$. $F(A \cap \eta)$ codes a cofinal branch, thus for some $\xi < \eta$ lying on a level above γ $\xi \in F(A \cap \eta)$, but $\eta \in C_\xi$. Contradiction!

A similar argument shows, that the branch is wellfounded.

We thus can use $(+)_I$ as a substitute for homogeneity, and thus arrive at the following result.

Theorem 4.4: *Assume $PFA(\aleph_2)$. Let I be a normal precipitous Ideal on ω_1, such that $(+)_I$ holds. Then $AD^{L(\mathbb{R})}$ holds.*

We will finally show, that $(+)$ is consistent with forcing axioms.

Lemma 4.5: *Assume, that all $A \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ are $<\omega_2$-universally baire. Then $(+)$ holds.*

Proof: Fix $F \in L(\mathbb{R})$,$A \subseteq \omega_1$ and $\xi < \omega_1$. Let $B \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ code F. Fix trees T, U witnessing the universal baireness of B.

Let θ be some sufficiently big cardinal. Let $X \prec H_\theta$ be countable, s.t. $\xi, T, U, A \in X$. Let H be the transitive collapse of X, $\bar{T}, \bar{U}$ the images of T, U under the collapse and $\eta := \omega_1 \cap X = \omega_1^H$.

Claim 1: *$F(A \cap \eta) \in H$. Furthermore it is uniformly definable in $\eta, \bar{T}, \bar{U}, A$.*

Proof of Claim: Let $h \subset \mathrm{Col}(\omega, \eta)$ be generic over H. Because T, U witness the universal baireness of B, we have by elementarity, that

$$(p\left[\bar{T}\right])^{H[h]} \cup (p\left[\bar{U}\right])^{H[h]} = \mathbb{R}^{H[h]}.$$

On the other hand we have

$$p\left[\bar{T}\right] \subseteq p\left[T\right], p\left[\bar{U}\right] \subseteq p\left[U\right].$$

Thus $(p\left[\bar{T}\right])^{H[h]} = B \cap H\left[h\right]$. This doesn't depend on the choice of the generic so $F(A \cap \eta)$ can be defined as the unique element such that $(x, y) \in p\left[\bar{T}\right]$ for all real codes y of it and all real codes x of $A \cap \eta$ in any $\mathrm{Col}(\omega, \eta)$-generic extension. $\Box$

By the claim if we have $X_0 \prec X_1 \prec H_\theta$ and $\pi : H_0 \to H_1$ is the concatenation of the uncollapse of H_0 and the collapse of X_1, $\pi(F(A \cap \eta_0)) = F(A \cap \eta_1)$ and thus $\xi \in F(A \cap \eta_0)$ iff $\xi \in F(A \cap \eta_1)$. So because

$$\{X \cap \omega_1 | X \prec H_\theta, \mathrm{Card}(X) = \aleph_0, (\xi, T, U, A) \in X\}$$

is club, we are done. $\dashv$

5. Determinancy in $V^{\mathrm{Col}(\omega,\omega_1)}$

Theorem 5.1: *Assume $PFA(\aleph_2)$ and that there is a precipitous Ideal I on ω_1, then inductive determinancy holds in $V^{\mathrm{Col}(\omega,\omega_1)}$, i.e. $J_\alpha(\mathbb{R}^{\mathrm{Col}(\omega,\omega_1)}) \models AD$ for α the least admissible ordinal of $V^{\mathrm{Col}(\omega,\omega_1)}$.*

The proof is by core model induction inside of $V^{\mathrm{Col}(\omega,\omega_1)}$, though we will have to go back to V for some of the arguments. Let us assume $PFA(\aleph_2)$ and let I be a precipitous ideal on ω_1.

Fix a $\mathrm{Col}(\omega,\omega_1)$-generic filter g and a $\mathcal{P}(\omega_1)/I$-generic Filter G over $V[g]$. Let us write N for the ultrapower of V by G and $j : V \to N$ for the ultrapower embedding.

In the course of the induction we will show that the universe is closed under operators $M_{n,\beta}$. For that we need to know the following about the operators:

- $M_{n,\beta}$ is extraordinarily nice,
- for all partial orders $\mathbb{P}$, all $\mathbb{P}$-names σ and all $h \subseteq \mathbb{P}$ generic over $M_{n,\beta}(\mathbb{P},\sigma)$ can $M_{n,\beta}(\sigma^G)$ be computed from the background construction of $M_{n,\beta}(\mathbb{P},\sigma)\,[h]$.

we have already seen in chapter 3 that all the operators appearing in the course of the core model induction have these properties, so we will not need to examine the exact nature of β in the following.

So let us fix some β less than α the least admissible, and assume that $M_{0,\beta}$ is total on the reals of $V[g]$ above some real a_β. We will show that $V[g]$ is closed under $M_{n,\beta}$ for all $n < \omega$.

It is shown in [SS] that $M_{0,\beta}$ gives rise to an operator $M^V_{n,\beta}$ in V, which is total on H_{ω_2} above some name τ_β for a_β, with the property that $M^V_{0,\beta}(\sigma)\,[g] = M_{0,\beta}(\sigma^g)$ for any name σ for a real above a_β.

When h is some Filter generic over $V[g]$ for some forcing in V and $M_{n,\beta}$ is sufficiently total on $V[g]\,[h] = V[h]\,[g]$, we can use the same "reverse of generic extension" construction to get an operator $M^{V[h]}_{n,\beta}$ in $V[h]$ extending $M^V_{n,\beta}$.

Our reflection lemma Lemma 2.2 can be applied to the operator $M^V_{n,\beta}$ and from there can be extended to all forcing extensions of $V[g]$ in the usual way. But because τ_β might be uncountable we cannot apply our core model arguments from chapter 3.

We will now define a model operator $M^N_{0,\beta}$ inside of N. Let T^θ_β be the tree searching for reals x, y and

- a countable model $\mathcal{P}$ with $\tau_\beta \in \mathcal{P}$,
- an elementary embedding $\sigma : \mathcal{P} \to j(M^V_{0,\beta}(H_\theta))$ such that $\sigma \upharpoonright (\tau_\beta \cup \{\tau_\beta\}) = j \upharpoonright (\tau_\beta \cup \{\tau_\beta\})$,

- a filter $h := h^0 \times h^1$ generic over $\mathrm{Col}(\omega, \omega_1^V) \times \mathbb{P}$, where $\sigma(\mathbb{P}) \in j(H_\theta)$

such that y codes $M_{0,\beta}^{\mathcal{P}[h^1]}(x)$ as computed from $\mathcal{P}[h]$.

We also define U_β^θ a tree searching for the same things, but such that y codes a structure different from $M_{0,\beta}^{\mathcal{P}[h^1]}(x)$. It is easy to see that $T_\beta^\theta, U_\beta^\theta \in N$.

Lemma 5.2: *Let $W := V[g][G][H]$ be a generic extension of $V[g][G]$ by $\mathbb{P}$, where $\mathbb{P} \in H_\theta$, then*

$$\begin{aligned} p\left[T_\beta^\theta\right]^W &\supseteq \{(x,y) \in (\mathbb{R}^2)^{V[G][H]} | y = M_{0,\beta}^{V[G][H]}(x)\}, \\ p\left[U_\beta^\theta\right]^W &\supseteq \{(x,y) \in (\mathbb{R}^2)^{V[G][H]} | y \neq M_{0,\beta}^{V[G][H]}(x)\}. \end{aligned}$$

Proof: Let y code $M_{0,\beta}^{V[G][H]}(x)$. Let $\mathcal{P}[h]$ be the collapse of a countable hull of $H_\theta[g \times G \times H]$ with y and τ_β in $\mathcal{P}[h]$. Let σ be the inverse of the collapse. Then $\mathcal{P}, j \circ \sigma \restriction \mathcal{P}, h$ witness that $(x,y) \in p\left[T_\beta^\theta\right]$. The proof for U_β^θ works the same way. ⊣

Lemma 5.3: $p\left[T_\beta^\theta\right] \cap p\left[U_\beta^\theta\right] = \emptyset$.

Proof: Because of absoluteness it suffices to show this inside of N. Assume not. This means we must have two models $\mathcal{P}_0, \mathcal{P}_1$, embeddings $\sigma_i : \mathcal{P}_i \to j(M_{0,\beta}^V(H_\theta))$ and generic filters $h_i := h_i^0 \times h_i^1$, such that $M_{0,\beta}^{\mathcal{P}_0[h_0^1]}(x)$ as computed by $\mathcal{P}_0[h_0]$ differs from $M_{0,\beta}^{\mathcal{P}_1[h_1^1]}(x)$ as computed by $\mathcal{P}_1[h_1]$.

Let us call these models $\mathcal{M}_0$ and $\mathcal{M}_1$ respectively. Because of the extraordinary niceness of $M_{0,\beta}$ both models believe they are the minimal model satisfying $\varphi(x, \tau_\beta)$, where φ is Σ_1. But both models inherit the iterability of $j(M_{0,\beta}^V(H_\theta))$, thus by comparison they are equal. Contradiction! ⊣

We can now define $M_{0,\beta}^N(A)$ as the unique set B such that $(x,y) \in p\left[T_{0,\beta}^\theta\right]$ whenever y codes B and x codes A inside of a generic extension by $\mathrm{Col}(\omega, A)$.

Lemma 5.4: *$M_{0,\beta}^N$ is welldefined and in fact $M_{0,\beta}^N = M_{0,\beta}^{V[G][H]} \cap N$ for all forcing extensions $V[g][G][H]$ of $V[g][G]$*

Proof: Let $A \in N$ be arbitrary and let $V[g][G][H]$ be a forcing extension of $V[g][G]$. W.l.o.g. H is generic for some $\mathrm{Col}(\omega, \lambda)$ such that $\lambda \geq \aleph_2$ is at least the N-cardinality of A.

We then have that H is generic over N and of course $N[H] \subseteq V[G][H]$. Let x be a real code for A in $N[H]$ and let y be a real code for $M_{0,\beta}^{V[G][H]}(A)$ in $V[G][H]$.

By Lemma 5.2 then $(x, y) \in p\left[T_\beta^\theta\right]$ for $\theta > \lambda$, so by absoluteness there is some y' in $N[H]$ such that $(x, y') \in p\left[T_\beta^\theta\right]$. If y' were not a code for $M_{0,\beta}^{V[G][H]}(A)$, then by Lemma 5.2 $(x, y') \in p\left[U_\beta^\theta\right]$, but this contradicts Lemma 5.3!

All this didn't depend on the choice of codes or generics (except of course the fixed choice of G and g) and thus $M_{0,\beta}^{V[G][H]}(A) \in N$ and it is the unique structure there, which fits the definition of $M_{0,\beta}^N$. $\dashv$

Using the fact that $M_{0,\beta}^N$ has a nice uniform definition we can find a sequence of mouse operators $\langle M_\xi : \xi < \omega_1\rangle$ such that $[M_\xi \cap H_\kappa : \xi < \omega_1]_G$ is equal to $M_{0,\beta}^N \cap H_{j(\kappa)}^N$ for all regular uncountable cardinals κ.

By Łoś's theorem each M_ξ is a mouse operator in V defined above some real with an universally baire representation.

Thus the arguments from chapter 3 can be applied to each of the M_ξ. Thus we have that $(M_n^{M_\xi})^\#$ exists for all $n < \omega$. Applying Łoś's theorem again we then have $(M_{n,\beta}^N)^\#$ for all $n < \omega$ inside of N.

We can now finish the argument. First step is to show, that the $M_{0,\beta}^\#$ are total. Let $A \in H_{\omega_2}$ be a name for a real in $V[g]$ above τ_β. Then $A \in N$, so $\mathcal{M} := (M_{0,\beta}^N)^\#(A)$ exists. By the proof of Lemma 5.2 $\mathcal{M}[g]$ is closed under $M_{0,\beta}$ and of course the top extender survives into small forcing extensions.

So $\mathcal{M}[g] = M_{0,\beta}^\#(A^g)$. By standard arguments it can then be pulled back to V, which will show that $(M_{0,\beta}^V)^\#$ is total on H_{ω_2} and thus by Lemma 2.2 total.

Now assume that $(M_{n,\beta}^V)^\#$ is total for some n. We can then build trees $T_{n,\beta}^\theta$ and $U_{n,\beta}^\theta$ in N, which will define an operator $(M_{n,\beta}^N)^\#$ that can be decoded to $M_{n,\beta}^\#$ using g. For any name $A \in H_{\omega_2}$ for a real in $V[g]$ over τ_β we will then have $\mathcal{M} := (M_{n+1,\beta}^N)^\#(A)$ inside of N, which we can then show to be iterable using the trees $T_{n,\beta}^\theta$ close to how we showed iterability for K_β^N on p.21.

Then $\mathcal{M}[g] = M_{n+1,\beta}^\#(A^g)$ and reversing the generic will give $(M_{n+1,\beta}^V)^\#$ total on H_{ω_2}. Using Lemma 2.2 then closes the argument.

Part II.

On the strength of Namba-like forcings on successors of regular cardinals

6. Introduction

Let $*_{\kappa,\nu}$ be the statement:

> There is a p.o. $\mathbb{P}$ such that $\mathbf{1}_{\mathbb{P}} \Vdash \mathrm{cof}(\check{\kappa^+}) = \nu$, but $\mathbb{P}$ doesn't change cofinalities and cardinalities $\leq \kappa$.

Our main result will be a generalization of the following result due to Peter Koepke:

Theorem 6.1 (Koepke): *Assume there is no inner model with a Woodin cardinal. Let κ be a regular infinite cardinal, where $\kappa \geq \aleph_2$. Let $\mu > \kappa$ be measurable. Then there exists a generic extension of the universe $V[G]$, such that $V[G] \models *_{\kappa,\omega}$.*

Our own result is the following:

Theorem 6.2: *Let $\nu < \kappa$ be regular infinite cardinals, where $\kappa \geq \aleph_2$. Let η be such that $\omega \cdot \eta = \nu$.*

(a) Assume GCH. Let $\mu > \kappa$ be a measurable cardinal with $\mathrm{o}(\mu) \geq \eta$. Then there exists a generic extension of the universe $V[G]$, such that $V[G] \models *_{\kappa,\nu}$.

(b) Assume $*_{\kappa,\nu}$. Then there exists an inner model M, such that $M \models \mathrm{o}(\mu) \geq \eta$ for $\mu = \kappa^+$.

The "$\Rightarrow$" direction follows by a standard argument from the following cover theorem (see [Cox09b]):

Theorem 6.3 (Cox): *Suppose $0^\P$ does not exist. Then the "core model for non-overlapping extenders" K exists and for all ordinals $\alpha > \aleph_2$, which are regular cardinals in K, if $\nu = \mathrm{cof}(\alpha) < \mathrm{Card}(\alpha)$ and η is such that $\eta \cdot \omega = \nu$, then $K \models \mathrm{o}(\alpha) \geq \eta$.*

Assume $*_{\kappa,\nu}$. If $0^\P$ exists, we are done as this gives us for any cardinal μ an inner model, in which μ is strong (just iterate $0^\P$ by the smallest measure on the strong and its images μ-many times). So assume, that $0^\P$ does not exist. Let $\mathbb{P}$ witness $*_{\kappa,\nu}$, and let $G \subset \mathbb{P}$ be a generic filter.

Note that forcing can not add $0^\P$, so the core model for non-overlapping extenders exists in both V and $V[G]$ and the two versions are equal. Let us call this model K. As $K \subseteq V$ we have $K \models \mu$ is regular , where $\mu = (\kappa^+)^V$. But then by the theorem we have $K \models \mathrm{o}(\mu) \geq \eta$.

Thus we will concentrate on the "$\Leftarrow$" direction. So let us fix regular cardinals $\nu < \kappa < \mu$ and η such that $\eta \cdot \omega = \nu$ and $\kappa \geq \aleph_2$, and let $\mathrm{o}(\mu) \geq \eta$.

One might be inclined to just collapse μ to be κ^+ and then apply the Prikry forcing in the resulting model to witness $*_{\kappa,\omega}$, but there are combinatorial reason, why this cannot possibly work. The problem lies with the following result by Shelah:

Lemma 6.4 (Shelah): *If μ is a regular cardinal and if a notion of forcing $\mathbb{P}$ makes $\mathrm{cof}(\mu) \neq \mathrm{Card}(\mu)$, then $\mathbb{P}$ collapses μ^+.*

See [Jec06] p. 451.

