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Coding into Inner Models at the Level of Strong Cardinals

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To the memory of my parents, Ioannis Koulakis and
Anastasia Chrysochoidou

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Introduction

Generic absoluteness is undoubtedly a popular topic in the field of set theory. This is not surprising, given the fact that forcing is the major tool in extending models of ZFC and also a very flexible one. Thus whenever the truth of a sentence remains unchanged by forcing, it is reasonable to claim that it has some form of logical stability. For simple formulas, such as $\mathbf{\Sigma}^1_2$, one can derive their generic absoluteness directly from ZFC [15], 13.14, but as more complex ones are considered, the need to use large cardinal axioms emerges. This begins from $\mathbf{\Sigma}^1_3$ formulas, whose absoluteness requires closure under sharps [15], 15.13, [2], and $\mathbf{\Sigma}^1_{n+3}$ formulas which require n strong cardinals [35]. The picture then expands up to the existence of a fully iterable $M^\sharp_\omega$ to get $L[\mathbb{R}]$ absoluteness [32], [34], [20] and further on.

Changing the range of the collection of formulas which should remain frozen under forcing is not the only parameter that alters the consistency strength requirements. Actually one can also reach different consistency strength levels by changing the range of the forcing notions that preserve those formulas. For example, suppose we fix the collection of formulas to $L(\mathbb{R})$. Then absoluteness for c.c.c. forcings is at the level of a weakly compact cardinal (Kunen) and for proper forcings at the level of a remarkable cardinal [27], [30]. The collection we will be interested in is the collection of stationary preserving forcings.

To force absoluteness for stationary preserving forcings, one can clearly use the results applying to the case of all possible notions. This way we get the same upper bounds on the consistency strength of the corresponding absolutenesses. The inverse direction, i.e. getting lower bounds, remains more unexplored. The main reference in this direction is [28]. There, the lower bound of the existence of a strong cardinal is reached for 2-step

$\mathbf{\Sigma}^1_4$ absoluteness of reasonable or stationary preserving forcings. It is also conjectured there that, consistency strength-wise, the 2-step absoluteness of stationary preserving, as well as reasonable forcings, behaves the same way as for all forcing notions. This is the case of interest for us and it will be proved that approaches like the one of Woodin and Wilson cannot produce absoluteness for more complicated formulas than the ones allowed by this conjecture. This is the content of theorems 7.8 and 7.9.

The main technique used in [28] as well as in this thesis is that of coding over canonical inner models[1]. This uses the combination of the almost disjoint coding and the reshaping forcing [13]. The idea behind using those tools, in order to get lower bounds, is quite straightforward. One codes a well-ordering of $\mathbb{R}$ into a real r by almost disjoint forcing and over some canonical inner model, say K. Then in the generic extension, this well-ordering can be defined using $K|\omega_2[r]$. The new definition depends on the complexity of K up to ω_2, which is known and relatively low, for the case we are below a Woodin cardinal and assuming just the existence of some strong cardinals [10], [26]. At the same time, using additional ideas from [31] one can show that the coding forcing is stationary preserving. Therefore we get a method of producing low-complexity defined well-orderings of the reals, while being stationary preserving, and this blocks 2-step absoluteness for stationary preserving forcings and formulas of complexity above that of the well-ordering.

Following the above reasoning, we produce two coding theorems. The first one is theorem 5.1 and it deals with coding a subset of ω_1 into a real over a sequence of mice whose complexity is the same as that of $K|\omega_1$. We assume here that we are below a Woodin cardinal and that there are no strong cardinals above ω_1 in K. The second one is theorem 6.1, where a subset of a successor cardinal $\kappa = \iota^+$, ι is regular, is coded to a subset of ι over $K[\vec{A}]$. The model $K[\vec{A}]$ has the flavor of Chang's model, as the sequence $\vec{A}$ is one which lets K view which ordinals in (ι, κ) have countable cofinalities. Those two theorems are the core of the work appearing in this thesis.

[1] Hence the title!

Overview

This thesis consists roughly of four components. In the beginning, chapters 1 and 2 provide a background on core models and almost disjoint coding. This material is basic knowledge needed for an understanding of the rest of the text. The next two chapters serve as an intermediate passing, as 3 motivates the strategy of our proofs and 4 deals with some preliminary steps of them. Right after, comes the main content of this thesis, which consist of the coding theorems of chapters 5 and 6. Finally, we finish with chapter 7, where we apply the results of chapter 5 to get information about 2-step generic absoluteness. Below we give more details for each individual chapter.

Chapter 1 contains a collection of basic results concerning the core model below a Wooding cardinal, as well as below a given number of strong cardinals in $K|\omega_1$. In the first case we include mainstream weak covering, maximality, correctness and absoluteness theorems and also a pinch of combinatorial properties of K. We spend afterwards some time in describing the definition of $K|\omega_1$ and making some remarks on how it gets more complicated moving from the environment below a measurable to that below a Woodin. Finally, we give some idea on how the complexity of this definition lies in-between the aforementioned cases and gets progressively more complicated while adding strong cardinals in $K|\omega_1$.

Chapter 2 provides some background on almost disjoint coding and reshaping, which are the basic tools for coding with sets of smaller size without collapsing certain cardinals. The forcings are tested on simple settings and on an inner model M which strongly resembles L. Almost disjoint coding is first tested with very restrictive assumptions on V and, later on, reshaping is used to relax them. After reshaping, it is possible to apply almost disjoint coding and perform the decoding inductively. In every case we are interested in the preservation of cardinals and also in examining which parts of the proofs are directly transferable to K. In the end, we give some motivation for the use of coding and reshaping, as well as mention some more general results, such as Jensen's coding theorem which enables one to get $V = L[x]$, $x \in \mathbb{R}$, in a class generic extension.

Chapter 3 serves mostly motivational purposes. Assuming the existence of an α-strong cardinal κ, where $\kappa^{++} \leq \alpha$, we produce embeddings into K which resemble the ones appearing in the proof of distributivity of reshaping. Those embeddings are actually counterexamples to the sort of condensation we required from M in chapter 1. The first case describes an embedding into $K||\alpha$ and the second one to $K|\theta$, for any regular cardinal $\theta \geq \alpha$. The latter indicates that we should avoid using K as a model to code over, as the distributivity proof, the way we presented it in chapter 2, will break down.