If now in our situation the Prikry forcing would witness $*_{\kappa,\omega}$, then we would have

$$\mathrm{cof}(\mu) = \omega \neq \kappa = \mathrm{Card}(\mu)$$

and thus μ^+ must be collapsed by the lemma, but Prikry forcing has the μ^+ c.c. giving a contradiction.

The actual solution will still involve Prikry forcing, but it won't be quite as simple as that.

The rest of this chapter will be dedicated to introducing most of the concepts, which will appear throughout this part of the thesis. In the second chapter we will present a slightly more general proof of the original Koepke result. This will also serve as a framework for the general theorem. In chapter 3 we will introduce a forcing of Gitik's, which will serve as the basis for the forcing in our main proof. Chapter 4 will contain the main proof. In chapter 5 we will discuss applications of iteration to our main result. Chapter 6 finally will feature a departure from the methods of the rest of the thesis, but will only maintain a thematic connection to the rest of this part of the thesis.

Miscellaneous

We will start by introducing notational shortcuts, we shall use heavily throughout the following chapters. All of these will deal with trees, and by that we mean sets of finite sequences of ordinals closed under initial segments. As a general rule our sequences will be increasing, and so we will confuse them with finite sets of ordinals whereever it may be convenient.

We shall use $\trianglelefteq$ for end extension, $\mathrm{lh}(\cdot)$ for the length of sequences, and $^\frown$ for concatenation.

Notation: Let γ be some ordinal and $T \subset [\gamma]^{<\omega}$ a tree (, i.e. it is closed under initial segments).

(a) We call $t \in T$ the stem of T, iff t is $\trianglelefteq$-maximal with the property that for all $s \in T$ $s \trianglelefteq t$ or $t \trianglelefteq s$.

(b) Let $s \in T$. We write T_s for $\{t \in T | t \trianglelefteq s \vee s \trianglelefteq t\}$.

(c) Let $s \in T$. We write T/s for $\{t \in [\gamma]^{<\omega} | s^\frown t \in T\}$.

(d) Let $s \in [\gamma]^{<\omega}$ be such, that for all non-empty $t \in T$ $\sup(s) < \min(t)$. We then write $s^\frown T$ for $\{s^\frown t | t \in T\}$.

(e) Let $s \in T$. We write $\mathrm{suc}_T(s)$ for $\{\xi < \gamma | s^\frown\langle\xi\rangle \in T\}$.

In the following we will make heavy use of Gitik's "Prikry-type forcing" notation. For the sake of readers, who might not be familiar with it, we will give the definitions here:

Definition 6.5: A triple $\langle \mathbb{P}, \leq, \leq^* \rangle$ is called a Prikry-type forcing, iff:

- $\langle \mathbb{P}, \leq \rangle$ and $\langle \mathbb{P}, \leq^* \rangle$ are partial orders.
- $\leq^* \subseteq \leq$
- Let $\varphi(\tau)$ be any statement in the forcing language of $\langle \mathbb{P}, \leq \rangle$ and $p \in \mathbb{P}$. Then there is $q \leq^* p$, such that $q \parallel_{\langle \mathbb{P}, \leq \rangle} \varphi(\tau)$.

In talking of Prikry type forcings we will often use the phrase "p is a direct extension of q". By this we simply mean $p \leq^* q$.

Definition 6.6: Let γ be a regular uncountable cardinal. A Prikry-type forcing $\langle \mathbb{P}, \leq, \leq^* \rangle$ is called weakly $<\gamma$-closed, iff $\langle \mathbb{P}, \leq^* \rangle$ is $<\gamma$-closed.

For a non-principal normal measure U on some cardinal μ the Prikry forcing for U would be an example of an $<\mu$-closed Prikry type forcing. The basic forcings appearing in the proof of the main theorem will be based on a variant of Prikry forcing, that might best be described as tree Prikry forcing.

The conditions of tree Prikry forcing are pairs (t, T), where t is a finite set of ordinals less than μ and $T \subseteq [\mu]^{<\omega}$ is tree with stem t, with the property that whenever $s \in T$ with $t \trianglelefteq s$, then $\mathrm{suc}_T(s) \in U$.

(s, S) extends (t, T), iff $S \subseteq T$, and (s, S) is a direct extension of (t, T), if additionally s equals t.

Remark: A weakly $<\gamma$-closed Prikry type forcing does not add new bounded subsets of γ. (The proof of this is the same, that is already known from Prikry forcing.)

Later we will use iterations of Prikry type forcings. These iterations differ from the usual concept of an iteration in one important point.

Definition 6.7: $\langle\langle\langle \mathbb{P}_\alpha, \leq_\alpha, \leq^*{}_\alpha \rangle : \alpha \leq \gamma \rangle, \langle\langle \dot{Q}_\beta, \dot{\leq}_\beta, \dot{\leq}^*_\beta \rangle : \beta < \gamma \rangle\rangle$ is a Gitik-iteration iff

(i) for all $\alpha \leq \gamma$ $\mathbb{P}_\alpha$ consists of sequences p of length α such that $p(\beta)$ is an $\mathbb{P}_\beta$-name and $p \restriction \beta \Vdash p(\beta) \in \dot{Q}_\beta$ for all $\beta < \alpha$,

(ii) $\langle \dot{Q}_\beta, \dot{\leq}_\beta, \dot{\leq}^*_\beta \rangle$ is a $\mathbb{P}_\beta$-name for a Prikry type forcing notion for all $\beta < \gamma$,

(iii) if $\alpha \leq \gamma$ is inaccessible then a p as in (i) is in $\mathbb{P}_\alpha$ iff $p \restriction \beta \Vdash p(\beta) = \mathbf{1}$ for all but boundedly many $\beta < \alpha$,

(iv) if $\alpha \leq \gamma$ is not inaccessible then $p \in \mathbb{P}_\alpha$ if it is as in (i),

(v) for all $\alpha \leq \gamma$ for all $p, q \in \mathbb{P}_\alpha$ $p \leq^*{}_\alpha q$ iff $p \restriction \beta \Vdash p(\beta) \dot{\leq}^*_\beta q(\beta)$ for all $\beta < \alpha$,

(vi) for all $\alpha \leq \gamma$ for all $p, q \in \mathbb{P}_\alpha$ $p \leq_\alpha q$ iff $p \restriction \beta \Vdash p(\beta) \dot{\leq}_\beta q(\beta)$ for all $\beta < \alpha$ *and* $p \restriction \beta \Vdash p(\beta) \dot{\leq}^*_\beta q(\beta)$ for all but finitely many $\beta < \alpha$.

Lemma 6.8 (Gitik): *Let $\langle\langle\langle \mathbb{P}_\alpha, \leq_\alpha, \leq^*{}_\alpha\rangle : \alpha \leq \gamma\rangle, \langle\langle \dot{Q}_\beta, \dot{\leq}_\beta, \dot{\leq}^*_\beta\rangle : \beta < \gamma\rangle\rangle$ be a Gitik-iteration, then $\langle \mathbb{P}_\alpha, \leq_\alpha, \leq^*{}_\alpha\rangle$ is a Prikry type forcing notion for all $\alpha \leq \gamma$.*

See [Git86] or alternatively [Git10] for a proof.

7. The countable case

We will first consider the case of $\nu = \omega$. In solving this we will introduce the basic method, which will also be applied in the uncountable case.

Fix a non-principal measure U on μ. The following forcing, which consists of trees right out of Prikry forcing, with collapses attached to each node, will be the key to forcing $*_{\kappa,\omega}$.

The forcing $\mathbb{P}$ for the measure U consists of triples (t, T, F), where $t \in [\mu]^{<\omega}$, a tree $T \subset [\mu]^{<\omega}$ (to be understood as a tree on increasing sequences), and a function $F : T \to \mathrm{Col}(\kappa, {<}\mu)$. A triple (t, T, F) is a condition in $\mathbb{P}$ iff

- $F(\emptyset) = \emptyset$,
- t is the stem of T,
- if $s \in T$ and $t \trianglelefteq s$, then $\mathrm{suc}_T(s) \in U$,
- $\forall t, t' \in T : t \neq \emptyset \Rightarrow (F(t) \in \mathrm{Col}(\kappa, < \max t) \wedge t \trianglelefteq t' \Rightarrow F(t) = F(t') \restriction (\kappa \times \max t))$.

Partially order $\mathbb{P}$ by

$$(t', T', F') \leq (t, T, F) \text{ iff } T' \subset T \text{ and } \forall t \in T' : F'(t) \supseteq F(t).$$

We say that (t', T', F') is a direct extension of (t, T, F), $(t', T', F') {\leq^*} (t, T, F)$, iff moreover $t = t'$.

Lemma 7.1: *$\langle \mathbb{P}, \leq, \leq^* \rangle$ is a weakly ${<}\kappa$-closed Prikry type forcing.*

Proof: The weak closure is easy, so let us concentrate on showing that it is of Prikry type.

Let $\varphi(\tau)$ be some statement and $(t, T, F) \in \mathbb{P}$. We want to show that there is some direct extension, which decides $\varphi(\tau)$. We will show, that if there exists some $A \subseteq \mathrm{suc}_T(t)$ in U, such that there exist direct extensions of $(t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \restriction T_{t^\frown\langle\xi\rangle})$, which decide $\varphi(\tau)$, for all $\xi \in A$, then there exists a direct extension of (t, T, F) which decides $\varphi(\tau)$.

Fix such an A and conditions $(t^\frown\langle\xi\rangle, T_\xi, F_\xi) \leq^* (t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \restriction T_{t^\frown\langle\xi\rangle})$, which decide $\varphi(\tau)$. Because U is an ultrafilter, we can assume all of them to decide in the same way, say positively. By further shrinking A we will furthermore assume all the F_ξ to agree on the value of $F_\xi(t)$. (Note that there are ${<}\mu$ many possible values.)

Now set $T^* := \bigcup_{\xi \in A} T_\xi$. Then define F^* by $F^*(s) = F_\xi(s)$ iff $s \trianglelefteq t^\frown\langle\xi\rangle \vee t^\frown\langle\xi\rangle \trianglelefteq s$. This is welldefined by the choice of A, and gives us a condition $(t, T^*, F^*) \leq^* (t, T, F)$, which forces $\varphi(\tau)$ as the set $\{p \in \mathbb{P} | \exists \xi \in A : p \leq (t^\frown\langle\xi\rangle, T_\xi, F_\xi)\}$ is dense below (t, T^*, F^*) and each of it's elements forces $\varphi(\tau)$.

Let us now assume that that there is no direct extension of (t, T, F) deciding $\varphi(\tau)$. We will then construct a sequence of trees $\langle T^m : m < \omega\rangle$ with stem t such that

- $\forall n \leq m < \omega : T^n \subseteq T^m$,
- for all $n \leq m < \omega$ for all $s \in T^n$, if $\mathrm{lh}(s) \leq \mathrm{lh}(t) + n$, then $s \in T^n \Leftrightarrow s \in T^m$,
- for all $m < \omega$ for all $s \in T^m$, if $\mathrm{lh}(s) \leq \mathrm{lh}(t) + m$, then there is no direct extensions of $(s, T^m_s, F \restriction T^m_s)$, which decides $\varphi(\tau)$. $\dashv$

We start with $T^0 := T$. Let us assume that we had already constructed T^m. Let $s \in T^m$ be arbitrary with $\mathrm{lh}(s) = \mathrm{lh}(t) + m$. By induction hypothesis there is no direct extension of $(s, T^m_s, F \restriction T^m_s)$, which decides $\varphi(\tau)$.

So by applying the argument from above to $(s, T^m_s, F \restriction T^m_s)$ we find that there is a measure one set of $\xi < \mu$ such that there is no direct extension of $(s^\frown\langle\xi\rangle, T^m_{s^\frown\langle\xi\rangle}, F \restriction T^m_{s^\frown\langle\xi\rangle})$, which decides $\varphi(\tau)$. Let us call this set A_s.

We can then define T^{m+1} as the set of $s \in T^m$ such that, if $\mathrm{lh}(s) \geq \mathrm{lh}(t) + n + 1$, then $s(\mathrm{lh}(t) + n) \in A_{s\restriction(\mathrm{lh}(t)+n)}$.

This finishes the construction of the $\langle T^m : m < \omega\rangle$. Set $T^* := \bigcup_{m<\omega} T^m$. This tree has the property, that for all $s \in T^*$, there is no direct extension of $(s, T^*_s, F \restriction T^*_s)$, which decides $\varphi(\tau)$.

But surely there is some extension of $(t, T^*, F \restriction T^*)$, which decides $\varphi(\tau)$, (s, S, G) say. But then $s \in T^*$ and (s, S, G) is a direct extension of $(s, T^*_s, F \restriction T^*_s)$. Contradiction!

We will now show, that $\mathbb{P}$ can be written in a form reminiscent of a product, whose first component is $\mathrm{Col}(\kappa, {<}\mu)$. Write $\langle \mathbb{Q}, \leq, \leq^*\rangle$ for the following forcing:

- $\mathbb{Q} = \{\langle p, (t, T, F)\rangle | p \in \mathrm{Col}(\kappa, {<}\mu), (t, T, F) \in \mathbb{P}, \forall \zeta \in \mathrm{suc}_T(t) F(t^\frown\langle\zeta\rangle) = p\}$
- $\langle p, (t, T, F)\rangle \leq \langle q, (s, S, G)\rangle \Leftrightarrow p \leq q \wedge (t, T, F) \leq (s, S, G)$
- $\langle p, (t, T, F)\rangle \leq^* \langle q, (s, S, G)\rangle \Leftrightarrow p \leq q \wedge (t, T, F) \leq^* (s, S, G)$

Lemma 7.2: *$\sigma : \mathbb{Q} \to \mathbb{P}$, where $\sigma(\langle p, (t, T, F)\rangle) = (t, T, F)$, is a dense embedding.*

Proof: Let $(t, T, F) \in \mathbb{P}$. The function

$$\begin{array}{rcl} \mathrm{suc}_T(t) & \to & \mathrm{Col}(\kappa, {<}\mu) \\ \xi & \mapsto & F(t^\frown\langle\xi\rangle) \end{array}$$

is basically regressive on a set in U. So on some $A \subseteq \mathrm{suc}_T(t)$ in U the function is constant. Call its constant value p. Set $T^* := \{s \in T | s \trianglelefteq t \vee \exists \xi \in A : t^\frown\langle\xi\rangle \trianglelefteq s\}$, then $(t, T^*, F \restriction T^*) \leq^* (t, T, F)$ and $\langle p, (t, T^*, F \restriction T^*)\rangle \in \mathbb{Q}$. So $\mathrm{ran}(\sigma)$ is dense in $\langle \mathbb{P}, \leq\rangle$ (in

$\langle \mathbb{P}, \leq^* \rangle$ even).

Let $\langle p_0, (t_0, T_0, F_0) \rangle, \langle p_1, (t_1, T_1, F_1) \rangle \in \mathbb{Q}$ with $(t_0, T_0, F_0) \parallel (t_1, T_1, F_1)$, so there is some $(s, S, G) \leq (t_0, T_0, F_0), (t_1, T_1, F_1)$. W.l.o.g. assume that $(s, S, G) \in \operatorname{ran}(\sigma)$, say $\sigma(\langle q, (s, S, G) \rangle) = (s, S, G)$ for some q.

It now suffices to show, that $q \leq p_0, p_1$. By symmetry it suffices to show $q \leq p_0$, so for sake of brevity we will omit the subscripts for the rest of the proof. We know $t \trianglelefteq s$. Consider two cases:

(a) Assume that $t \neq s$, say $t^\frown\langle \xi \rangle \trianglelefteq s$ for some $\xi \in \operatorname{suc}_T(t)$. Then by the definition of the partial orders

$$p = F(t^\frown\langle \xi \rangle) \subseteq G(t^\frown\langle \xi \rangle) \subseteq G(s) \subseteq G(s^\frown\langle \xi' \rangle) = q$$

where $\xi' \in \operatorname{suc}_S(s)$ is arbitrary.

(b) Assume that $t = s$. Then $\operatorname{suc}_S(s) \subseteq \operatorname{suc}_T(t)$, so by the definition of the partial orders

$$p = F(t^\frown\langle \xi \rangle) \subseteq G(t^\frown\langle \xi \rangle) = q$$

where $\xi \in \operatorname{suc}_S(s)$ is arbitrary. $\dashv$

Remark: Because $\operatorname{ran}(\sigma)$ was dense in $\langle \mathbb{P}, \leq^* \rangle$, $\langle \mathbb{Q}, \leq, \leq^* \rangle$ inherits the Prikry type. (Say $p \in \mathbb{Q}$. Let $q^* \leq^* \sigma(p)$ decide some statement $\varphi(\tau)$. Take some q with $\sigma(q) \leq^* q^*$, then $q \leq^* p$ and it decides $\varphi(\tau)$.) So there is no harm in confusing $\mathbb{P}$ with $\mathbb{Q}$ from now on.