Chapter 4 prepares the ground for the coding theorem in chapter 5. We prove the existence of an ordinal δ, where weak covering holds and the overlap of extenders in K is as small as possible in order to allow enough condensation for the proof of distributivity. The latter actually uses the assumption that the strong cardinals of K are all strictly below ω_1. After picking such a δ, we collapse it to ω_1 and show that resulting extension is "coding friendly" in the interval (ω_1, ω_2).

Chapter 5 is the place where the first coding theorem is proved. As always, we assume we are below a Woodin cardinal, and we additionally require that there are no strong cardinals above ω_1 in K, in order to start from the extension produced in chapter 2. In this setting we code a subset A of ω_1 to a real over some sequence of nicely definable mice. Due to this coding, the definition of A in the final extension is $\mathbf{\Delta}^1_{n+3}$ if there are at most n strong cardinals in $K|\omega_1$, or in $J_{1+\alpha}(\mathbb{R})$ if the order type of the strong cardinals in $K|\omega_1$ is strictly less than $\omega\alpha$. The proof consists of three parts. The first one is coding H_{ω_2} over $K|\omega_2$, the second one reshaping ω_1 over the sequence of mice and the last one coding down to a real over this sequence.

Chapter 6 contains our second coding theorem. This time we shift the situation up and we code below a regular cardinal ι, working in the intervals (κ, κ^+) and (ι, κ), where $\kappa = \iota^+$. Our assumption is the non-existence of an inner model with a Woodin cardinal and we code an $A \subset \kappa$ to a set $C \subset \iota$ over $K[\vec{A}]$. $\vec{A}$ is a collection of countable ordinal sequences enabling K to see the real cofinality of ω-cofinal ordinals in the interval (ι, κ). This time we deal with the distributivity of reshaping by using the condensation from the proof of weak covering as well as the ability of the mixed model $K[\vec{A}]$ to spot ω-cofinal sequences[2].

Chapter 7 is where we use our result from chapter 5 to force definable well-orderings of the reals which block specific forms of 2-step generic absoluteness. The first part here, describes how to force the well-orderings. Right after, we describe how the ability to force definable well-orderings prevents us from forcing 2-step absoluteness for stationary preserving forcings. Finally, in the third section, we show that restricting the existence of strong cardinals in K, it is not possible to force generic absoluteness by the usual technique of collapsing a cardinal which lies a bit above a strong cardinal.

[2]Remember that those where the ordinals creating problematic situations in the results of [18].

Chapter 1

Some properties of the core model

This chapter serves as a short exposition of the basic properties of the core model below a Woodin cardinal and also of the additional definability properties it acquires in the case we are below a fixed number of strong cardinals in $K|\omega_1$. The material is divided into two parts. The first focuses on the treatment of the core model in the general case, where the only assumption is that there is no inner model with a Woodin cardinal. The highlight of this section is the weak covering lemma, which will be used in a crucial way later on. Towards the end of this part, we talk about the complexity of the definition of K and $K|\omega_1$, which is unfortunately too high for our goals. The second part deals with the case where $K|\omega_1$ contains at most a given number of strong cardinals. When we add this stronger anti-large cardinal assumption, the definition of $K|\omega_1$ becomes simpler and depends on the order type of the strong cardinals we allow to exist. The rest of this section deals with the analysis of this complexity.

1.1 Below a Woodin cardinal

The core model below a Woodin cardinal has been well studied and two basic references are the monographs [19] and [33]. The first one describes the construction of canonical $L[\vec{E}]$ models below a Woodin cardinal and the second one -building on K^c- analyzes the structure of the core model

K. Simplified versions of those two monographs exist in [36] and [23] respectively, appearing as consecutive chapters of the handbook of set theory. [23], also contains a proof of the weak covering theorem for K, a list of properties of K as well as some applications.

The full-detailed proof of weak covering can be found in [18] for the case of countably closed cardinals, and [17] adds the ingredients needed to remove this assumption. Combinatorial principles of K such as square and diamond can be found in [21], [22] and [25]. [24] contains some theorems which deal with the maximality of K, namely the fact that it is as close as possible to V consistency strength-wise while remaining canonical. Though a first order definition of K already exists in [33], there is a significant simplification of it at the levels above $\aleph_2$. This appears in [9]. Finally, we have to note that in all the above references, K^c and K are built with the additional assumption of the existence of a measurable cardinal Ω. For the sake of a simpler exposition, we also assume its existence and we will fix a corresponding measure U. The reader should keep in mind that K can be defined without this additional assumption and [14] contains a method of removing it.

1.1.1 Weak covering theorems

Covering theorems form one of the most popular category of core model properties. In our setting below a Woodin cardinal, they express the closeness of the core model to V by requiring that for specific V-cardinals κ, there are no extra cardinals of K in the interval (κ, κ^+). One of their weakest forms already appears in the construction of K^c and K.

Theorem 1.1. *([33], 3.5, 5.10, 5.18) Assume there is no inner model with a Woodin cardinal. Then the following hold:*

(a) $\{\kappa < \Omega : (\kappa^+)^{K^c} = \kappa^+\} \in U$.

(b) $\{\kappa < \Omega : (\kappa^+)^{K} = \kappa^+\} \in U$.

Though quite direct to prove, the above covering property already ensures that K^c and K are universal weasels. This means that they capture all the consistency strength of V which is contained in extenders that appear in mice below Ω. It is actually enough for this statement, to assume that weak covering holds on a stationary set of cardinals below Ω.

Definition 1.2. A premouse W is a *weasel* if it is $(\Omega+1)$-iterable and its height is Ω. A weasel W is called *universal* if it wins the comparison with any mouse of height smaller or equal to Ω.