We want to prove something very much like a product lemma for $\mathbb{Q}$, but we will need a technical lemma first.

Lemma 7.3: *Let $\langle p, (t, T, F) \rangle \in \mathbb{Q}$ and $q \leq p$, then there exist $(t, T^*, F^*) \leq^* (t, T, F)$, such that $\langle q, (t, T^*, F^*) \rangle \in \mathbb{Q}$.*

Proof: Take $\xi < \mu$, $\xi > \max t$ so that $q \in \operatorname{Col}(\kappa, {<}\xi)$.

Define $T^* := \{ s \in T | s \trianglelefteq t \vee \exists \zeta \in \operatorname{suc}_T(t) \cap (\xi, \mu) : t^\frown\langle \zeta \rangle \trianglelefteq s \}$ and

$$F^*(s) = (F(s) \cup q) \restriction (\kappa \times \max s)$$

for $s \in T^*$. Then $(t, T^*, F^*) \leq^* (t, T, F)$ and for $\zeta \in \operatorname{suc}_{T^*}(t)$

$$F^*(t^\frown\langle \zeta \rangle) = F(t^\frown\langle \zeta \rangle) \cup q = p \cup q = q$$

so $\langle q, (t, T^*, F^*) \rangle \in \mathbb{Q}$. $\dashv$

For any $G \subset \operatorname{Col}(\kappa, {<}\mu)$ a generic filter over V, we will write $\mathbb{P}^G := \{ (t, T, F) | \exists p \in G : \langle p, (t, T, F) \rangle \in \mathbb{Q} \}$.

Lemma 7.4: *(a) Let $I \subset \mathbb{Q}$ be generic over V. Then*

$$G := \{p | \exists (t, T, F) : \langle p, (t, T, F) \rangle \in I\}$$

is generic over V for $\mathrm{Col}(\kappa, {<}\mu)$ *and*

$$H := \{(t, T, F) | \exists p : \langle p, (t, T, F) \rangle \in I\}$$

is generic over $V[G]$ for $\mathbb{P}^G$.

(b) Let $G \subset \mathrm{Col}(\kappa, {<}\mu)$ be a generic filter over V and $H \subseteq \mathbb{P}^G$ be generic over $V[G]$. Then $I := \{\langle p, (t, T, F) \rangle \in \mathbb{Q} | p \in G, (t, T, F) \in H\}$ is generic over V.

Proof: (a) With Lemma 7.3 it is trivial to show the genericity of G. So let us fix a name $\dot{D}$ and some $p^* \in G$, such that

$$p^* \Vdash \dot{D} \subseteq \mathbb{P}^{\dot{G}} \text{ is open dense.}$$

It is then enough to show that $D := \{(p, \langle t, T, F \rangle) \in \mathbb{Q} | p \Vdash (\check{t}, \check{T}, \check{F}) \in \dot{D}\}$ is dense below $\langle p^*, (t^*, T^*, F^*) \rangle$ for some (t^*, T^*, F^*) with $\langle p^*, (t^*, T^*, F^*) \rangle \in I$.

So let $\langle p, (t, T, F) \rangle \leq \langle p^*, (t^*, T^*, F^*) \rangle$ be arbitrary. There is some name τ, such that

$$p \Vdash \tau \leq (\check{t}, \check{T}, \check{F}) \wedge \tau \in \dot{D}$$

For some $q \leq p$ we then get an $(s, S, J) \leq (t, T, F)$, such that $q \Vdash (\check{s}, \check{S}, \check{J}) = \tau$. Because "$q \Vdash \dot{D}$ is open" we can assume by Lemma 7.3, that $\langle q, (s, S, J) \rangle \in \mathbb{Q}$. Thus $\langle q, (s, S, J) \rangle \in D$.

(b) Let $D \subseteq \mathbb{Q}$ be dense. It suffices to show, that $\{(t, T, F) \in \mathbb{P}^G | \exists p \in G \langle p, (t, T, F) \rangle \in D\}$ is dense. So let $(t, T, F) \in \mathbb{P}^G$ be arbitrary. Let $p \in G$ be such that $\langle p, (t, T, F) \rangle \in \mathbb{Q}$. Let $p^* \leq p$ be arbitrary, then there is some $(t^*, T^*, F^*) \leq^* (t, T, F)$, such that $\langle p^*, (t^*, T^*, F^*) \rangle \in \mathbb{Q}$. There is now some $\langle q, (s, S, J) \rangle \leq \langle p^*, (t^*, T^*, F^*) \rangle$ in D. So we have shown, that below p there are densely many q, for which some $(s, S, J) \leq (t, T, F)$ exists with $(q, (s, S, J)) \in D$. So there is some $q \in G$ and some $(s, S, J) \leq (t, T, F)$ with $(q, (s, S, J)) \in D$. $\dashv$

Corollary 7.5: *Let $G \subseteq \mathrm{Col}(\kappa, {<}\mu)$ be generic over V. Then $\mathbb{P}^G$ is a ${<}\kappa$ weakly closed Prikry type forcing in $V[G]$.*

Proof: Let $\varphi(\tau)$ be a statement of the forcing language in $V[G]$ and let $p \in G$ force this. Let $(t, T, F) \in \mathbb{P}^G$ be arbitrary. We can assume by Lemma 7.3, that there is some $q \leq p$, such that $\langle q, (t, T, F) \rangle \in \mathbb{Q}$. As $\mathbb{Q}$ is of Prikry type, we can find some $(t^*, T^*, F^*) \leq^* (t, T, F)$ and some $p^* \leq q$, such that

$$\langle p^*, (t^*, T^*, F^*) \rangle \parallel \varphi(\tau),$$

let us assume that $\langle p^*, (t^*, T^*, F^*)\rangle \Vdash \varphi(\tau)$, by the lemma we can write this as

$$p^* \Vdash (\check{t}^*, \check{T}^*, \check{F}^*) \Vdash \varphi(\tau).$$

This proves, that the set of conditions p^*, which force, that there is some direct extension of (t, T, F) deciding $\varphi(\tau)$, is dense below p. (Please excuse our abuse of notation in this proof.) $\dashv$

We have seen how $\mathbb{P}$ can be seen as the combination of a collapse first and a singularizing forcing second. Now we will switch our viewpoint and show that it is equally valid to think of it as the conbination of a singularizing forcing first and a collapse second.

Let $\mathbb{Q}_P$ refer to the tree Prikry forcing. For any condition $(t, T, F) \in \mathbb{P}$ we will define a $\mathbb{Q}_P$-name:

$$\tau_F := \{((s, S), \ulcorner \alpha, \beta, \gamma \urcorner) | (s, S) \leq (t, T) \wedge (\alpha, \beta, \gamma) \in F(s)\}$$

The important thing to note here is that

$$(s, T_s) \Vdash \tau_F \cap (\check{\kappa} \times \max \check{s}) = \check{F(s)} \tag{7.1}$$

for any $s \in T$ with $t \trianglelefteq s$. The exact definition of τ_F doesn't matter as long as this is true. Write $\dot{C}$ for $(\mathrm{Col}(\kappa, {<}\mu))^{\mathbb{Q}_P}$, i.e. a $\mathbb{Q}_P$-name for the collapse of everything below μ to κ in the Prikry generic extension.

Lemma 7.6: *$\sigma : \mathbb{P} \to \mathbb{Q}_P * \dot{C}$ where $\sigma((t, T, F)) = ((t, T), \tau_F)$ is a dense embedding.*

Proof: Let $((t, T), \tau) \in \mathbb{Q}_P * \dot{C}$ be arbitrary. We will now begin pruning the tree T. In the first step we will take some subtree T^0 of T with stem t that fixes the value of τ up to $\max t$ using the Prikry property. Now for any $\xi \in \mathrm{suc}_T(t)$ let T^1_ξ be a subtree of $T^0_{t^\frown\langle\xi\rangle}$ with stem $t^\frown\langle\xi\rangle$ that fixes the value of τ up to ξ. Then let T^1 be the amalgamation of the T^1_ξ. Continuing this for all levels of T we will have defined trees T^n with stem t for all $n < \omega$.

Let $T^* := \bigcap_{n<\omega} T^n$. This tree has the important property that for all $s \in T^*$ with $t \trianglelefteq s$ T^*_s fixes the value of τ upto $\max s$. Call this value $F(s)$.

Claim 1: $\sigma((t, T^*, F)) \leq ((t, T^*), \tau)$

Proof of Claim: It is enough to show, that $(t, T^*) \Vdash \tau = \tau_F$. Assume not. Then there is some $(s, S) \leq (t, T^*)$ that forces that τ and τ_F disagree somewhere before $\alpha < \mu$.

W.l.o.g. $\max s > \alpha$. But then we have

$$(s, S) \Vdash \tau_F \cap (\check{\kappa} \times \max \check{s}) = \check{F(s)}$$

by (7.1) and

$$(s, S) \Vdash \tau \cap (\check{\kappa} \times \max \check{s}) = \check{F(s)}$$

by the definition of F. Contradiction! $\square$

So this shows, that $\mathrm{ran}(\sigma)$ is dense. Now let $(t_0, T_0, F_0), (t_1, T_1, F_1) \in \mathbb{P}$ with $\sigma((t_0, T_0, F_0))||\sigma((t_1, T_1, F_1))$. There must then be some $(s, S) \leq (t_0, T_0), (t_1, T_1)$ such that (s, S) forces τ_{F_0} and τ_{F_1} to be compatible.

Now assume that $F_0(r)$ and $F_1(r)$ are incompatible for some $r \in S$. But then

$$(r, S_r) \Vdash \tau_{F_0} \cap (\check{\kappa} \times \max \check{r} = \check{F_0(r)})$$

and

$$(r, S_r) \Vdash \tau_{F_1} \cap (\check{\kappa} \times \max \check{r} = \check{F_1(r)})$$

by (7.1). Contradiction! ⊣

We now have everything we need to handle the proof of Theorem 6.2 in the countable case:

Let I be a $\mathbb{P}$-generic Filter. By Lemma 7.1 $V[I] \cap \mathcal{P}(\lambda) = V \cap \mathcal{P}(\lambda)$ for every $\lambda < \kappa$. So forcing with $\mathbb{P}$ doesn't change cofinalities or cardinalities strictly below κ and neither does it collapse κ.

We furthermore want κ to remain regular. To see this we only have to switch our viewpoint to the one of Lemma 7.6. We can consider our forcing to consist of two components, the first of which will not add bounded subsets to μ, the second of which will not add $<\kappa$-sequences. So κ must remain regular. On the other hand μ will have countable cofinality, as our forcing contains the tree Prikry forcing.

Take now a $\mathrm{Col}(\kappa, <\mu)$-generic Filter G. Because μ is inaccessible, then $\mu = (\kappa^+)^{V[G]}$. By Lemma 7.4 taking a $\mathbb{P}^G$ generic filter over $V[G]$ then lands us in a $\mathbb{P}$ generic extension. By Lemma 7.6 in there $\mathrm{cof}(\mu) = \omega$ and by the preceding argument neither cofinalities nor cardinalities less than or equal to κ have been changed.

So in fact $\mathbb{P}^G$ witnesses $(*)_{\kappa,\omega}$ in $V[G]$.

8. Changing to uncountable cofinalities

For the uncountable case we will need to replace Prikry forcing by some other forcing construction. We will use the forcing from section 3 of Gitik's paper [Git86]. In many ways this forcing is similar to the tree-Prikry-forcing, so we can apply the same basic idea as in section 2. (The forcing actually is a iteration of tree forcings, but we can easily break it down to the last step.)

For the sake of brevity we will only discuss the things, which we will need for the next section, and blackbox the rest. Gitik's forcing is rather complicated, and we recommend studying [Git86] before continuing.

For this and the next section let $\omega < \eta = \nu < \kappa < \mu$. Let $\vec{U} := \langle U(\alpha, \beta) : \alpha \leq \mu, \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha)\rangle$ be a coherent sequence of normal measures such that $\mathrm{o}^{\tilde{\mathrm{U}}}(\mu) = \nu$, i.e. $U(\alpha, \beta)$ is a non principal normal measure on α and

$$j^\alpha_\beta(\vec{U}) \restriction (\alpha + 1) \times \mathrm{On} = \vec{U} \restriction (\alpha \times \mathrm{On} \cup \{\alpha\} \times \beta$$

for all $(\alpha, \beta) \in \mathrm{dom}(\vec{U})$, where $j^\alpha_\beta : V \to \mathrm{Ult}(V; U(\alpha, \beta))$ is the ultrapower embedding. Let us assume w.l.o.g. that $\mathrm{o}^{\tilde{\mathrm{U}}}(\alpha) = 0$ for all $\alpha \leq \kappa$. Furthermore assume GCH to hold. (Assuming $\mathrm{o}(\mu) \geq \nu$ this can be realized in some model of the form $L\,[E]$. See [Mit74])

We write N^μ_β for the ultrapower by $U(\mu, \beta)$ and j^μ_β for the canonical embedding.

We recursively construct a Gitik-iteration of Prikry-type forcings $\langle \mathbb{P}_\alpha : \alpha \leq \mu\rangle$ using Lemma 6.8.

At each step α of the iteration we will define a sequence of names for forcings $\langle \dot{\mathbb{P}}(\alpha, \beta) : \beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha)\rangle$, of which $\dot{\mathbb{P}}(\alpha, \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha))$ will be the next step of the iteration. The interpretation of these names in an extension by $\mathbb{P}_\alpha$ will add a club subset of order type ω^β (if $\beta > 0$) to α.

Let us assume, that $\mathbb{P}_\mu$ was already constructed. Fix a $\mathbb{P}_\mu$-generic filter G'. Let b_α denote the closure of the clubset, which was added by $\mathbb{P}(\alpha, \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha))$ (so just throw in α too). For α with $\mathrm{o}^{\tilde{\mathrm{U}}}(\alpha) = 0$ let b_α be $\{\alpha\}$. For some finite subset t of μ write b_t for $\bigcup\limits_{\alpha \in t} b_\alpha$. We will now introduce a notion of coherence for finite subsets of μ (which in the following we will often identify with their increasing enumeration).

Definition 8.1: Let $\beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$. We call $t \in [\mu]^{<\omega}$ having the incr. enum. $\langle \delta_0, \ldots, \delta_{n-1}\rangle$ β-coherent iff:

(i) $\beta = 0$ implies $t = \varnothing$

(ii) $\forall i < n : \mathrm{o}^{\tilde{\mathrm{U}}}(\delta_i) < \beta$

(iii) $\forall i < n : b_{\langle \delta_{i^*}, \ldots, \delta_{i-1} \rangle} \trianglelefteq b_{\delta_i}$, where

$$i^* = \begin{cases} \min\{k < i | \forall k \leq j < i : \mathrm{o}^{\tilde{\mathrm{U}}}(\delta_j) < \mathrm{o}^{\tilde{\mathrm{U}}}(\delta_i)\} & \text{if the minimum exists} \\ i & \text{else} \end{cases}$$

(iv) $\forall i < n : \max(b_{\langle \delta_k : k < i^* \rangle}) < \min(b_{\langle \delta_k : i^* \leq i \rangle})$, note, that the maximum always exists.

The set of β-coherent Sequences in $[\mu]^{<\omega}$ is denoted by $\mathrm{Koh}(\mu, \beta)$.

Remark: (a) Let $s, t \in \mathrm{Koh}(\mu, \beta)$, then $s \trianglelefteq t \Rightarrow b_s \trianglelefteq b_t$.

(b) Let $t^\frown \langle \delta_0, \ldots, \delta_{n-1} \rangle^\frown \langle \delta \rangle \in \mathrm{Koh}(\mu, \beta)$, where $\forall i < n : \mathrm{o}^{\tilde{\mathrm{U}}}(\delta_i) < \mathrm{o}^{\tilde{\mathrm{U}}}(\delta)$. Then $b_{t^\frown \langle \delta_0, \ldots, \delta_{n-1} \rangle^\frown \langle \delta \rangle} = b_{t^\frown \langle \delta \rangle}$ (by (iii)).

Definition 8.2: Let $\beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $t \in \mathrm{Koh}(\mu, \beta)$.Note that there are μ^+ many $\mathbb{P}_\mu$-nice-names for subsets of μ. Let $\langle \dot{A}_\xi : \xi < \mu^+ \rangle$ be an enumeration of those nice names. So for every $\mathbb{P}_\mu$-generic Filter G:

$$\langle \dot{A}_\xi^G : \xi < \mu^+ \rangle = \mathcal{P}^{V[G]}(\mu)$$

Write $\mathbb{P}_\gamma^{N_\beta^\mu}$ for the γ-th stage of N_β^μ's iteration, i.e.

$$\mathbb{P}_\gamma^{N_\beta^\mu} = j_\beta^\mu(\langle \mathbb{P}_\alpha : \alpha \leq \mu \rangle)(\gamma)$$

Working in N_β^μ we then find a sequence of $\mathbb{P}_{\mu+1}^{N_\beta^\mu}$-names $\langle \dot{p}_\xi^\beta : \xi < \mu^+ \rangle$ for conditions in $\mathbb{P}_{j_\beta^\mu(\mu)}^{N_\beta^\mu} / \mathbb{P}_{\mu+1}^{N_\beta^\mu}$, such that for all $\xi < \mu^+$

$$\mathbf{1}_{\mathbb{P}_{\mu+1}} \Vdash (\dot{p}_\xi^\beta \parallel \check{\mu} \in j_\beta^\mu(\dot{A}_\xi)) \wedge \dot{p}_{\xi+1}^\gamma \leq^* \dot{p}_\xi^\gamma$$
$$\mathbf{1}_{\mathbb{P}_{\mu+1}} \Vdash (\forall \sigma < \check{\xi} : \dot{p}_\xi^\beta \leq^* \dot{p}_\sigma^\beta) \text{ for a limit } \xi$$

Then we define a filter on μ in the generic ext. $V[G']$, by $A \in U(\mu, \beta, t)$, $A \subseteq \mu$, iff

$$\exists r \in G' \exists \xi < \mu^+ N_\beta^\mu \models \exists \dot{T} \in V^{\mathbb{P}_\mu} : r^\frown (\check{t}, \dot{T})^\frown \dot{p}_\xi^\beta \Vdash \check{\mu} \in j_\beta^\mu(\dot{A})$$

where $\dot{A}$ be some name for A.

Notation: Let $\beta^* < \beta$ and $t \in \mathrm{Koh}(\mu, \beta)$. By $t \upharpoonright \beta^*$ we refer to the longest endsegment of t, which is in $\mathrm{Koh}(\mu, \beta^*)$.

Definition 8.3: Let $T \subseteq \mathrm{Koh}(\mu, \beta)$ be a tree and $\beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$. We call T a β-tree with stem t, iff:

- t is the stem of T, i.e. t is $\trianglelefteq$-maximal with $\forall s \in T : s \trianglelefteq t \vee t \trianglelefteq s$.

- For all $s \in T$
$$\mathrm{suc}_T(s) = \bigcup_{\xi<\beta} \mathrm{suc}_{T,\xi}(s)$$
where $\mathrm{suc}_{T,\xi}(s) \in U(\mu, \xi, t \upharpoonright \xi)$ for all $\xi < \beta$.

The set of all β-Trees with Stem t on μ will be denoted by $\mathrm{Tr}(\mu, \beta, t)$.

Remark: We can and will from now on always assume that $\alpha \in \mathrm{suc}_{T,\xi}(t)$ implies $\mathrm{o}^{\tilde{\mathrm{U}}}(\alpha) = \xi$.

From these terms we will define a Prikry-type notion of forcing, which will add a cofinal sequence to μ.

Definition 8.4: Let $\beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$. Define a partial order $\mathbb{P}(\mu, \beta)$:

- The domain of $\mathbb{P}(\mu, \beta)$ consists of pairs (t, T) where $t \in \mathrm{Koh}(\mu, \beta)$ and $T \in \mathrm{Tr}(\mu, \beta, t)$.
- For $(t_1, T_1), (t_2, T_2) \in \mathbb{P}(\mu, \beta)$ $(t_1, T_1) \leq (t_2, T_2)$ holds, iff:
  - $\exists s \in T_2 : t_2 \trianglelefteq s \wedge b_s = b_{t_1}$
  - $s^\frown(T_1/t_1) \subseteq T_2$
- For $(t_1, T_1), (t_2, T_2) \in \mathbb{P}(\mu, \beta)$ $(t_1, T_1) \leq^* (t_2, T_2)$ holds, iff $(t_1, T_1) \leq (t_2, T_2)$ and $t_1 = t_2$.

Theorem 8.5: *Let $\gamma < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$.*

(a) $\forall \beta \leq \gamma : (\mathrm{Koh}(\mu, \beta))^{N^\mu_\gamma[G']} = \mathrm{Koh}(\mu, \beta)$

(b) $\forall \beta < \gamma \forall t \in \mathrm{Koh}(\mu, \beta) : (U(\mu, \beta, t))^{N^\mu_\gamma[G']} = U(\mu, \beta, t)$

(c) $\forall \beta \leq \gamma : (\mathbb{P}(\mu, \beta))^{N^\mu_\gamma[G']} = \mathbb{P}(\mu, \beta)$

(d) Let $\beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $\dot{Q}$ be a $\mathbb{P}_\mu$-name for the μ-th component of $j^\mu_\beta(\mathbb{P}_\mu)$. Then $\dot{Q}^{G'} = \mathbb{P}(\mu, \beta)$.

(e) Let $\beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $t \in \mathrm{Koh}(\mu, \beta)$. Then $U(\mu, \beta) \subseteq U(\mu, \beta, t)$ and $U(\mu, \beta, t)$ is a $<\mu$-closed Ultrafilter in $V[G']$.

(f) Let $\beta^* < \beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $t \in \mathrm{Koh}(\mu, \beta)$, then $\{\delta < \mu | t^\frown\langle\delta\rangle \in \mathrm{Koh}(\mu, \beta)\} \in U(\mu, \beta^*, t \upharpoonright \beta^*)$.

(g) Let $\beta \leq \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $t_1, t_2 \in \mathrm{Koh}(\mu, \beta)$ be such that $b_{t_1} = b_{t_2}$. Then $U(\mu, \beta, t_1) = U(\mu, \beta, t_2)$. This implies, that for any $T_1 \in \mathrm{Tr}(\mu, \beta, t_1)$ the tree $T_2 := t_2{}^\frown(T_1/t_1)$ is in $\mathrm{Tr}(\mu, \beta, t_2)$ and obviously $(t_1, T_1) \leq (t_2, T_2) \leq (t_1, T_1)$.

(h) $\mathbb{P}(\mu, \gamma)$ is a $<\mu$-weakly closed prikry type forcing notion.

(i) *Let H be a $\mathbb{P}(\mu,\gamma)$-generic filter. Then*

$$\bigcup\{t|\exists T \in \mathrm{Tr}(\mu,\gamma,t) : (t,T) \in H\}$$

is a clubset of ordertype ω^γ.

See [Git86] Lemmata 3.7 - 3.11 for proofs.

Remark: (a) In N^μ_β $j^\mu_\beta(\mathbb{P}_\mu)$ is the iteration of the $\langle\dot{\mathbb{P}}(\alpha, \mathrm{o}^{j^\mu_\beta(\vec{U})}(\alpha) : \alpha < j^\mu_\beta(\mu)\rangle$. Looking at it's μ-th component we get:

$$\mathbb{P}^{N^\mu_\beta[G']}(\mu, \mathrm{o}^{j^\mu_\beta(\vec{U})}(\mu)) = \mathbb{P}^{N^\mu_\beta[G']}(\mu,\beta) = \mathbb{P}(\mu,\beta)$$

(b) Working in N^μ_β let $r^\frown\langle t,\dot{T}\rangle^\frown p^\mu_\alpha$ be a condition as appearing in the definition of "$A \in U(\mu,\beta,t)''$. It follows from (a), that $\dot{T}^{G'} \in \mathrm{Tr}(\mu,\beta,t)$.

Note that the $U(\mu,\beta,t)$ can't be assumed to be all normal (for positive β they definitely won't be).But we can prove this:

Lemma 8.6: *Let $\beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $t \in \mathrm{Koh}(\mu,\beta)$. Take some $A \in U(\mu,\beta,t)$ and some regressive $f : A \to \mu$. Then there exists some $s \in \mathrm{Koh}(\mu,\beta)$ end-extending t and $B \subseteq A$, such that $f \restriction B$ is constant and $B \in U(\mu,\beta,s)$.*

Proof: Fix $T \in N^\mu_\beta$ some β-tree with stem t, such that for some $\xi < \mu^+$:

$$N^\mu_\beta\left[G'\right] \models (t,T)^\frown\dot{p}^\beta_\xi \Vdash \check{\mu} \in j^\mu_\beta(\dot{A})$$

Now fix first a name $\dot{f}$ for f and nice names for the set A and the sets $\{\delta \in A|f(\delta) = \eta\}$. Then there is a $\xi < \mu^+$ by which all of these nice names have been enumerated in our fixed enumeration. Fix one of those too.

Work in $N^\mu_\beta[G']$. Take some $(s,S)^\frown\dot{p} \leq (t,T)^\frown\dot{p}^\beta_\xi$ that decides the value of $j^\mu_\beta(\dot{f})(\check{\mu})$, say it forces it to be some $\eta < \mu$. Then we get, that

$$N^\mu_\beta\left[G'\right] \models (s,S)^\frown\dot{p} \Vdash \check{\mu} \in \{\delta \in j^\mu_\beta(\dot{A})|j^\mu_\beta(\dot{f})(\delta) = \check{\eta}\}$$

but by the choice of $\dot{p}^\beta_\xi$ we can find some $S^* \subseteq S$ such that

$$N^\mu_\beta\left[G'\right] \models (s,S^*)^\frown\dot{p}^\beta_\xi \Vdash \check{\mu} \in \{\delta \in j^\mu_\beta(\dot{A})|j^\mu_\beta(\dot{f})(\delta) = \check{\eta}\}$$

so $\{\delta \in A|f(\delta) = \eta\} \in U(\mu,\beta,s)$. W.l.o.g. $t \trianglelefteq s$, so we're done. ⊣

Corollary 8.7: *$U(\mu,0,\varnothing)$ is normal.*

9. The uncountable case

Let $\mu, \kappa, \nu, \langle U(\alpha, \beta) : \alpha \leq \mu, \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha)\rangle, G'$ be as in section 3. Working in $W := V[G']$, we define a partial order $\mathbb{P}$:

The forcing $\mathbb{P}$ consists of triples (t, T, F), where $t \in [\mu]^{<\omega}$, a tree $T \subset [\mu]^{<\omega}$ (to be understood as a tree on increasing sequences), and a function $F : T \to \mathrm{Col}(\kappa, {<}\mu)$. A triple (t, T, F) is a condition in $\mathbb{P}$ iff

- $F(\emptyset) = \emptyset$,
- $T \in \mathrm{Tr}(\mu, \nu, t)$,
- $\forall t, t' \in T : t \neq \emptyset \Rightarrow (F(t) \in \mathrm{Col}(\kappa, < \max t) \wedge t \trianglelefteq t' \Rightarrow F(t) = F(t') \restriction (\kappa \times \max t))$,
- $b_t = b_{t'} \Rightarrow F(t) = F(t')$ (observe $\max t = \max t'$).

Partially order $\mathbb{P}$ by $(t', T', F') \leq (t, T, F)$ iff

$$\exists s \in T \Big[b_s = b_{t'} \wedge t \trianglelefteq s \wedge s^\frown(T'/t') \subset T \Big] \text{ and } \forall r \in s^\frown(T'/t') : F''(r) \supseteq F(r),$$

where F'' is defined on $s^\frown(T'/t')$ from F' in the obvious way (with $F''(s) = F'(t')$). We say that (t', T', F') is a direct extension of (t, T, F), $(t', T', F') \leq^* (t, T, F)$, iff moreover $t = t'$.

Lemma 9.1: $\langle \mathbb{P}, \leq, \leq^* \rangle$ *is a weakly* ${<}\kappa$*-closed Prikry type forcing.*

Proof: The weak closure is easy, so let us concentrate on the Prikry type.

Let $\varphi(\tau)$ be some statement and $(t, T, F) \in \mathbb{P}$. We want to show that there is some direct extension, which decides $\varphi(\tau)$. For this it suffices to show, that if there exists some $\beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and some $A \subseteq \mathrm{suc}_{T,\beta}(t)$ in $U(\mu, \beta, t \upharpoonright \beta)$, such that for all $\xi \in A$ there exist direct extensions of $(t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \restriction T_{t^\frown\langle\xi\rangle})$, which decide $\varphi(\tau)$, then there exists a direct extension of (t, T, F) which decides $\varphi(\tau)$.

If this were to hold, and we had no direct extension of (t, T, F), which decides $\varphi(\tau)$, then by pruning the tree level-by-level, we would arrive at some condition $(t, T^*, F \restriction T^*)$, such that for all $s \in T^*$ there would be no direct extension of $(s, T^*_s, F \restriction T^*_s)$ which decides $\varphi(\tau)$, but that's nonsense. We used pretty much the same argument in the proof of Lemma 7.1.

So let us fix some $\beta^* < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ and $A_{\beta^*} \subseteq \mathrm{suc}_{T,\beta^*}(t)$ in $U(\mu, \beta^*, t \upharpoonright \beta^*)$ and direct extensions of $(t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \restriction T_{t^\frown\langle\xi\rangle})$ $(t^\frown\langle\xi\rangle, T_\xi, F_\xi)$ for $\xi \in A_{\beta*}$ which decide $\varphi(\tau)$. By shrinking A_{β^*} we can assume the $(t^\frown\langle\xi\rangle, T_\xi, F_\xi)$ to decide $\varphi(\tau)$ the same way (let's just assume positively). By further shrinking we can assume the F_ξ to agree upto $\max(t)$.

As a first step we want to construct for $\beta^* < \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ $A_\beta \subseteq \mathrm{suc}_{T,\beta}(t)$ in $U(\mu, \beta, t \upharpoonright \beta)$ and direct extensions of $(t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \upharpoonright T_{t^\frown\langle\xi\rangle})$ $(t^\frown\langle\xi\rangle, T_\xi, F_\xi)$ for $\xi \in A_\beta$ which force $\varphi(\tau)$.

Claim 1: $\mathrm{suc}_{T,\beta}(t) \cap \bigcup_{\xi \in A_{\beta^*}} \mathrm{suc}_{T_\xi,\beta}(t^\frown\langle\xi\rangle) \in U(\mu, \beta, t \upharpoonright \beta)$

Proof of Claim: Assume not. $A := \mathrm{suc}_{T,\beta}(t) \setminus \bigcup_{\xi \in A_{\beta^*}} \mathrm{suc}_{T_\xi,\beta}(t^\frown\langle\xi\rangle) \in U(\mu, \beta, t \upharpoonright \beta)$, so we can fix some $S \in \mathrm{Tr}(\mu, \beta, t \upharpoonright \beta)$, such that for some $\alpha < \mu^+$:

$$N^\mu_\beta \left[G'\right] \models (t \upharpoonright \beta, S)^\frown p^\beta_\alpha \Vdash \check{\mu} \in j^\mu_\beta(\dot{A})$$

Then $\mathrm{suc}_{S,\beta^*} \in U(\mu, \beta^*, (t \upharpoonright \beta) \upharpoonright \beta^*)$. $(t \upharpoonright \beta) \upharpoonright \beta^* = t \upharpoonright \beta^*$ and thus $A_{\beta^*} \cap \mathrm{suc}_{S,\beta^*}$ is nonempty, so fix some ξ in it. Then $(t \upharpoonright \beta^\frown\langle\xi\rangle, S_{t\upharpoonright\beta^\frown\langle\xi\rangle}) \leq (t \upharpoonright \beta, S)$, so

$$N^\mu_\beta \left[G'\right] \models (t \upharpoonright \beta^\frown\langle\xi\rangle, S_{t\upharpoonright\beta^\frown\langle\xi\rangle})^\frown p^\beta_\alpha \Vdash \check{\mu} \in j^\mu_\beta(\dot{A})$$

which implies, that both A and $\mathrm{suc}_{T_\xi,\beta}(t^\frown\langle\xi\rangle)$ are in $U(\mu, \beta, t^\frown\langle\xi\rangle \upharpoonright \beta)$ (note that $(t^\frown\langle\xi\rangle) \upharpoonright \beta = t \upharpoonright \beta^\frown\langle\xi\rangle)$,and thus have nonempty intersection. But this contradicts the choice of A! $\square$

So set $A_\beta := \mathrm{suc}_{T,\beta}(t) \cap \bigcup_{\xi \in A_{\beta^*}} \mathrm{suc}_{T_\xi,\beta}(t^\frown\langle\xi\rangle)$. By Theorem 8.5 (f) the $(t^\frown\langle\xi\rangle, T_\xi, F_\xi)$ then induce direct extensions of $(t^\frown\langle\xi\rangle, T_{t^\frown\langle\xi\rangle}, F \upharpoonright T_{t^\frown\langle\xi\rangle})$ for $\xi \in A_\beta$ which force $\varphi(\tau)$.