Theorem 1.3. *([33], 3.5) Suppose W is a weasel. If $\{\kappa < \Omega : (\kappa^+)^W = \kappa^+\}$ is stationary, then W is universal.*

With significantly more effort, one can prove for K that weak covering holds for all singular cardinals above ω_2. This is the main result of [18], which is completed in [17].

Theorem 1.4. *([17], 0.1) Assume there is no inner model with a Woodin cardinal and that κ is a cardinal of K such that $\omega_2 \leq \kappa < \Omega$. Then $cf(\kappa^+)^K \geq \overline{\overline{\kappa}}$, i.e. $(\kappa^+)^K = \kappa^+$ or $cf(\kappa^+)^K = \overline{\overline{\kappa}}$.*

Corollary 1.5. *In the above setting, if κ is singular cardinal of V, then $(\kappa^+)^K = \kappa^+$.*

1.1.2 Maximality, correctness and absoluteness

There exist several theorems that express the closeness of K to V in terms of consistency strength. Several of them appear on [24], which deals with the maximality of K. Here we mention two of them. The first one consists of the extension of the universality of K to lower levels. Note that for all the results below, the non-existence of inner models with a Woodin cardinal is assumed.

Theorem 1.6. *([24], 3.4) For every cardinal $\omega_2 \leq \kappa < \Omega$, $K||\kappa$*[1] *is universal for all premice of height $< \kappa$.*

In the statement of the above theorem, in [24], it is claimed that $K|\kappa$ is universal for all premice of height $\leq \kappa$. Unfortunately there was a gap in the proof which rendered it non-functional for premice of height κ.

[1] Remember that the $||$ notation means that the extender indexed on α is contained in the model, i.e. $K||\alpha = (K|\alpha, E_\alpha)$.

The second theorem is the main maximality result appearing in [24], which describes the absorption of weakly countable certified extenders that cohere with K. Moreover the same is true for normal iterates of it.

Theorem 1.7. *([24], 2.3) Suppose $\mathcal{T}$ is a normal iteration tree on K with last model $\mathcal{M}^{\mathcal{T}}_{\infty}$. Let F be an extender such that $(\mathcal{M}^{\mathcal{T}}_{\infty}, F)$ is weakly countably certified, with $lh(E^{\mathcal{T}}_{\alpha}) < lh(F)$ for each $\alpha < lh(\mathcal{T})$ and such that $(\mathcal{M}^{\mathcal{T}}_{\infty}|lh(F), F)$ is a premouse. Then F is actually on the $\mathcal{M}^{\mathcal{T}}_{\infty}$-sequence.*

Another significant property expressing the closeness of K to V is its Σ^1_3-correctness, i.e. its agreement with V on Σ^1_3 formulas.

Theorem 1.8. *([33], 7.9) Assume that for every $x \in \mathbb{R}$, $x^{\dagger}$ exists. Then K is Σ^1_3-correct, i.e. $K \prec_{\Sigma^1_3} V$.*

While achieving to be close enough to V, K behaves at the same time in a way that it resembles L. This is already evident from its canonical definition. Probably its most significant and useful property in this direction is its absoluteness under set forcing.

Theorem 1.9. *([33], 5.18) If $\mathbb{P} \in V_{\Omega}$, then $\Vdash_{\mathbb{P}} K = K^V$*

1.1.3 Combinatorial principles

In order to give some more evidence of the correspondence of K with L, we mention a couple of combinatorial principles that hold true for K. Actually the theorem below is more general as it applies not only to K, but to any weasel.

Theorem 1.10. *Every weasel W satisfies the following statements.*

(a) If κ is a cardinal, then $\diamondsuit^+_{\kappa^+}$ holds. ([21], 1.2, I)

(b) If κ is an inaccessible cardinal, then $\diamondsuit^+_{\kappa}$ holds $\Leftrightarrow$ κ is not ineffable. ([21], 1.2, II)

(c) If κ is a cardinal, then $\square_{\kappa}$ holds $\Leftrightarrow$ κ is not subcompact. ([25], 15)

1.2 Complexity of the definition of K

Since K is built in a way that it should resemble L, it is of great interest to have a level-by-level inductive definition of it. It turns out that the complexity of this definition depends on the large cardinal strength that K is called to absorb. To get a flavor of what we opt for, we first have a look at the case where there is no inner model with a measurable cardinal[2].

Definition 1.11. *([4])* Suppose there is no inner model with a measurable cardinal. Then the *core model* K can be defined in the following way:

(a) $K|\omega = J_1$.

(b) $K|(\kappa^+)^K = \bigcup\{\mathcal{M} : \mathcal{M}$ is an iterable premouse which is ω-sound, $\rho_\omega(\mathcal{M}) = \kappa$ and $K|\kappa \trianglelefteq \mathcal{M}\}$.

(c) $K|\lambda = \bigcup_{\kappa<\lambda} K|\kappa$, for limit cardinals λ of K.

The main part of the definition is (b), where one needs to verify that the premice with the specific properties line up and form $K|(\kappa^+)^K$. To check this, compare two premice $\mathcal{M}$, $\mathcal{N}$ with those properties. Since there is no inner model with a measurable cardinal, no extender should strictly overlap κ in either $\mathcal{M}$ or $\mathcal{N}$, because otherwise we would have two measures on different cardinals and the biggest one could be used to iterate $\mathcal{M}$ or $\mathcal{N}$ to an inner model with a measurable. This non-overlap means that the extenders used in the comparison will have critical points at least κ. But since both $\mathcal{M}$ and $\mathcal{N}$ project at κ, each side of the comparison either drops or doesn't move.

If both mice iterate to the same model, then neither side moves, thus $\mathcal{M} = \mathcal{N}$. Otherwise, assume without loss of generality that $\mathcal{M}$ wins the comparison. Then $\mathcal{N}$ does not move and there is a last model $\mathcal{M}^{\mathcal{T}}_\infty \rhd \mathcal{N}$ on the $\mathcal{M}$-side iteration. If $\mathcal{M}$ moved, then there would be at least one cardinal in $\mathcal{M}^{\mathcal{T}}_\infty$, inside the interval $(\kappa, ht(\mathcal{N}))$, created by some extender used along the final branch. But $\mathcal{M}^{\mathcal{T}}_\infty$ strictly contains $\mathcal{N}$, which projects to κ, thus it sees no cardinals in this interval, contradiction.