To see this first note that, if $\xi' \in A_\beta$ then there is a unique $\xi \in A_{\beta^*}$, such that $\xi' \in \mathrm{suc}_{T_\xi,\beta}(t^\frown\langle\xi\rangle)$ (ξ is the smallest ordinal $> \max(t)$ of order β^* in $b_{\xi'}$). For some $t^\frown\langle\xi'\rangle^\frown s \in T$ we can then set $t^\frown\langle\xi'\rangle^\frown s \in T_{\xi'}$ iff $t^\frown\langle\xi, \xi'\rangle^\frown s \in T_\xi$ for the unique ξ such that $t^\frown\langle\xi, \xi'\rangle \in T_\xi$, and similarly $F_{\xi'}(t^\frown\langle\xi'\rangle^\frown s) := F_\xi(t^\frown\langle\xi, \xi'\rangle^\frown s)$.

Note that $b_{t^\frown\langle\xi'\rangle^\frown s} = b_{t^\frown\langle\xi,\xi'\rangle^\frown s}$, so indeed $(t^\frown\langle\xi'\rangle, T_{\xi'}, F_{\xi'}) \leq^* (t^\frown\langle\xi'\rangle, T_{t^\frown\langle\xi'\rangle}, F \upharpoonright T_{t^\frown\langle\xi'\rangle})$. Here we use, that

$$F(t^\frown\langle\xi'\rangle^\frown s) = F(t^\frown\langle\xi, \xi'\rangle^\frown s) \subseteq F_\xi(t^\frown\langle\xi, \xi'\rangle^\frown s) = F_{\xi'}(t^\frown\langle\xi'\rangle^\frown s).$$

On the other hand

$$(t^\frown\langle\xi'\rangle, T_{\xi'}, F_{\xi'}) \leq (t^\frown\langle\xi, \xi'\rangle, (T_\xi)_{t^\frown\langle\xi,\xi'\rangle}, F_\xi \upharpoonright (T_\xi)_{t^\frown\langle\xi,\xi'\rangle}) \leq (t^\frown\langle\xi\rangle, T_\xi, F_\xi),$$

so $(t^\frown\langle\xi'\rangle, T_{\xi'}, F_{\xi'}) \Vdash \varphi(\tau)$.

We now have all the material we need to construct T^*. We will call $s := t^\frown\langle\delta_i : i < n\rangle \in T$ *short*, iff $\mathrm{o}^{\tilde{\mathrm{U}}}(\delta_i) < \beta^*$ for all $i < n$. If $r \in T$ and $t \trianglelefteq r$, we will write $r_{<\beta^*}$ for the longest initial segment of r, which is short.

Define S as the set of initial segments of $(t \upharpoonright \beta^*)^\frown\langle\delta_i : i < n\rangle$, where $t^\frown\langle\delta_i : i < n\rangle \in T$ is short. It is easy to see, that S is a β^*-tree with stem $t \upharpoonright \beta^*$. If $s \in S$ write s^β for $s \cup (t \upharpoonright \beta)$. Note that for $\beta^* \leq \beta < \beta'$ $s^{\beta'} \upharpoonright \beta = s^\beta$.

We will now construct in an induction by level (note that we start counting from the end of the stem on, also remember $\nu = \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$):

- An β^*-tree $S^* \subseteq S$ with stem $t \upharpoonright \beta^*$,
- for every $s \in S^*$ sets $\langle A_\beta^{s^\nu} : \beta^* \leq \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)\rangle$ with $A_\beta \supseteq A_\beta^{s^\nu} \in U(\mu, \beta, s^\beta)$,
- for every ξ in some $A_\beta^{s^\nu}$ a ν-tree $T_\xi^{s^\nu}$ with stem $r := s^\nu \frown \langle \xi \rangle$ and $F_\xi^s : T_\xi^{s^\nu} \to \mathrm{Col}(\kappa, {<}\mu)$, such that $(r, T_\xi^{s^\nu}, F_\xi^s) \leq (r, T_r, F \upharpoonright T_r)$ and $(r, T_\xi^{s^\nu}, F_\xi^s) \Vdash \varphi(\tau)$ and $F_\xi^{s^\nu}(r)$ only depends on $\mathrm{o}^{\tilde{\mathrm{U}}}(\xi)$,
- for every $s \in S^*$ and $\beta^* \leq \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\mu)$ an $S_\beta^{s^\nu} \in \mathrm{Tr}(\mu, \beta, s^\beta)$, which witnesses that $A_\beta^{s^\nu} \in U(\mu, \beta, s^\beta)$, i.e.

  $$N_\beta^\mu \left[G'\right] \models (s^\beta, S_\beta^{s^\nu})^\frown p_\alpha^\beta \Vdash \check{\mu} \in j_\beta^\mu(\dot{A}_\beta^{s^\nu})$$

  for some $\alpha < \mu^+$.

Begin by setting $A_\beta^t := A_\beta$,$T_\xi^t := T_\xi$,$F_\xi^t := F_\xi$. Note that the F_ξ^t all agree up to $\max(t)$ because they all directly derive from the original set of functions.

So let $s \in S$ and assume that everything is already constructed for all strict initial segments of it. We then require s^β to be in $S_\beta^{r^\nu}$ for all strict initial segments r of s and all $\beta^* \leq \beta < \nu$.

Claim 2: $A_\beta \in U(\mu, \beta, s^\beta)$

Proof of Claim: By choice $s^\beta \in S_\beta^t$, so $(s^\beta, (S_\beta^t)_{s^\beta}) \leq ((t \upharpoonright \beta), S_\beta^t)$, but then by choice of S_β^t:

$$N_\beta^\mu \left[G'\right] \models (s^\beta, (S_\beta^t)_{s^\beta})^\frown p_\alpha^\beta \Vdash \check{\mu} \in j_\beta^\mu(\dot{A}_\beta)$$

But this means that $A_\beta \in U(\mu, \beta, s^\beta)$. □

So we can set $A_\beta^{s^\nu} := A_\beta \cap \mathrm{suc}_{T,\beta}(s^\nu)$.

Then we can for $\xi \in A_\beta^{s^\nu}$ set $T_\xi^{s^\nu} := ((s^\nu \frown \langle \xi \rangle)^\frown (T_\xi / t^\frown \langle \xi \rangle)) \cap T$. $F_\xi^{s^\nu}$ is then defined from F_ξ by setting

$$F_\xi^{s^\nu}(r) = \begin{cases} F_\xi(t^\frown \langle \xi \rangle) \cap (\kappa \times \max r) & r \trianglelefteq s^\nu, \\ F_\xi(t^\frown r') & r = s^\nu \frown r'. \end{cases}$$

We can assume that $F_\xi^{s^\nu}$ and $F_\eta^{s^\nu}$ agree up to $\max(s)$ for ξ, η of the same order, simply by shrinking the corresponding $A_\beta^{s^\nu}$ further. Finally note that $(s^\nu \frown \langle \xi \rangle, T_\xi^{s^\nu}, F_\xi^{s^\nu}) \Vdash \varphi(\tau)$, because it is a stronger condition than $(t^\frown \langle \xi \rangle, T_\xi, F_\xi)$. (Note that $b_{s^\nu \frown \langle \xi \rangle} = b_{t^\frown \langle \xi \rangle}$.)

We can now pick some $S_\beta^{s^\nu}$ which witness $A_\beta^{s^\nu} \in U(\mu, \beta, s^\beta)$. Thus ends the construction.

From all this we can now derive a conditon (t, T^*, F^*). Let $r \in T^*$, iff

$r \in T$

and $r_{<\beta^*} \upharpoonright \beta^* \in S^*$

and $\xi := \min r \backslash r_{<\beta^*} \in A^{r_{<\beta^*}}_{\mathrm{o}^{\tilde{\mathrm{U}}}(\xi)}$ (if r is short, this condition is empty)

and $r \in T^{r_{<\beta^*}}_{\xi}$ (if r is short, this condition is empty)

For $r \in T^*$ we set $F^*(r) = F^{r_{<\beta^*}}_{\min r \backslash r_{<\beta^*}}(r)$ if r is not short. Otherwise we set $F^*(r) = F^r_\xi(r)$, where $\xi \in A^r_{\beta^*}$ is arbitrary.

Note that if $r \in T^*$ is not short, then by breaking down the definitions $t^\frown(r \backslash r_{<\beta^*}) \in T_\xi$ and

$$F^{r_{<\beta^*}}_{\xi}(r) = F_\xi(t^\frown(r \backslash r_{<\beta^*})), \tag{9.1}$$

where $\xi := \min(r \backslash r_{<\beta^*})$.

Claim 3: $(t, T^*, F^*) \in \mathbb{P}$.

Proof of Claim: It is easy to see, that T^* is a ν-tree, so it remains to be seen, that for any $r, r' \in T^*$:

(a) if $b_r = b_{r'}$, then $F^*(r) = F^*(r')$,

(b) if $r \trianglelefteq r'$, then $F^*(r') \cap (\kappa \times \max r) = F^*(r)$.

Let us start with (a). There are two cases:

1st case:
Assume that both r, r' are short. Then $A^r_{\beta^*} \in U(\mu, \beta^*, r \upharpoonright \beta^*)$ and $A^{r'}_{\beta^*} \in U(\mu, \beta^*, r' \upharpoonright \beta^*)$. By assumption and Theorem 8.5 (g) we have $U(\mu, \beta^*, r \upharpoonright \beta^*) = U(\mu, \beta^*, r' \upharpoonright \beta^*)$, and thus there is some $\xi \in A^r_{\beta^*} \cap A^{r'}_{\beta^*}$. We then have

$$F^*(r) = F^r_\xi(r) = F^r_\xi(r^\frown\langle\xi\rangle) \cap (\kappa \times \max(r)) \overset{(9.1)}{=} F_\xi(t^\frown\langle\xi\rangle) \cap (\kappa \times \max(r))$$

and

$$F^*(r') = F^{r'}_\xi(r') = F^{r'}_\xi(r'^\frown\langle\xi\rangle) \cap (\kappa \times \max(r')) \overset{(9.1)}{=} F_\xi(t^\frown\langle\xi\rangle) \cap (\kappa \times \max(r'))$$

as $\max r = \max r'$ we are done.

2nd case:
Assume that neither r nor r' are short. Let $\xi = \min r \backslash r_{<\beta^*}$ and $\xi' = \min r' \backslash r'_{<\beta^*}$. If $\xi = \xi'$ we are done, as then $t^\frown(r \backslash r_{<\beta^*}), t^\frown(r' \backslash r'_{<\beta^*}) \in T_\xi$ and thus

$$F^*(r) \overset{(9.1)}{=} F_\xi(t^\frown(r \backslash r_{<\beta^*})) = F_\xi(t^\frown(r' \backslash r'_{<\beta^*})) \overset{(9.1)}{=} F^*(r').$$

So let us assume $\xi \neq \xi'$. It follows that $\mathrm{o}^{\tilde{\mathrm{U}}}(\xi) \neq \mathrm{o}^{\tilde{\mathrm{U}}}(\xi')$. Otherwise

$$\xi = \min\{\zeta \in b_r \backslash (\max t{+}1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \mathrm{o}^{\tilde{\mathrm{U}}}(\xi)\} = \min\{\zeta \in b_{r'} \backslash (\max t{+}1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \mathrm{o}^{\tilde{\mathrm{U}}}(\xi')\} = \xi'.$$

Let us first assume that $\beta^* = \mathrm{o}^{\tilde{\mathrm{U}}}(\xi) < \mathrm{o}^{\tilde{\mathrm{U}}}(\xi') := \beta$. Then

$$\xi = \min\{\zeta \in b_r \backslash (\max t+1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \beta^*\} = \min\{\zeta \in b_{r'} \backslash (\max t+1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \beta^*\}$$

and thus $t^\frown\langle\xi\rangle^\frown(r'\backslash r'_{<\beta^*}) \in T_\xi$ by construction of $T_{\xi'}$. We then get

$$F^*(r) \overset{(9.1)}{=} F_\xi(t^\frown(r\backslash r_{<\beta^*})) = F_\xi(t^\frown\langle\xi\rangle^\frown(r'\backslash r'_{<\beta^*}))$$

and

$$F^*(r') \overset{(9.1)}{=} F_{\xi'}(t^\frown(r'\backslash r'_{<\beta^*})) = F_\xi(t^\frown\langle\xi\rangle^\frown(r'\backslash r'_{<\beta^*})).$$

So finally assume $\beta^* < \mathrm{o}^{\tilde{\mathrm{U}}}(\xi), \mathrm{o}^{\tilde{\mathrm{U}}}(\xi')$, then set

$$\xi'' := \min\{\zeta \in b_r \backslash (\max t+1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \beta^*\} = \min\{\zeta \in b_{r'} \backslash (\max t+1) \mid \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) = \beta^*\}.$$

and we will have $t^\frown\langle\xi''\rangle^\frown(r\backslash r_{<\beta^*}), t^\frown\langle\xi''\rangle^\frown(r'\backslash r'_{<\beta^*}) \in T_{\xi''}$. By definition

$$F^*(r) \overset{(9.1)}{=} F_\xi(t^\frown(r\backslash r_{<\beta})) = F_{\xi''}(t^\frown\langle\xi''\rangle^\frown(r\backslash r_{<\beta^*}))$$

and

$$F^*(r') \overset{(9.1)}{=} F_\xi(t^\frown(r'\backslash r'_{<\beta})) = F_{\xi''}(t^\frown\langle\xi''\rangle^\frown(r'\backslash r'_{<\beta^*}))$$

, so we can finish the proof of (a), by noticing that $b_{t^\frown\langle\xi''\rangle^\frown(r\backslash r_{<\beta^*})} = b_{t^\frown\langle\xi''\rangle^\frown(r'\backslash r'_{<\beta^*})}$ and thus

$$F_{\xi''}(t^\frown\langle\xi''\rangle^\frown(r\backslash r_{<\beta^*})) = F_{\xi''}(t^\frown\langle\xi''\rangle^\frown(r'\backslash r'_{<\beta^*})).$$

We will now continue with the proof of (b) (w.l.o.g. assume $r \neq r'$). There are three cases:

1st case:
Assume that both r, r' are short. By definition $r' \restriction \beta^* \in S^r_{\beta^*}$. $S^r_{\beta^*}$ was chosen such that

$$N^\mu_{\beta^*}\left[G'\right] \models (r \restriction \beta^*, S^r_{\beta^*})^\frown p^{\beta^*}_\alpha \Vdash \check{\mu} \in j^\mu_{\beta^*}(\dot{A}^r_{\beta^*})$$

for some $\alpha < \mu^+$. (This means that $A^r_{\beta^*} \in U(\mu, \beta^*, r \restriction \beta^*)$.) We then have

$$N^\mu_\beta\left[G'\right] \models (r' \restriction \beta^*, (S^r_{\beta^*})_{r'\restriction\beta^*})^\frown p^\beta_\alpha \Vdash \check{\mu} \in j^\mu_{\beta^*}(\dot{A}^r_{\beta^*})$$

thus $A^r_{\beta^*} \in U(\mu, \beta^*, r' \restriction \beta^*)$. So there is some $\xi \in A^r_{\beta^*} \cap A^{r'}_{\beta^*}$. For this ξ we have

$$F^*(r) \overset{(9.1)}{=} F^r_\xi(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r) = F_\xi(t^\frown\langle\xi\rangle) \cap (\kappa \times \max r)$$

and

$$F^*(r') \overset{(9.1)}{=} F^{r'}_\xi(r'^\frown\langle\xi\rangle) \cap (\kappa \times \max r') = F_\xi(t^\frown\langle\xi\rangle) \cap (\kappa \times \max r').$$

2nd case:
Assume that neither r nor r' are short. Let

$$\xi := \min\{\zeta \in r \backslash (\max t + 1) \,|\, \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) \geq \beta^*\} = \min\{\zeta \in r' \backslash (\max t + 1) \,|\, \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) \geq \beta^*\}.$$

Then $t^\frown(r \backslash r_{<\beta^*}), t^\frown(r' \backslash r'_{<\beta^*}) \in T_\xi$ and $t^\frown(r \backslash r_{<\beta^*}) \trianglelefteq t^\frown(r' \backslash r'_{<\beta^*})$. So

$$F^*(r) \overset{(9.1)}{=} F_\xi(t^\frown(r \backslash r_{<\beta^*})) = F_\xi(t^\frown(r' \backslash r'_{<\beta^*})) \cap (\kappa \times \max r)$$

and

$$F^*(r') \overset{(9.1)}{=} F_\xi(t^\frown(r' \backslash r'_{<\beta^*}))$$

3rd case:
Assume that r is short, but r' isn't. Utilizing the 1st case, we can assume that $r = r'_{<\beta^*}$. Let $\xi := \min\{\zeta \in r' \backslash (\max t + 1) \,|\, \mathrm{o}^{\tilde{\mathrm{U}}}(\zeta) \geq \beta^*\}$, set $\beta := \mathrm{o}^{\tilde{\mathrm{U}}}(\xi)$. Utilizing the second case it is enough to show $F^*(r) = F^*(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r)$. First assume $\beta = \beta^*$. Then by definition:

$$F^*(r) = F^r_\xi(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r) = F^*(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r)$$

So assume $\beta > \beta^*$. Notice then that $\mathrm{suc}_{T^*,\beta^*}(r) \cap \mathrm{suc}_{S^r_\beta,\beta^*}(r \restriction \beta) \in U(\mu, \beta^*, r \restriction \beta^*)$. So there is some ξ' in the intersection. By the argument from above we have that $A^r_\beta \in U(\mu, \beta, (r \restriction \beta)^\frown\langle\xi'\rangle)$. So there is some $\xi'' \in A^r_\beta \cap \mathrm{suc}_{T^*,\beta}(r^\frown\langle\xi'\rangle)$. We then get

$$F^*(r^\frown\langle\xi', \xi''\rangle) = F^*(r^\frown\langle\xi''\rangle)$$

using (a). By the choice of F^* we furthermore have

$$F^*(r^\frown\langle\xi''\rangle) \cap (\kappa \times \max r) = F^r_{\xi''}(r^\frown\langle\xi''\rangle) \cap (\kappa \times \max r) = F^r_\xi(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r) = F^*(r^\frown\langle\xi\rangle) \cap (\kappa \times \max r).$$

To tie all this together we have

$$F^*(r) = F^r_{\xi'}(r^\frown\langle\xi'\rangle) \cap (\kappa \times \max r) = F^*(r^\frown\langle\xi'\rangle) \cap (\kappa \times \max r)$$

and

$$F^*(r^\frown\langle\xi', \xi''\rangle) \cap (\kappa \times \xi') = F^*(r^\frown\langle\xi'\rangle). \qquad \square$$

Claim 4: $(t, T^*, F^*) \leq^* (t, T, F)$

Proof of Claim: It is obvious, that $T^* \subseteq T$. So it remains to show, that

$$\forall s \in T^* : F(s) \subseteq F^*(s)$$

We can assume $t \trianglelefteq s$, because it suffices to show this for a cofinal subset. There are two cases:

1st case:
Let s be short. Let $\xi \in A^s_{\beta^*}$ be arbitrary. Then $b_{s^\frown\langle\xi\rangle} = b_{t^\frown\langle\xi\rangle}$ and thus $F(s^\frown\langle\xi\rangle) = F(t^\frown\langle\xi\rangle)$. So

$$\begin{aligned} F^*(s) =& F^s_\xi(s^\frown\langle\xi\rangle) \cap (\kappa \times \max s) \\ \overset{(9.1)}{=}& F_\xi(t^\frown\langle\xi\rangle) \cap (\kappa \times \max s) \\ \supseteq& F(t^\frown\langle\xi\rangle) \cap (\kappa \times \max s) \\ =& F(s^\frown\langle\xi\rangle) \cap (\kappa \times \max s) = F(s). \end{aligned}$$

2nd case:
Assume s is not short. Let $\xi := \min(s \backslash s_{<\beta^*})$. Then $t^\frown(s\backslash s_{<\beta^*}) \in T_\xi$ and $b_s = b_{t^\frown(s\backslash s_{<\beta^*})}$, thus $F(s) = F(t^\frown(s\backslash s_{<\beta^*}))$, and we get:

$$F^*(s) \overset{(9.1)}{=} F_\xi(t^\frown(s\backslash s_{<\beta^*}) \supseteq F(t^\frown(s\backslash s_{<\beta^*})) = F(s) \qquad \square$$

So it remains to see, that $(t, T^*, F^*) \Vdash \varphi(\tau)$. For that look at an arbitrary $(s, T^{**}, F^{**}) \leq (t, T^*, F^*)$. Note that we can assume $s \in T^*$.

1st case:
Let us first assume that s is short. Then take any $\xi \in A^s_{\beta^*}$ such that $s^\frown\langle\xi\rangle \in T^{**}$. Then $(s^\frown\langle\xi\rangle, T^{**}_{s^\frown\langle\xi\rangle}, F^{**} \upharpoonright T^{**}_{s^\frown\langle\xi\rangle}) \leq (s, T^{**}, F^{**}), (s^\frown\langle\xi\rangle, T^s_\xi, F^s_\xi)$. Remember that $(s^\frown\langle\xi\rangle, T^s_\xi, F^s_\xi) \Vdash \varphi(\tau)$ and we are done.

2nd case:
Assume that s is not short. Let $\xi := \min(s\backslash s_{<\beta^*})$. Then

$$(s, T^{**}, F^{**}) \leq (s, (T^*)_s, F^* \upharpoonright (T^*)_s) \leq (s_{<\beta^*}{}^\frown\langle\xi\rangle, T^{s_{<\beta^*}}_\xi, F^{s_{<\beta^*}}_\xi)$$

Remember that $(s_{<\beta^*}{}^\frown\langle\xi\rangle, T^s_\xi, F^s_\xi) \Vdash \varphi(\tau)$ and we are done. $\dashv$

We will now show, that $\mathbb{P}$ can be written in a form reminiscent of a product, whose first component is $\mathrm{Col}(\kappa, <\mu)$. Write $\langle \mathbb{Q}, \leq, \leq^* \rangle$ for the following forcing:

- $\mathbb{Q} = \{\langle p, (t, T, F)\rangle | p \in \mathrm{Col}(\kappa, <\mu), (t, T, F) \in \mathbb{P}, \forall \zeta \in \mathrm{suc}_{T,0}(t) F(t^\frown\langle\zeta\rangle) = p \wedge \forall 0 < \beta < \nu\, \mathrm{suc}_{T,\beta}(t) \subseteq \bigcup_{\zeta \in \mathrm{suc}_{T,0}(t)} \mathrm{suc}_{T,\beta}(t^\frown\langle\zeta\rangle)\}$

- $\langle p,(t,T,F)\rangle \le \langle q,(s,S,G)\rangle \Leftrightarrow p \le q \wedge (t,T,F) \le (s,S,G)$
- $\langle p,(t,T,F)\rangle \le^* \langle q,(s,S,G)\rangle \Leftrightarrow p \le q \wedge (t,T,F) \le^* (s,S,G)$

Lemma 9.2: *$\sigma : \mathbb{Q} \to \mathbb{P}$, where $\sigma(\langle p,(t,T,F)\rangle) = (t,T,F)$, is a dense embedding.*

Proof: Let $(t,T,F) \in \mathbb{P}$. The function

$$\begin{array}{rcl} \mathrm{suc}_{T,0}(t) & \to & \mathrm{Col}(\kappa,{<}\mu) \\ \xi & \mapsto & F(t^\frown\langle\xi\rangle) \end{array}$$

is basically regressive on a set in $U(\mu,0,\emptyset)$. So on some $A_0 \subseteq \mathrm{suc}_{T,0}(t)$ in U the function is constant. Call it's constant value p. Define $A_\beta := \mathrm{suc}_{T,\beta}(t) \cap \bigcup_{\zeta\in A_0} \mathrm{suc}_{T,\beta}(t^\frown\langle\zeta\rangle)$. Note that by the proof of Lemma 9.1 $A_\beta \in U(\mu,\beta,t \upharpoonright \beta)$. Set $T^* := \{s \in T | s \trianglelefteq t \vee \exists\beta < \nu \exists\xi \in A_\beta : t^\frown\langle\xi\rangle \trianglelefteq s\}$ then $(t,T^*,F \upharpoonright T^*) \le (t,T,F)$ and $\langle p,(t,T^*,F \upharpoonright T^*)\rangle \in \mathbb{Q}$. So $\mathrm{ran}(\sigma)$ is dense in $\langle\mathbb{P},\le\rangle$ (in $\langle\mathbb{P},\le^*\rangle$ even).

Let $\langle p_0,(t_0,T_0,F_0)\rangle, \langle p_1,(t_1,T_1,F_1)\rangle \in \mathbb{Q}$ with $(t_0,T_0,F_0) \parallel (t_1,T_1,F_1)$, so there is some $(s,S,G) \le (t_0,T_0,F_0),(t_1,T_1,F_1)$. W.l.o.g. assume that $(s,S,G) \in \mathrm{ran}(\sigma)$ say $\sigma(\langle q,(s,S,G)\rangle) = (s,S,G)$ for some q.

It now suffices to show, that $q \le p_0, p_1$. By symmetry it suffices to show $q \le p_0$, so for sake of brevity we will omit the subscripts for the rest of the proof. We can assume $t \trianglelefteq s$. Consider three cases:

(a) Assume that $t \ne s$, say $t^\frown\langle\xi\rangle \trianglelefteq s$ for some $\xi \in \mathrm{suc}_T(t)$. Let us furthermore assume $\mathrm{o}^{\tilde{\mathrm{U}}}(\xi) = 0$. Then by the definition of the partial orders

$$p = F(t^\frown\langle\xi\rangle) \subseteq G(t^\frown\langle\xi\rangle) \subseteq G(s) \subseteq G(s^\frown\langle\xi'\rangle) = q$$

where $\xi' \in \mathrm{suc}_{S,0}(s)$ is arbitrary.

(b) Assume that $t \ne s$, say $t^\frown\langle\xi\rangle \trianglelefteq s$ for some $\xi \in \mathrm{suc}_T(t)$. Let us furthermore assume $\mathrm{o}^{\tilde{\mathrm{U}}}(\xi) > 0$. Then there is some $\zeta \in \mathrm{suc}_{T,0}(t)$, such that $t^\frown\langle\zeta,\xi\rangle \in T$. Note that $b_{t^\frown\langle\xi\rangle} = b_{t^\frown\langle\zeta,\xi\rangle}$ and thus

$$p = F(t^\frown\langle\zeta\rangle) \subseteq F(t^\frown\langle\zeta,\xi\rangle) = F(t^\frown\langle\xi\rangle) \subseteq G(t^\frown\langle\xi\rangle) \subseteq G(s) \subseteq G(s^\frown\langle\xi'\rangle) = q$$

where $\xi' \in \mathrm{suc}_{S,0}(s)$ is arbitrary.

(c) Assume that $t = s$. Then $\mathrm{suc}_{S,0}(s) \subseteq \mathrm{suc}_{T,0}(t)$, so by the definition of the partial orders

$$p = F(t^\frown\langle\xi\rangle) \subseteq G(t^\frown\langle\xi\rangle) = q$$

where $\xi \in \mathrm{suc}_{S,0}(s)$ is arbitrary. $\dashv$

Remark: Because $\operatorname{ran}(\sigma)$ was dense in $\langle \mathbb{P}, \leq^* \rangle$, $\langle \mathbb{Q}, \leq, \leq^* \rangle$ inherits the Prikry type. (Say $p \in \mathbb{Q}$. Let $q^* \leq^* \sigma(p)$ decide some statement $\varphi(\tau)$. Take some q with $\sigma(q) \leq^* q^*$, then $q \leq^* p$ and it decides $\varphi(\tau)$.) So there is no harm in confusing $\mathbb{P}$ with $\mathbb{Q}$ from now on.

We want to prove something very much like a product lemma for $\mathbb{Q}$, but we will need a technical lemma first.

Lemma 9.3: *Let $\langle p, (t, T, F) \rangle \in \mathbb{Q}$ and $q \leq p$, then there exist $(t, T^*, F^*) \leq^* (t, T, F)$, such that $\langle q, (t, T^*, F^*) \rangle \in \mathbb{Q}$.*

Proof: Take $\xi < \mu$, $\xi > \max t$ so that $q \in \operatorname{Col}(\kappa, <\xi)$. Set $A_0 := \operatorname{suc}_{T,0}(t) \cap (\xi, \mu)$ and $A_\beta := \operatorname{suc}_{T,\beta} \cap \bigcup_{\zeta \in A_0} \operatorname{suc}_{T,\beta}(t^\frown \langle \zeta \rangle)$. Define $T^* := \{s \in T | s \trianglelefteq t \vee \exists \beta < \nu \exists \zeta \in A_\beta : t^\frown \langle \zeta \rangle \trianglelefteq s\}$ and

$$F^*(s) = (F(s) \cup q) \restriction (\kappa \times \max s)$$

for $s \in T^*$. Then $(t, T^*, F^*) \leq^* (t, T, F)$ and for $\zeta \in \operatorname{suc}_{T^*,0}(t)$

$$F^*(t^\frown \langle \zeta \rangle) = F(t^\frown \langle \zeta \rangle) \cup q = p \cup q = q$$

so $\langle q, (t, T^*, F^*) \rangle \in \mathbb{Q}$. ⊣

Let $G \subset \operatorname{Col}(\kappa, <\mu_\xi)$ be a generic filter over W. Write $\mathbb{P}^G := \{(t, T, F) | \exists p \in G : \langle p, (t, T, F) \rangle \in \mathbb{Q}\}$.

Lemma 9.4: *(a) Let $I \subset \mathbb{Q}$ be generic over V. Then $G := \{p | \exists (t, T, F) : \langle p, (t, T, F) \rangle \in I\}$ is generic over W for $\operatorname{Col}(\kappa, <\mu)$ and $H := \{(t, T, F) | \exists p : \langle p, (t, T, F) \rangle \in I\}$ is generic over $W[G]$ for $\mathbb{P}^G$.*

(b) Let $G \subset \operatorname{Col}(\kappa, <\mu)$ be a generic filter over W and $H \subseteq \mathbb{P}^G$ be generic over $W[G]$. Then $I := \{< p, (t, T, F) \rangle \in \mathbb{Q} | p \in G, (t, T, F) \in H\}$ is generic over W.

Corollary 9.5: *Let $G \subseteq \operatorname{Col}(\kappa, <\mu)$ be generic over V. Then $\mathbb{P}^*$ is a $<\kappa_\xi$ weakly closed forcing in $V[G]$.*

The proofs are exactly the same as in section 2 (see p.41f).

We have seen how $\mathbb{P}$ can be seen as the combination of a collapse first and a singularizing forcing second. Now we will switch our viewpoint and show that it is equally valid to think of it as the conbination of a singularizing forcing first and a collapse second.

Remember $\mathbb{P}(\mu, \nu)$ as the forcing from the preceding chapter that adds a club set of order type ω^ν to μ using trees. For any condition $(t, T, F) \in \mathbb{P}$ we will define a $\mathbb{P}(\mu, \nu)$-name:

$$\tau_F := \{((s, S), \ulcorner \alpha, \beta, \gamma \urcorner) | s \in T \wedge (s, S) \leq (t, T) \wedge (\alpha, \beta, \gamma) \in F(s)\}$$

The important thing to note here is that

$$(s, T_s) \Vdash \tau_F \cap (\check{\kappa} \times \max \check{s}) = \check{F(s)} \tag{9.2}$$

for any $s \in T$ with $t \trianglelefteq s$. The exact definition of τ_F doesn't matter as long as this is true. Write $\dot{C}$ for $(\mathrm{Col}(\kappa, {<}\mu))^{\mathbb{P}(\mu,\nu)}$, i.e. a $\mathbb{P}(\mu, \nu)$-name for the collapse of everything below μ to κ in the Prikry generic extension.