[2]Note that here the measurable Ω that simplifies the construction is not needed at all. The definition of K simply consists of 1.11.

Equipped with this simple definition, one can get low complexity definitions of initial segments of K. For example:

Corollary 1.12. *([4]) Suppose there is no inner model with a measurable cardinal. Then,*

(a) $K|\omega_1^K$ *is* Σ^1_3*, when coded by reals.*

(b) There is a Σ^1_3 *well-ordering of* $\mathbb{R}^K$*.*

(c) $K|\omega_1$ *is* Σ^1_4*, when coded by reals.*

Sketch of proof.

(a) $a \in K|\omega_1^K$ iff there exists an M satisfying 1.11(b) for $\kappa = \omega$ and such that $x \in M$. The complexity of 1.11(b) is exactly the complexity of a countable premouse being iterable which is Π^1_2, thus $K|\omega_1^K$ is Σ^1_3. In order to be precise, one would have to restate the whole definition above by replacing the countable objects by reals coding them. As this does not raise the complexity, $K|\omega_1^K$ with its elements coded by reals is literally Σ^1_3

(b) Every real of K is already in $K|\omega_1^K$ by acceptability. So the constructibility ordering of $K|\omega_1^K$ defines a well-ordering of $\mathbb{R}^K$ which is Σ^1_3.

(c) $a \in K|\omega_1$ iff:

$$\begin{aligned}\exists \alpha < \omega_1 \exists f\ (f(\alpha) = a \wedge\ dom(f) = \alpha\ \wedge\ f(0) = J_1\ \wedge\ \forall \beta \in \alpha + 1 \\ [\beta\ limit \to f(\beta) = \bigcup_{\gamma \in \beta} f(\gamma)\ \wedge\ \beta = \gamma + 1 \to (x \in f(\beta) \leftrightarrow \\ \exists M\ satisfying\ 1.11(b)\ for\ \kappa = \gamma\ and\ such\ that\ x \in M)])\end{aligned}$$

By (a), the last line is a conjunction of a Σ^1_3 and a Π^1_3 formula, therefore the whole expression is Σ^1_4.

□

The definition of K gets more and more complicated as we weaken our anti-large cardinal hypothesis. The problem is that, in general, there will be extenders overlapping κ, thus we will need stronger restrictions imposed to our premice to show that they line up. Those extra requirements are what is actually raising the complexity. We can view those extra requirements as a demand of a stronger iterability property on the premice and, below a Woodin cardinal, this is implemented by the notion of $\alpha-$*strongness*, $\alpha \in \Omega$ ([33], 6.1). Actually α-strongness can be defined simultaneously with the new levels of K using the notion of phalanges (see [33], 6.6 for the definition and [19], 8.1 to get an idea of how it was first introduced).

Theorem 1.13. *([33], 6.11, 6.14) Suppose there is no inner model with a Woodin cardinal. Then K can be inductively defined in the following way:*

(a) $K|\omega = J_1$.

(b) $\mathcal{M}$ *is κ-strong iff* $M|\kappa = K|\kappa$ *and for every premouse $\mathcal{N}$ which is α-strong for every $\alpha < \kappa$ and* $\overline{\overline{\mathcal{N}}} = \kappa$*, the phalanx* $(\mathcal{N}, \mathcal{M}, \kappa)$ *has no countable bad iteration trees.*

(c) $K|(\kappa^+)^K = \bigcup\{\mathcal{M} : \mathcal{M}$ *is a κ-strong premouse which is ω-sound and* $\rho_\omega(\mathcal{M}) = \kappa\}$.

(d) $K|\lambda = \bigcup_{\kappa<\lambda}$*, for limit cardinals λ of K.*

Just as in the case below a measurable, one can derive complexity bounds for the levels of K below ω_1.

Corollary 1.14. *([33], 6.15) Suppose there is no inner model with a Woodin cardinal. Then,* $K|\omega_1$ *is* Σ_1*-definable over* $J_{\omega_1}(\mathbb{R})$.

Though we needed the more complicated definition of K, involving the iterability of certain phalanges, it turns out that this is not actually needed above ω_2. In fact after that layer of K, we can return to the inductive definition which was used in the case below a measurable. Just to get an idea why this phenomenon takes place, one should consider that above ω_2 the powerful machinery appearing in the proof of covering can be used to carry out this simplification.

Theorem 1.15. *([9], 3.5) Suppose there is no inner model with a Woodin cardinal. If $\kappa \geq \omega_2$, then*

$$K|(\kappa^+)^K = \bigcup\{\mathcal{M} : \mathcal{M} \text{ is an iterable premouse which is } \omega\text{-sound above } \kappa,\ \rho_\omega(\mathcal{M}) = \kappa \text{ and } K|\kappa \trianglelefteq \mathcal{M}\}.$$

1.3 Below a fixed number of strong cardinals

As we saw in the previous paragraph, K can be quite complicated below ω_2 and in particular below ω_1. The latter level is important because it is the place where all the elements of K which can be coded by reals appear. If we strengthen the anti-large cardinal hypothesis to the non-existence of more than $n \in \omega$ or, respectively, $\alpha < \omega_1$ many strong cardinals in $K|\omega_1$, we can get definitions of lower complexities. Actually the complexity can be directly computed from n or α. Those results appear in [10] and [26].

First we consider the case of at most $n \in \omega$ many strong cardinals in $K|\omega_1$. This is true if we assume there is no inner model with more than n strong cardinals.

Theorem 1.16. *([10], 3.4, 3.6) Assume there are at most n strong cardinals in $K|\omega_1$. Then $K|\omega_1$, coded by reals, is $\boldsymbol{\Delta}^1_{n+5}$.*

In a setting higher above, where we have at most $\alpha < \omega_1$ strongs below ω_1 in K, we get the following:

Theorem 1.17. *([26], 1.1) Assume the strong cardinals in $K|\omega_1$ have order type $< \omega\alpha$. Then $K|\omega_1$, coded by reals, is definable over $J_\alpha(\mathbb{R})$.*

The above two results, reveal a clear dependence of the level of $L(\mathbb{R})$ where $K|\omega_1$ belongs and the number of strongs inside it. Actually what is happening is that one can simplify the definition of Steel for mice below a Woodin, in a way that the complexity does not raise while being in-between two strong cardinals. The idea is to relativize the notions appearing in the second order definition of K to the the number of strong cardinals appearing at different times. Then it is enough to give an inductive definition of the relativized structures whose successor steps move on the strong cardinals below ω_1. Below we give a very short sketch of the finite case.

We define the notion of an n-cutpoint and n-fullness just to give a flavor of the relativization to the current number of strongs. We remind the reader that a *cutpoint* in a premouse $\mathcal{M}$ is an ordinal such that the extenders in the sequence of $\mathcal{M}$ do not overlap it.

Definition 1.18. Suppose $\mathcal{M}$, $\mathcal{N}$ are premice. $\mathcal{M}$ is an *n-cutpoint* in $\mathcal{N}$ iff:

(a) $\mathcal{M} \trianglelefteq \mathcal{N}$, $\mathcal{M}$ is passive.

(b) $\mathcal{M}$ and $\mathcal{N}$ agree on the cardinals of $\mathcal{M}$ and $\mathcal{M} \models$ "λ is the largest cardinal".

(c) For every extender $E_\nu^{\mathcal{N}}$ with critical point $\kappa \leq \lambda$ and natural length $> OR \cap \mathcal{M}$, either

   (a) $d^{\mathcal{M}}(\kappa) < n$, where $d^{\mathcal{M}}(\kappa)$ denotes the order type of the strong cardinals of $\mathcal{M}|\kappa$[3], or

   (b) $E_\nu^{\mathcal{N}}$ is partial.

The notion of an n-cutpoint will be used everywhere in order to relativize to n and focus only on extenders that drag models with less than n strongs. The same happens with n-fullness. Remember that the original notion of fullness requires that the definability property holds for a universal weasel containing a mouse, as well as for its iterates, at the points above the extenders which are used.

Definition 1.19. $\mathcal{M}$ is *n-full* iff there is a universal weasel $W \triangleright \mathcal{M}$ such that:

(a) $\mathcal{M}$ is an n-cutpoint of W.

(b) Suppose W^* is an iterate of W with iteration map $i : W \to W^*$ and such that for every extender E used in the iteration, $d^{i(\mathcal{M})}(crit(E)) \geq n$. Then W^* has the definability property at all W^*-cardinals $\kappa \in$

[3]Note that in the finite case, the number of strongs is $d^{\mathcal{M}}(\kappa) - 1$ according to this definition.

$[\nu, OR \cap i(\lambda))$ with the property $d^{W^*}(\kappa) < n$, where ν is the least ordinal above the generators of extenders used in the iteration and λ is the largest cardinal of $\mathcal{M}$.

$\mathcal{M}$ is *strongly n-full* if additionally $cf(\lambda)^{\mathcal{M}} = \omega$ and $OR \cap \mathcal{M} = (\lambda^+)^W$.

Using the notions of being strongly n-full, n-full and n-collapsing, one can define K in a way similar to its usual second order definition.

Remark 1.20. (derived from [10], 3.4, 3.6) Suppose $K|\omega_1 \models$ "There are at most n strong cardinals". Then $K|\omega_1$ can be defined using:

(a) n-fullness and strong n-fullness, if ω_1 is inaccessible in K.

(b) n-collapsing premice, if ω_1 is a successor in K.

The above remark reduces the definition of K to the definition of n-fullness. The way to get a low complexity for the latter is by the use of n-beavers. Those are auxiliary premice whose purpose is to enable us to prove that certain extenders lie on given mice or their iterates. This further leads to the definition of internal n-fullness. We will avoid defining those concepts here and the interested reader can find this material in [10] and [26]. The only information we will add is the dependency of those notions according to 2.15-2.17 of [10]. Just keep in mind that the definition of K^c is altered a bit in this context in order to absorb extenders coming from $< n$ beavers. This dependency shows how to get an inductive definition by climbing up the strong cardinals and thus eventually being projective in case they are finitely many.

Remark 1.21. (Derived from [10], 2.15-2.17) Assume $\mathcal{M}$ is a premouse. The properties of being strongly n-full, n-full, n-collapsing and an n-beaver can be inductively defined using the steps below. Each step adds one real quantifier to the definition and eventually the complexity of each notion is Π^1_3.

(a) Strong i-fullness can be defined by internal i-fullness and $< i$-beavers.

(b) i-beavers can be defined by internal $(i-1)$-fullness and $< i$-beavers.

(c) Being i-collapsing can be defined by i-fullness.

Note here that the definition of internal i-fullness is independent of i, thus the induction can be carried along. We finally get the complexity in 1.16 combining 1.20 and 1.21, respectively for the cases where ω_1 is an inaccessible or a successor.

Chapter 2

Some Coding

In this chapter we describe the procedure of coding a subset A of κ^+ to a subset B of κ over some inner model M, where κ is a regular cardinal which (weakly) bounds the continuum function for ordinals below it. By that, we mean the process which, given A, produces a generic extension where $A \in M[B]$. This roughly breaks down into two parts. The first one is applying almost disjoint coding to shrink A to B over M. Ideally this forcing would do the trick, but unfortunately it has the heavy requirement that M should see enough almost disjoint subsets of κ. In order to achieve this, we need the second part which is a forcing called reshaping. In this case, reshaping means adding more information to A in order to give $M[A]$ the ability to see, for all $\xi < \kappa^+$, that $\overline{\overline{\xi}} = \kappa$, using only the information from $A \cap \xi$. Afterwards, the decoding procedure works by inductively recovering A in $M[B]$. We will focus here on the case where M behaves just like L, following the first chapter of [1]. We consider three simple situations where we describe respectively almost disjoint coding, reshaping and the inductive procedure of decoding after having reshaped. In each one of the forcings we will prove the preservation of cardinals, which is crucial in making sense of coding[1]. Finally, at the end of each case we give a flavor of how it could be extended to more complicated models like K and indicate the difficulties that one faces in this task.