Lemma 9.6: *$\sigma : \mathbb{P} \to \mathbb{P}(\mu, \nu) * \dot{C}$ where $\sigma((t, T, F)) = ((t, T), \tau_F)$ is a dense embedding.*

The proof is basically the same as Lemma 7.6. From here on the proof of Theorem 6.2 can be finished using the exact same argument from the end of chapter 7 on p.43. Just substitute W for V, Lemma 9.4 for Lemma 7.4 and Lemma 9.6 for Lemma 7.6.

10. Singularizing multiple cardinals

In this section we will discuss a method to get $(*)_{\kappa,\nu}$ for several cardinals at once. First we want to establish some limits of what we can hope to achieve in the context of this chapter[1].

Lemma 10.1: *Let κ and $\nu_0, \nu_1 < \kappa$ be regular cardinals, where $\kappa \geq \aleph_2$. Assume both $(*)_{\kappa,\nu_0}$ and $(*)_{\kappa^+,\nu_1}$. Then there exists an inner model with a Woodin cardinal.*

Proof: We will once again make use of the Steel core model and weak covering. Assume that there is no inner model with a Woodin cardinal. Then the core model K exists. Let $\mathbb{P}_0$ witness $(*)_{\kappa,\nu_0}$ and $\mathbb{P}_1$ witness $(*)_{\kappa^+,\nu_1}$. Fix $\langle G_i : i < 2\rangle$ V-generic filters for $\mathbb{P}_i$. Remember $K = K^{V[G_i]}$. Let $\lambda = ((\kappa^+)^+)^K$.

Claim 1: $\lambda < \kappa^{++}$

Proof of Claim: Assume that $\lambda = \kappa^{++}$. Using weak covering in $V[G_1]$ we get:

$$\mathrm{cof}^{V[G_1]}(\lambda) \geq \mathrm{Card}^{V[G_1]}(\kappa^+) = \kappa^+$$

But $\mathrm{cof}^{V[G_1]}(\lambda) = \nu_1 < \kappa^+$. Contradiction! $\square$

Using this claim and weak covering in V we immediatly get the following.

Claim 2: $\mathrm{cof}^V(\lambda) = \kappa^+$

So by this claim $\mathrm{cof}^{V[G_0]}(\lambda) = \nu_0$, but applying weak covering in $V[G_0]$ yields

$$\mathrm{cof}^{V[G_0]}(\lambda) \geq \mathrm{Card}^{V[G_0]}(\kappa^+) = \kappa. \qquad \dashv$$

Contradiction!

In light of this, the following theorem is nearly optimal.

Theorem 10.2: *Assume GCH. Let $\langle \kappa_\xi, \nu_\xi : \xi < \gamma\rangle$ be a sequence of regular cardinals, such that the κ_ξ are strictly increasing, $\nu_\xi < \kappa_\xi$, and $\kappa_0 \geq \aleph_2$. Assume that there are measurable cardinals $\langle \mu_\xi : \xi < \gamma\rangle$ such that:*

- $\forall \xi < \gamma : \kappa_\xi < \mu_\xi$,
- $\forall \xi + 1 < \gamma : \kappa_{\xi+1} > \mu_\xi$,

[1] We would like to thank Peter Koepke for the argument

- *for all limit $\xi < \gamma$ is $\kappa_\xi > \sup\limits_{\zeta<\xi} \mu_\zeta$,*
- *$\forall \xi < \gamma : \mathrm{o}(\mu_\xi) \geq \eta_\xi$, where η_ξ is unique with $\nu_\xi = \omega \cdot \eta_\xi$,*
- *$\forall \xi < \gamma \forall \zeta < \xi : \nu_\xi \notin (\kappa_\zeta, (\mu_\zeta)^+]$*

Then there exists a generic extension, in which $(*)_{\kappa_\xi,\nu_\xi}$ holds for all $\xi < \gamma$.

For each $\langle \kappa_\xi, \nu_\xi, \mu_\xi \rangle$ we will need to have a $<\kappa_\xi$ weakly closed Prikry type forcing $\mathbb{P}_\xi$, which will change the cofinality of μ_ξ to ν_ξ and incorporates a collapse. Fortunately we have just constructed such a forcing. The problem is that this forcing might not exist in the ground model.

So let us first assume that there is a coherent sequence of measures $\vec{U} := \langle U(\alpha,\beta\rangle : \alpha < \delta, \beta < \mathrm{o}^{\tilde{\mathrm{U}}}(\alpha) \rangle$, such that for all ξ $\mathrm{o}^{\tilde{\mathrm{U}}}(\mu_\xi) \geq \eta_\xi$ and w.l.o.g. $\mathrm{o}^{\tilde{\mathrm{U}}}(\beta) = 0$ for all $\sup\limits_{\zeta<\xi} \mu_\zeta < \beta \leq \kappa_\xi$. Then we can define like in Gitik's paper an iteration $\mathbb{Q}$ of length δ, such that for every $\xi < \gamma$ $\mathbb{Q} \restriction \xi$ is the iteration, that will add $\mathbb{P}_\xi$ and $\mathbb{Q}/\mathbb{Q} \restriction \xi$ is a sufficiently weakly closed prikry type forcing notion in $V^{\mathbb{Q}\restriction\xi}$. (To achieve this we will have to leave the μ_ξ-th spot in the iteration blank.)

So if W is a $\mathbb{Q}$-generic extension of V, Corollary 7.5 and Corollary 9.5 tell us, that if G_β were to be $\mathrm{Col}(\kappa_\beta, <\mu_\beta)$-generic over W, then in $W[G_\beta]$ there would be some $<\kappa_\beta$ weakly closed Prikry type forcing $\mathbb{P}_\beta^{G_\beta}$ changing μ_β's cofinality to η_β.

Proof (of Theorem 10.2): Let $G \subseteq \prod\limits_{\xi<\gamma} \mathrm{Col}(\kappa_\xi, <\mu_\xi)$ be generic over W. We write $G_\beta := \{f(\beta) | f \in G\}$ and similarly $G_{<\beta} := \{f \restriction \beta | f \in G\}, G_{>\beta} := \{f \restriction (\beta,\gamma) | f \in G\}$. We notice that $\mathbb{P}_\beta$ is still well behaved in $V[G_{>\beta}]$ as no subsets of μ_β were added. Let $\mathbb{P}_\beta^G$ be the rest forcing, that exists in the extension by G_β given by Lemma 7.4 or Lemma 9.4. (By the notation of the preceding chapters we would have to name it $\mathbb{P}_\beta^{G_\beta}$, but there is no danger of confusion and this way makes it more readable.) We want to show, that $\mathbb{P}_\beta^G$ doesn't add any new bounded subsets to κ_β over $W[G]$ for any $\beta < \gamma$. We do this by a product analysis.

By Corollary 7.5 or Corollary 9.5 we know that $\mathbb{P}_\beta^G$ is a $<\kappa_\beta$ weakly closed Prikry type forcing in $W[G_{>\beta}][G_\beta]$.

Now let us assume for a contradiction that for some $H \subset \mathbb{P}_\beta^G$ generic over $W[G]$, there is some bounded subset $A \in W[G][H]$, which is not in $W[G]$. We can rearrange $W[G][H]$ as $W[G_{>\beta}][G_\beta][H][G_{<\beta}]$.

Let σ be a $\prod\limits_{\xi<\beta} \mathrm{Col}(\kappa_\beta, <\mu_\beta)$-name and $p \in G_{<\beta}$, such that $A = \sigma^{G_{<\beta}}$ and $p \Vdash \sigma \subseteq \check{\alpha}$ for some $\alpha < \kappa_\beta$.Set

$$\tau := \{(\check{\xi}, q) | q \leq p \wedge q \Vdash \check{\xi} \in \sigma\}.$$

Then $\sigma^{G_{<\beta}} = \tau^{G_{<\beta}}$ and τ can be coded by a bounded subset of κ_β. So by weak closure of $\mathbb{P}_\beta^G$ we have $\tau \in W[G_{>\beta}][G_\beta]$ and thus $A = \tau^{G_{<\beta}} \in W[G]$. Contradiction!

Lastly to see that κ_β will remain regular, again write $W[G][H]$ as $W[G_{>\beta}][G_\beta][H][G_{<\beta}]$, then the leftmost filter belongs to a sufficiently closed forcing, the middle two "combine" into a $\mathbb{P}_\beta$ generic and we have seen that this forcing will not singularize κ_β, and the rightmost filter belongs of course to a small forcing, so none of these filters will add a singularizing sequence.

This shows that $\mathbb{P}^G_\beta$ witnesses $(*)_{\kappa_\beta,\nu_\beta}$ in $W[G]$. As β was arbitrary we are done! $\dashv$

We can in fact show more. We can construct a forcing, which will singularize all the μ_ξ at the same time without adding any unwanted sets. The situation is the same as before. We will work in the model $W[G]$. As before we then have the forcings $\mathbb{P}^G_\xi$ inside of $W[G]$. We will then consider the Gitik-iteration of these forcing notions. Obviously this will add cofinal sequences to all the μ_ξ at once. What remains to be seen, is that this iteration is well-behaved.

For every $\beta \leq \gamma$ let $\mathbb{Q}_\beta$ be the iteration with $\mathbb{P}^G_\xi$ for $\xi < \beta$ as its component forcings.

Lemma 10.3: *$\mathbb{Q}_\beta$ is a Prikry type forcing for all $\beta \leq \gamma$.*

Proof: Assume not. For this proof let us work in some universe, where everything relevant is countable. In there construct a sequence of ordinals $\langle \eta_n : n < \omega\rangle$ and a sequence $\langle \bar{G}_n : n < \omega\rangle$ such that

(a) $\forall n < \omega : \eta_{n+1} < \eta_n$,

(b) $\bar{G}_n \subset \mathbb{P}^G_{\eta_n}$ is generic over $W\left[\bar{G}_0\right] \ldots \left[\bar{G}_{n-1}\right]$,

(c) $\mathbb{Q}_{\eta_n}$ is not of Prikry type in $W\left[\bar{G}_0\right] \ldots \left[\bar{G}_n\right]$.

This obviously gives a contradiction. So set $\eta_0 = \gamma$ and $\bar{G}_0 = \{\emptyset\}$ (we abuse notation a little bit here, by setting $\mathbb{P}^G_\gamma = \{\emptyset\}$).

Let us now assume, that η_n and $\bar{G}_n$ are already constructed. By induction hypothesis (c) and Lemma 6.8 there must be some $\eta < \eta_n$, such that

$$\mathbb{Q}_\beta \not\Vdash \check{\mathbb{P}}^G_\eta \text{ is of Prikry type.}$$

Let η_{n+1} be the minimal such η.

Claim 1: *$\mathbf{1}_{\mathbb{P}^G_{\eta_{n+1}}} \not\Vdash \check{\mathbb{Q}}_{\eta_{n+1}}$ is of Prikry type.*

Proof of Claim: Assume not. Let $\varphi(\tau)$ be any statement in the forcing language of $\mathbb{Q}_{\eta_{n+1}}$ and let $p \in \mathbb{P}^G_{\eta_{n+1}}$ and $q \in \mathbb{Q}_{\eta_{n+1}}$ any condition. As

$$p \Vdash \mathbb{Q}_{\eta_{n+1}} \text{ is of Prikry type,}$$

there is some $\mathbb{P}^G_{\eta_{n+1}}$-name σ for a condition in $\mathbb{Q}_{\eta_{n+1}}$ directly extending q, which decides $\varphi(\tau)$ (slight abuse of notation here). Such conditions are basically subsets of $\sup\langle \mu_\xi : \xi < \eta_{n+1}\rangle$, thus there exists some $p' \leq^*_{\eta_{n+1}} p$ and some $q^* \in \mathbb{Q}_{\eta_{n+1}}$, such that $p' \Vdash \check{q}^* = \sigma$.

Here we used both that $\mathbb{P}^G_{\eta_{n+1}}$ is sufficiently weakly-closed Prikry type forcing in $W[G_{>\beta}][G_\beta]$ and that $W[G]$ is a sufficiently closed extension of that model.

We will abuse notion further and regard

$$\psi(\tau) \equiv \check{q}^* \Vdash \varphi(\tau)$$

as a statement in the forcing language of $\mathbb{P}^G_{\eta_{n+1}}$. So take some $p^* \leq^*_{\eta_{n+1}} p'$, which decides $\psi(\tau)$.

Then $q^{*\frown}\langle \check{p}^* \rangle$ decides $\varphi(\tau)$.

First case: Assume $p^* \Vdash \psi(\tau)$. This immediately implies $q^{*\frown}\langle \check{p}^* \rangle \Vdash \varphi(\tau)$.

Second case: Assume $p^* \Vdash \neg\psi(\tau)$. But $p^* \Vdash \check{q}^* \parallel \varphi(\tau)$ so we have

$$p^* \Vdash \check{q}^* \Vdash \neg\varphi(\tau),$$

which implies $q^{*\frown}\langle \check{p}^* \rangle \Vdash \neg\varphi(\tau)$.

But this gives, that $q^* \Vdash \check{p}^* \check{\leq}^*_{\eta_{n+1}} \check{p} \wedge \check{p}^* \parallel \varphi(\tau)$. As $\varphi(\tau)$ and q were arbitrary, this is a contradiction! $\square$

So there exists some condition $p \in \mathbb{P}^G_{\eta_{n+1}}$, such that

$$p \Vdash \mathbb{Q}_{\eta_{n+1}} \text{ is not of Prikry type.}$$

Let $\bar{G}_{n+1}$ then be some generic filter containing p, and we are done! $\dashv$

11. Simultaneous singularizing of subsequent successors

At the end of last chapter we discussed the simaltaneous singularizing of several successors of regular cardinals. We learned, that it is not fundamentally more difficult than singularizing a single successor as long as there are cardinals between all the successors involved.

We will now shortly talk about singularizing subsequent cardinals. We know by the covering lemma that this brings us up to the level of a Woodin cardinal. Our main tool at this level is the *stationary tower*.

Definition 11.1: Let κ be an inaccessible cardinal. Let $\mathbb{P}_\kappa$ be the poset of $S \in V_\kappa$ that are stationary in $\mathcal{P}(\bigcup S)$ ordered by

$$S \leq T :\Leftrightarrow \bigcup S \supseteq \bigcup T \wedge S \subseteq \{Y \subseteq \bigcup S | Y \cap \bigcup T \in T\}.$$

Lemma 11.2 (Woodin): *Let δ be a Woodin cardinal. Let $G \subseteq \mathbb{P}_\delta$ be generic over V. Then in $V[G]$ there exists an inner model M and an elementary embedding $j : V \to M$ such that*

- $\forall S \in V_\delta : S \in G \Leftrightarrow j\text{"}\,[\bigcup S] \in j(S)$,
- *M is wellfounded and in fact ${}^{<\delta}M \cap V[G] \subseteq M$.*

See [Lar04] for details.

With this it is now easy to simultaneously singularize a regular cardinal κ and its successor. Say we want to change κ's cofinality to η and κ^+'s cofinality to ξ, where $\eta, \xi < \kappa$ are regular cardinals.

We will need the following technical but immensely useful lemma:

Lemma 11.3: *Let λ be a regular cardinal. Let $H \supseteq \lambda$ and let $\mathfrak{a}$ be a sufficiently strong skolemized structure on H. Let $X \prec \mathfrak{a}$ with* $\mathrm{Card}(X) < \lambda$ *and let $A \subset \lambda$ be bounded inside of X, i.e. there is some $\delta < \lambda$ in X with $A \subseteq \delta$. Then* $\sup(X \cap \lambda) = \sup(\mathrm{Sk}^{\mathfrak{a}}(X \cup A) \cap \lambda)$.

Here $\langle H_\lambda^+; \in, \trianglelefteq\rangle$, where $\trianglelefteq$ is a wellorder would be a good example of sufficiently strong. The proof will show, what is needed.

Proof: Assume not. Then there is some $\gamma \in \mathrm{Sk}^{\mathfrak{a}}(X \cup A)$ with $\sup(X \cap \lambda) < \gamma < \lambda$. Say $\gamma = \tau(x_0, \ldots, x_{n-1}, \gamma_0, \ldots, \gamma_{m-1})$, where τ is a term, $x_i \in X$ for all $i < n$ and $\gamma_j \in A$ for all $j < m$.

Let $\delta \in X$ be some ordinal $<\lambda$ that bounds A. We assume then

$$\alpha := \sup\{\tau(x_0, \dots, x_{n-1}, \beta_0, \dots, \beta_{m-1}) < \lambda | \forall j < m\ \beta_j < \delta\}$$

is definable over $\mathfrak{a}$ from τ,δ and the $x_0, \dots, x_{n-1}$ and is thus in X. But surely $\gamma < \alpha$. Contradiction! $\dashv$

Now we can prove:

Lemma 11.4:

$$S := \{X \subseteq H_{\kappa^{++}} | X \cap \kappa \in \kappa \cap \mathrm{cof}(\eta) \wedge \mathrm{cof}(\sup(X \cap \kappa^+)) = \xi\}$$

is stationary.