[1]Otherwise why not just collapse κ^+ to κ?

2.1 Almost disjoint coding

Assume M is an inner model of ZFC, κ is a regular cardinal such that $(\kappa^+)^M = \kappa^+$, $2^\alpha \leq \kappa$, for all $\alpha < \kappa$, and $A \subset \kappa^+$. As mentioned in the introduction, there is a forcing coding A to a $B \subset \kappa$. In order to shrink A in size, we use κ^+ many subsets of κ in M to code its content. The next lemma ensures their existence.

Definition 2.1. Two subsets A, B of κ are called *almost disjoint (a.d.)* if $\overline{\overline{A}} = \overline{\overline{B}} = \kappa$ and $\overline{\overline{A \cap B}} < \kappa$.

Lemma 2.2. *There exist at least κ^+ pairwise a.d. subsets of κ.*

Proof. To begin with, there are at least κ many a.d. subsets of κ. Just pick a bijection $f : \kappa \times \kappa \to \kappa$ and $A_i = f''(\{i\} \times \kappa)$ will witness this fact. Now suppose $\vec{A} = (A_i : i < \kappa)$ is any sequence of a.d. sets. Then by the regularity of κ, $\vec{B} = (B_i : i < \kappa)$, where $B_i = A_i \setminus \bigcup_{j<i} A_j$, is a sequence of disjoint sets of size κ. The set $B = \{\min(B_i) : i < \kappa\}$ is a.d. from every element of $\vec{A}$ as each intersection $B \cap A_i$ has size at most $\overline{\overline{i}} < \kappa$. This allows us to construct a sequence $\vec{A} = (A_i : i < \kappa^+)$ of a.d. sets by transfinite induction. □

Our assumption that $(\kappa^+)^M = \kappa^+$ and the above lemma allow us to fix a sequence $\vec{A} \in M$ of a.d. sets. After applying a.d. forcing, this sequence will code A into B in the following way:

$$i \in A \leftrightarrow \overline{\overline{B \cap A_i}} < \kappa.$$

Remark 2.3. In order to keep things simple and precise, we use $\breve{p} = \{\beta \in \alpha : p(\beta) = 1\}$ to denote the subset of α with characteristic function $p : \alpha \to 2$. In the same fashion, $\hat{a}$ will denote the characteristic function p of a truncated on the length of a, i.e. $dom(p) = \sup(a)$ and $p(\alpha) = 1$ if $\alpha \in a$ else $p(\alpha) = 0$. We also use the $\oplus$ operator to glue sets of ordinals together. If $a, b \subset \alpha$, then $a \oplus b = \breve{p}$, where $p(2\beta + 1) = \hat{a}(\beta)$ and $p(2\beta) = \hat{b}(\beta)$, for $\beta < \alpha$. We set $(a \oplus b)_{even} = a$ and $(a \oplus b)_{odd} = b$.

Definition 2.4 (Almost disjoint coding, [13])**.** The *almost disjoint coding* $\mathbb{P}$ consists of tuples (p, p^*) such that:

1. $p : \alpha \to 2$, $\alpha < \kappa$.
2. $p^* \subset \kappa^+$, $\overline{\overline{p^*}} < \kappa$.
3. $(p, p^*) \leq (q, q^*)$ iff $p \supseteq q$, $p^* \supseteq q^*$ and for all $i \in q^*$, $(\check{p} \setminus \check{q}) \cap A_i = \emptyset$.

The idea is that the "p" parts of the tuples approximate the set B while the "p^*" parts make the promises that for some elements of A, the sets of $\vec{A}$ with the corresponding indices will not be met any more while extending p. Suppose now that G is generic over $\mathbb{P}$. We set $B = proj_1(\cup G)$.

Lemma 2.5. *The following hold for* $\mathbb{P}$*:*

(a) $A \in M[B]$.

(b) $\mathbb{P}$ *has the* κ^+*-cc property.*

(c) $\mathbb{P}$ *is* $< \kappa$*-closed.*

Proof.

(a) We just need to prove that $i \in A \Leftrightarrow \overline{\overline{B \cap A_i}} < \kappa$. For the $\Rightarrow$ direction, notice that given an $i \in A$, every condition (p, p^*) can be extended to $(p, p^* \cup \{i\})$, thus G contains a condition (p, p^*) such that $i \in p^*$. This in turn implies that $B \cap A_i \subset \check{p}$ which is enough since $\overline{\overline{\check{p}}} < \kappa$.

For the $\Leftarrow$ direction, assume we are given a condition (p, p^*) and an ordinal $\alpha < \kappa$. Since $\overline{\overline{p^*}} < \kappa$, $\overline{\overline{A_i \setminus \bigcup_{j \in p^*} A_j}} = \kappa$. Given that $i \notin A \Rightarrow i \notin p^*$, we may extend (p, p^*) by adding some element of A_i above α to p. This implies that there are cofinally many elements of A_i in B, hence those two sets can not be almost disjoint as κ is a regular cardinal.

(b) First we observe that one can freely extend a condition by just adding elements to the second coordinate. Namely for every pair of conditions, $(p, p^*), (p, q^*) \in \mathbb{P}$, $(p, p^* \cup q^*)$ witnesses their compatibility. This way every anti-chain must contain elements with pairwise different first coordinates, and given that the possibilities are at most $\sum_{\alpha \in \kappa} 2^\alpha = \kappa$, we get κ^+-cc.

(c) As κ is regular, for every decreasing sequence $((p_\beta, p^*_\beta) : \beta < \alpha)$, where $\alpha < \kappa$, $(\bigcup_{\beta<\alpha} p_\alpha, \bigcup_{\beta<\alpha} p^*_\alpha)$ is a lower bound in $\mathbb{P}$.

□

What happens over K?

The above lemma shows that in this simple setting we were able to code over M while preserving cardinalities. Since M could be any inner model, the same results hold true for K, as long as κ has the required properties. To find such a cardinal, we have to make sure that weak covering holds at κ and also that for every $\alpha < \kappa$, $2^\alpha \leq \kappa$. The second property holds on a club of cardinals below Ω and the first one on the stationary set of singular cardinals, provided that there is no inner model with a Woodin cardinal. Therefore we are able to apply this coding procedure at any of those stationarily many available cardinals.

2.2 Reshaping

In the previous sections we coded at cardinals where M satisfied weak covering. This gives us some options when M is close to V, but still if our goal is to code below a specific cardinal, e.g. ω_1, then we need some stronger technique. This technique is called reshaping and what it does, is to add some more information to the set $A \subset \kappa^+$ to assist the coding. The information is just enough to help M slowly collapse the possible cardinals in (κ, κ^+) while at the same time κ^+ remains intact in the forcing extension.

We also introduce this forcing using a simple example (see [1], 1.3). This time we will make M more specific since we need to add some condensation properties. The setting is the following:

(a) $M = L[E]$, where E is a class predicate[2].

[2]There is no harm for the reader to imagine E as a sequence of extenders, like in the case of K. It will be evident though, from the condensation property below, that this is not a good candidate in case it contains large cardinals which have enough consistency strength.

(b) A is a subset of κ^+, $P(\kappa^+) \subset M[A]$ and $2^{<\kappa} = \kappa$.

(c) M condenses, in the sense that for every model N of size κ and every elementary embedding $j : N \to M|\mu$, where μ is a cardinal $\geq \kappa^+$, $N = M|\alpha$ for some $\alpha < \kappa^+$.

We continue with the forcing of this section:

Definition 2.6 (Reshaping)**.** The *reshaping* forcing $\mathbb{P}$ consists of conditions p such that:

(a) $p : \alpha \to 2$, $\alpha < \kappa^+$.

(b) For every $\xi \leq \alpha$, $M[A \cap \xi, p \restriction \xi] \vDash \overline{\overline{\xi}} = \kappa$.

The idea behind this forcing, is to add a generic G which gives the ability to M of collapsing level-by-level possible cardinals which are not cardinals in $V[G]$. Thus it can see more almost disjoint subsets of κ each time some cardinal is collapsed to κ. Afterwards we may code using those a.d. sets which appear at the right time. The decoding procedure will then be inductive, as we will use at each step all the information we decoded up to that point, in order to fetch the next a.d. set which enables us to further decode and so on. Of course we have to check first that reshaping adds such a generic without collapsing any cardinals. Here are the basic properties of $\mathbb{P}$.

Lemma 2.7. *Supposing G is a generic for $\mathbb{P}$, the following hold true:*

(a) For every ordinal $\alpha < \kappa^+$, $M[A \cap \alpha, G \restriction \alpha] \vDash \overline{\overline{\alpha}} = \kappa$.

(b) $\mathbb{P}$ has the κ^+-cc.

(c) $\mathbb{P}$ is $< \kappa^+$-distributive.

Proof.

(a) Fixing α, we notice that every condition p can be extended to one of domain containing α. Simply set $q = p^\frown f^\frown 0^\alpha$, where $f : \kappa \to 2$ codes a function collapsing $dom(p) + \kappa + \alpha$ to κ. Therefore there is a $p \in G$ such that $dom(p) \geq \alpha$. But then by the definition of being a condition, $M[A \cap \alpha, p \restriction \alpha] \vDash \overline{\overline{\alpha}} = \kappa$ and since $G \restriction \alpha = p \restriction \alpha$, we are done.

(b) The size of the forcing is $\sum_{\alpha<\kappa^+} 2^\alpha = 2^\kappa \cdot \kappa^+ = \kappa^+$.

(c) Suppose $(D_i : i < \kappa)$ is a sequence of dense open subsets of $\mathbb{P}$ and p_0 a given condition. We will produce a decreasing sequence $(p_i : i \leq \kappa)$ of conditions such that $p_{i+1} \in D_i$ and eventually p_κ is in the intersection of the elements of $(D_i : i < \kappa)$. The successor steps can be dealt with, using the trick from (a) but at limit steps we take unions and thus need to ensure they will still be conditions. To do this we will define each extension in a minimal over M way so that M will be able to see enough of the $(p_i : i \leq \kappa)$ sequence on the limit cases.

Since every subset of κ^+ is in $M[A]$, we may use the condensation of M to get an embedding $j : M|\alpha[A] \to M|\mu[A]$, $\alpha < \kappa^{++}$, which witnesses that $\mathbb{P}$, p_0, $(D_i : i < \kappa) \in M|\kappa^{++}[A]$. We proceed by defining a strictly increasing tower of structures X_i, which are all elementary embedded inside the model $\mathfrak{A} = (M|\kappa^{++}[A], \mathbb{P}, p_0, (D_i : i < \kappa), <)$. This tower will assist us in picking the decreasing sequence of conditions in a canonical way. We define for $i \leq \kappa$,

(a) $X_0 = Hull^{\mathfrak{A}}(\kappa + 1)$;

(b) $X_{i+1} = Hull^{\mathfrak{A}}(X_i \cup \{X_i\})$;

(c) $X_i = \bigcup_{j<i} X_j$, for limit i;

(d) $f_i : \mathfrak{A}_i \overset{tr.coll.}{\simeq} X_i$;

By the condensation of M and the fact that $\kappa + 1 \subset \mathfrak{A}$, there exist α_i, κ_i, $i \leq \kappa$, such that $\mathfrak{A}_i = M|\alpha_i[A \cap \kappa_i]$. In fact κ_i is the critical point of f_i for $i \leq \kappa$. Now that we have constructed the canonical tower of structures, we may also pick the p_i's:

(a) p_0 is the initial condition.

(b) p_{i+1} = the $M|\kappa^+[A]$-least p, such that $p < p_i$, $otp(dom(p)) \geq \alpha_i$ and $p_{i+1} \in D_i$.

(c) $p_i = \bigcup_{j<i} p_j$, for limit ordinals i.