Proof: Let $\mathfrak{a}$ be any sufficiently strong skolemized structure on $H_{\kappa^{++}}$. Start with some $Y \prec \mathfrak{a}$ such that $\mathrm{cof}(\sup(Y \cap \kappa^+)) = \xi$ and $\mathrm{Card}(Y) < \kappa$. Construct recursively an elementary chain $\langle X_\alpha : \alpha < \eta \rangle$:

- set $X_0 := Y$,
- if X_β is already defined, let $X^*_\beta := \mathrm{Sk}^{\mathfrak{a}}(X_\beta \cup \{\sup(X_\beta \cap \kappa)\})$ and then put $X_{\beta+1} := \mathrm{Sk}^{\mathfrak{a}}(X^*_\beta \cup \sup(X^*_\beta \cap \kappa))$,
- if $\lambda < \eta$ is a limit, set $X_\lambda := \bigcup\limits_{\beta<\lambda} X_\beta$.

Set $X := \bigcup\limits_{\alpha<\eta} X_\alpha$. We then have $X \cap \kappa \in \kappa \cap \mathrm{cof}(\eta)$, furthermore $\sup(X \cap \kappa^+) = \sup(Y \cap \kappa^+)$ by Lemma 11.3 so $X \in S$. $\dashv$

With the set S we can now singularize κ and κ^+ simultaneously using the stationary tower under some Woodin cardinal δ.

For that take some $G \subset \mathbb{P}_\delta$ generic over V such that $S \in G$. By Lemma 11.2 we then have some elementary embedding $j : V \to M$ such that $j\text{"}[H_{\kappa^{++}}] \in j(S)$. This has two consequences of note

- $j\text{"}[\kappa] \in j(\kappa) \cap \mathrm{cof}(j(\eta))$; this implies, that $j\text{"}[\kappa]$ is an ordinal and thus $j \restriction \kappa = \mathrm{id}$; in fact $\mathrm{crit}(j) = \kappa$; furthermore $\kappa \in \mathrm{cof}(\eta)$ as wanted.
- $\mathrm{cof}(\sup(j\text{"}[\kappa^+])) = j(\xi) = \xi$; furthermore κ^+ can be embedded into $\sup(j\text{"}[\kappa^+])$, so $\mathrm{cof}(\kappa^+) = \xi$.

We showed this inside of M of course, but Lemma 11.2 guarantees enough agreement between M and $V[G]$ that these statements are absolute between these two models.

It is easy to see, that this approach generalizes to arbitrarily finite sequences of subsequent regular cardinals. So we will now discuss the case of infinitely many cardinals. To simplify our notation we will restrict ourselves to the cardinals $\langle \aleph_n : n < \omega \rangle$ and cofinalities $\aleph_0$ and $\aleph_1$.

Let Γ be a class of forcings and $f : \omega \to 2$ a function. By $(*)^\Gamma_f$ we refer to the statement:

There is a p.o. $\mathbb{P} \in \Gamma$ such that $\mathbf{1}_{\mathbb{P}} \Vdash \operatorname{cof}(\check{\aleph}_{m+3}) = \aleph_{\check{f(m)}}$ for all $m < \omega$, but $\mathbb{P}$ doesn't change cofinalities and cardinalities $\leq \aleph_2$.

Now if we wanted to try proving $(*)^{\Gamma}_{f}$ for some reasonable class of Γ, say stationary set preserving forcings, by the above method, then we would need the set

$$\{X \subset \aleph_\omega | \aleph_2 \subset X \wedge \forall m < \omega \ \sup(X \cap \aleph_{m+3}) \in S^{f(m)}_{m+3}\}$$

to be stationary. Here S^k_n refers to the set $\{\xi < \aleph_n | \operatorname{cof}(\xi) = \aleph_k\}$.

This can be considered a property of the sequence $\langle S^{f(m)}_{m+3} : m < \omega \rangle$. This property was first discussed by Foreman and Magidor in [FM01] and was dubbed mutual stationarity.

Definition 11.5: Let $\langle \kappa_n : n < \omega \rangle$ be a strictly increasing sequence of uncountable regular cardinals. Write $\lambda := \sup\limits_{n<\omega} \kappa_n$. The sequence $\langle S_n : n < \omega \rangle$, where $S_n \subseteq \kappa_n$, is called mutually stationary, iff the set

$$\{X \subseteq \lambda | \forall n < \omega \ \sup(X \cap \kappa_n) \in S_n\}$$

is stationary.

Note here that mutual stationarity implies the stationarity of all the members of the sequence, so we don't have to explicitely mention it in the definition. From now on write $\mathrm{MS}(S_n : n < \omega)$ for "the sequence $\langle S_n : n < \omega \rangle$ is mutually stationary".

The mutual stationarity properties we would need to have $(*)^{\Gamma}_{f}$ can not be proven in ZFC unless f is trivial.

Theorem 11.6: *Let $\mathcal{M}$ be a mouse, such that $\mathcal{M} \models (*)^{\Gamma}_{f}$ for some $f : \omega \to 2$. Then f is eventually constant.*

Here we will need the following technical lemma:

Lemma 11.7: *Let $\mathcal{M}$ be a mouse. Let κ be a cardinal of $\mathcal{M}$, such that $\mathcal{M}$ is sound above κ. Suppose that $\rho_{n+1}(\mathcal{M}) \leq \kappa < \rho_n(\mathcal{M})$. Then $\operatorname{cof}^V(\rho_{n+1}(\mathcal{M})^{+\mathcal{M}}) = \operatorname{cof}^V(\kappa^{+\mathcal{M}}) = \operatorname{cof}^V(\rho_n(\mathcal{M}))$.*

See [JSSS09] p.4f for proof.

Proof (of Theorem 11.6): Fix an ordinal α, such that

$$J^{\mathcal{M}}_{\alpha} \models (*)^{\Gamma}_{f} \Leftrightarrow \mathcal{M} \models (*)^{\Gamma}_{f}.$$

Let $\mathcal{P}$ be countable and transitive and fully elementary embeddable in $J^{\mathcal{M}}_{\alpha}$ with $f \in \mathcal{P}$. Let $\mathbb{P}$ be a partial order, which witnesses $(*)^{\Gamma}_{f}$ in $\mathcal{P}$ and let $G \subseteq \mathbb{P}$ be a generic filter over $\mathcal{P}$.

Working in $\mathcal{P}[G]$ let g_m be a witness to $\operatorname{cof}(\aleph^{\mathcal{P}}_{m+3}) = \aleph_{f(m)}$. Define $\mathcal{N} := \operatorname{Hull}^{\mathcal{P}|\aleph^{\mathcal{P}}_{\omega}}_{\omega}(\bigcup\limits_{m<\omega} g_m)$ and let $j : \mathcal{N} \to P|\aleph^{\mathcal{P}}_{\omega}$ be the uncollapsing map. Obviously

$$\operatorname{cof}^{\mathcal{P}[G]}(\aleph^{\mathcal{N}}_{m+3}) = \aleph_{f(m)}$$

as g_m is cofinal, and thus j maps $\aleph^{\mathcal{N}}_{m+3}$ contiously into $\aleph^{\mathcal{P}}_{m+3}$. Further note that $\mathcal{N} \neq \mathcal{P}|\aleph^{\mathcal{P}}_{\omega}$, because the former has cardinality $<\aleph^{\mathcal{P}}_2$ in $\mathcal{P}[G]$, while the latter has cardinality $\aleph^{\mathcal{P}}_2$ in $\mathcal{P}[G]$.

Claim 1: $\mathcal{N} \lhd \mathcal{P}$

Proof of Claim: We will proof the claim by coiterating $\mathcal{N}$ with $\mathcal{P}|\aleph^{\mathcal{P}}_{\omega}$ (in V). Let $(\mathcal{T},\mathcal{U})$ be the result of this coiteration. Let us denote by $M^{\mathcal{T}}_{\infty}$ the last model of $\mathcal{T}$ and by $M^{\mathcal{U}}_{\infty}$ the last model of $\mathcal{U}$. First note that neither $\mathcal{N}$ nor $\mathcal{P}$ have inaccessible cardinals, and thus no total extenders either.

Assume $M^{\mathcal{T}}_{\infty} \trianglelefteq M^{\mathcal{U}}_{\infty}$ (the argument for the other case is symmetric). The branch through $\mathcal{T}$ to $M^{\mathcal{T}}_{\infty}$ cannot drop, and thus no extender was applied along it. In other words $M^{\mathcal{T}}_{\infty} = \mathcal{N}$. Let $i : \mathcal{P}|\aleph^{\mathcal{P}}_{\omega} \to M^{\mathcal{U}}_{\infty}$ be the iteration embedding. Let us assume that i has a critical point κ. κ will then be inaccessible in $M^{\mathcal{U}}_{\infty}$ and futhermore $\kappa < \mathrm{On}^{\mathcal{N}}$ as it must be below the least disagreement. But $\mathcal{N} \trianglelefteq M^{\mathcal{U}}_{\infty}$ and therefore κ is inaccessible in $\mathcal{N}$. Contradiction!

So we showed, that $\mathrm{lh}(\mathcal{T}) = \mathrm{lh}(\mathcal{U}) = 1$, which proves the claim by a cardinality argument. $\square$

Let us now go back to $\mathcal{P}$.

Claim 2: $\forall m < \omega : \mathrm{cof}^{\mathcal{P}}(\aleph^{\mathcal{N}}_{m+3}) = \aleph_{f(m)}$

Proof of Claim: Fix some $m < \omega$. Assume that $\mathrm{cof}^{\mathcal{P}}(\aleph^{\mathcal{N}}_{m+3}) \neq \aleph_{f(m)}$, and thus

$$\mathrm{cof}^{\mathcal{P}}(\aleph^{\mathcal{N}}_{m+3}) \neq \mathrm{cof}^{\mathcal{P}[G]}(\aleph^{\mathcal{N}}_{m+3})$$

This implies that $\mathrm{cof}^{\mathcal{P}}(\aleph^{\mathcal{N}}_{m+3})$ will get collapsed by $\mathbb{P}$. But as it is certainly less than $\aleph^{\mathcal{P}}_2$, we get a contradiction! $\square$

By our first claim $\mathcal{N} = J^{\mathcal{P}}_{\beta}$ for some $\beta < \aleph^{\mathcal{P}}_2$. Thus there must be some initial segment of $\mathcal{P}$ projecting below it. Let $\mathcal{N}^*$ be the minimal initial segment of $\mathcal{P}$ projecting below $\mathrm{On}^{\mathcal{N}}$. Let n be minimal, such that $\rho_{n+1}(\mathcal{N}^*) < \mathrm{On}^{\mathcal{N}}$. It then follows, that $\mathrm{cof}^{\mathcal{P}}(\aleph^{\mathcal{P}}_{m+3}) = \mathrm{cof}^{\mathcal{P}}(\mathrm{On}^{\mathcal{N}^*})$ for m big enough. Together with the second claim this finishes the proof. $\dashv$

A somewhat simplified proof can be used to prove the following.

Theorem 11.8: *Fix some function $f : \omega \to 2$. Let $\mathcal{M}$ be a mouse, such that the sets $\langle S^{f(m)}_{m+3} : m < \omega\rangle$ are mutually stationary in $\mathcal{M}$. Then f is eventually constant.*

So this means, that $(*)^{\Gamma}_{f}$ is not in general provable from any large cardinal property, that can be captured by current inner model theory. And neither is $\mathrm{MS}(S^{f(m)}_{m+2} : m < \omega)$. That doesn't mean much in terms of consistency strength though.

In [CFM06] it is shown that a model of $MS(S^{f(m)}_{m+3} : m < \omega)$, where f is such that $f(n) = 0$ implies $f(n+1) \neq 0$ can be constructed starting from a model with infinitely many measurable cardinals. That paper also includes a much more general result, showing the existence of a model, where $MS(S^{f(m)}_{m+2} : m < \omega)$ holds for all f simultaneously. The consistency strength is significant though, weighing in at infintely many supercompact cardinals.

For our purposes, this means that the consistency strength of $(*)^{\Gamma}_{f}$ for certain f is exactly one Woodin cardinal. In general it might be above a supercompact even (,though we don't believe this to be true).

So far we have seen that mutual stationarity allows us to construct forcings that witness $(*)^{\Gamma}_{f}$. We want to finish with a result in the opposite direction, which has an interesting corollary.

Lemma 11.9: *Assume* $\mathrm{MA}_{\aleph_1}(\Gamma)$ *and* $(*)^{\Gamma}_{f}$ *for some* $f : \omega \to 2$. *Then* $MS(S^{f(m)}_{m+3} : m < \omega)$ *holds.*

Proof: Let $\mathbb{P}$ be a witness to $(*)^{\Gamma}_{f}$. By $\mathrm{MA}_{\aleph_1}(\Gamma)$ we get some $\theta >> \aleph_\omega$, such that the set M of substructures X of H_θ of size $\aleph_1$, containing $\aleph_1$ with a X-generic filter is stationary. We show:

Claim: *Let* $X \in M$, *then* $\mathrm{cof}(\sup(X \cap \aleph_{m+3})) = \aleph_{f(m)}$ *for all* $m < \omega$.

Proof of Claim: Let H be a X-generic filter for $\mathbb{P}$. Because

$$X \models \mathbf{1}_{\mathbb{P}} \Vdash \mathrm{cof}(\check{\aleph}_{m+3}) = \aleph_{f(m)}$$

for all $m < \omega$ and $\aleph_1 \subset X$ we have

$$\forall m < \omega : \mathrm{cof}(\sup(X[H] \cap \aleph_{m+3})) = \aleph_{f(m)}$$

The next step is to show that $\mathrm{On} \cap X = \mathrm{On} \cap X[H]$. So let $\alpha \in X[H] \cap \mathrm{On}$. So there is some name $\sigma \in X$, s.t $\sigma^H = \alpha$ and w.l.o.g. $\Vdash \sigma \in \mathrm{On}$. So the set $D := \{p \in \mathbb{P} | \exists \xi : p \Vdash \sigma = \check{\xi}\}$ is dense and it is certainly in X.

By the choice of H we have $G \cap D \cap X \neq \emptyset$. For some $p \in H \cap D \cap X$ $p \Vdash \sigma = \check{\alpha}$ must then hold true. This gives $\alpha \in X$.

So then $\mathrm{cof}(\sup(X \cap \aleph_{m+3}) = \mathrm{cof}(\sup(X[H] \cap \aleph_{m+3})) = \aleph_{f(m)}$ for all $m < \omega$. □

The claim shows that

$$\{X \cap H_{\aleph_\omega} | X \in M\} \subseteq \{X \prec H_{\aleph_\omega} | \forall m < \omega : \mathrm{cof}(\sup(X \cap \aleph_{m+3})) = \aleph_{f(m)}\}.$$

But the lefthand side is stationary. ⊣

Note that it was not necessary for the proof to know that $\mathbb{P}$ doesn't collapse $\aleph_2$. This is interesting because such a forcing would not violate covering and could thus exist in ZFC. In fact Jensen has constructed under the assumption of GCH for any $f : \omega \to 2$ just such a forcing $\mathbb{P}_f$ that will change the cofinality of $\aleph_{m+2}$ to $\aleph_{f(m)}$ while preserving stationary sets (see [Jena]).

Theorem 11.10 (Jensen): *Assume MM and $2^\kappa = \kappa^+$ for any $\kappa \geq \aleph_1$. Then $MS(S^{f(m)}_{m+2} : m < \omega)$ holds for all f.*

PROOF: Start by forcing with $\mathrm{Col}(\omega_1, \omega_2)$. CH will hold in the forcing extension and thus $\mathbb{P}_{f^*}$ exists there, where $f^* := [m \mapsto f(m+1)]$. The iteration of these two forcings will then be stationary set preserving and will give any of the $\aleph_n$ the appropriate cofinality for $n \geq 3$. By the preceding lemma we have $MS(S^{f(m+1)}_{m+3} : m < \omega)$. We can then insert the missing set using the technical lemma from the beginning of this chapter. $\dashv$

Jensen's original proof can be found in [Jenb] chapter 3. Jensen does use the "subcomplete forcing axiom" instead of MM, the principle is the same though.

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