Since $\mathfrak{A}_i \in \mathfrak{A}_{i+1}$, there is already a p in $\mathfrak{A}_{i+1}$ with the properties of (b) maybe without minimality. But this means that $p_{i+1} \in \mathfrak{A}_{i+1}$, as $\mathfrak{A}_{i+1}$ has the form $M|\alpha_{i+1}[A \cap \kappa_{i+1}]$ and the conditions appear below κ_{i+1}. The idea now is to prove that this also holds for limit ordinals

$i < \kappa$ and that the unions considered are already conditions. Given that this is true below some limit i, $(p_j : j < i)$ is definable in $\mathfrak{A}_{i+1}$ exactly the same way as in V, therefore $\bigcup_{j<i} p_j = p_i \in \mathfrak{A}_{i+1}$. In fact the aforementioned sequence is definable over $\mathfrak{A}_i = M|(\alpha_i+1)[A \cap \kappa_i]$. By the requirement that the conditions' lengths each time exceed α_j and the fact that i is limit, we have that $\sup_{j<i} \alpha_j = \kappa_i = dom(p_i)$. Thus $M|(\alpha_i + 1)[A \cap \kappa_i] \vDash \overline{\overline{\alpha}}_i \leq i \cdot \kappa = \kappa$ implying directly that p_i is a condition.

By the above we have that p_κ is a condition below all the dense open sets and p_0, thus we have κ-distributivity.

□

What happens over K?

In contrast with almost disjoint coding, reshaping cannot be applied to K, at least not in the way it is defined in our simple example. The reason for that is that we cannot always get for K the condensation we used here. In the next chapter we will see such a failure of condensation assuming consistency strength a little below a strong cardinal. Exactly this problem is forcing us to switch from K to other structures in chapters 5 and 6 where the main coding results are presented.

2.3 Almost disjoint coding on a reshaped set

We will see now how to apply almost disjoint coding after we have reshaped below κ^+. The situation is exactly as we left it at the end of the previous section. Thus κ is a regular cardinal that bounds the continuum function and M has the properties we mentioned. The interval (κ, κ^+) is reshaped, i.e. for every $\kappa < \alpha < \kappa^+$, $M[A \cap \alpha] \vDash \overline{\overline{\alpha}} = \kappa$, where $A \subset \kappa^+$ is the set we want to code.

The forcing used is the same as the one in definition 2.4. The only difference is that the choice of the sequence $\vec{A} = (A_i : i < \kappa^+)$ of almost disjoint sets, is now dictated by the reshaping. More specifically, the sequence is defined as follows:

$$A_i = \textit{the } M[A \cap i] - \textit{least set a.d. from all the elements}$$
$$\textit{of } (A_j : j < i) \textit{ and such that } \overline{\overline{\kappa \setminus \bigcup_{j \le i} A_j}} = \kappa.$$

The existence of each A_i is directly guaranteed by the reshaping condition. The κ^+-cc property and the $< \kappa$-closeness of the forcing can be proved the same way as before. The part which needs checking is that the decoding of A can be carried out successfully.

Lemma 2.8. *Given the reshaping assumptions mentioned above, in the generic extesion of a.d. forcing $A \in M[B]$.*

Proof. Assume that $B = proj_1(\cup G)$, like before. We prove inductively that for every $i < \kappa^+$, $M[A \cap i] \subset M[B]$, thus $A \in M[B]$. For a successor ordinal $i+1$, if $M[A \cap i] \subset M[B]$, then $A_i \in M[B]$. Thus $M[A \cap (i+1)] \subset M[B]$, as $i \in A$ is equivalent to $\overline{\overline{A_i \cap B}} < \kappa$. If i is a limit ordinal and $M[A \cap j] \subset M[B]$ for every $j < i$, then for every $j < i$, $A_i \in M[B]$ and it is additionally defined in a canonical way, thus $(A_j : j < i) \in M[B]$. The latter sequence contains enough information to define $A \cap i$ by checking the cardinalities of $A_j \cap B$, so finally $M[A \cap i] \subset M[B]$.

□

What happens over K?

Just like the first case of a.d. coding, everything mentioned in this section can be directly transferred to K. The main difficulty again lies in finding or creating the interval (κ, κ^+) which is reshaped. After this interval is fixed, the coding forcing does not produce any complications.

2.4 Why force, why reshape and coding the universe

Finally, we give justification for what we discussed in the previous sections as well as present some general coding results. We first prove that there are generic extensions of M, where forcing is needed in order to reshape. Those extensions exist if there is a Mahlo cardinal in M. If there is no Mahlo cardinal in M, one can actually run the reshaping by just using sets of V. The second part provides some arguments on why reshaping makes the coding procedure easier and helps avoid collapsing cardinals. The last part focuses on Jensen's coding theorem and some generalizations of it.

Why force?

Let's have a look first at the simple case of reshaping, where there are no inaccessible cardinals in our interval of interest. Suppose we have an $A \subset (\kappa, \kappa^+)$ which we wish to reshape. Furthermore, assume that there are no inaccessible cardinals in M in the interval (κ, κ^+) and M has the properties (a) and (b) mentioned in section 2.2. Define the set $B \subset \kappa^+$, so that for every cardinal $\kappa < \mu < \kappa^+$ of M, $B \cap [\mu, \mu + \kappa)$ codes a function that collapses μ to κ. Set $C = A \oplus B$. Then C is reshaped at successor cardinals of M in $[\kappa, \kappa^+)$. For any limit cardinal μ in the aforementioned interval, let $cf(\mu) = \nu < \mu$. Then in $M[C \cap \mu]$, the cardinality of μ is $\nu \cdot \kappa = \kappa$, assuming that C is reshaped below μ. Thus inductively C is reshaped in the whole interval $[\kappa, \kappa^+)$.

In the above case we didn't need to force in order to reshape. The same idea would work in the case where κ^+ is not Mahlo in M. In this situation, some inaccessible cardinals might exist, but they are non-stationarily many. Thus extending B to include those cardinals will still do the trick (see 5.12). Unfortunately this idea fails in case κ^+ is Mahlo in M. Moreover, we may produce a generic extension of M, where there is no reshaped subset of κ^+. Therefore forcing is indeed needed in order to perform reshaping.

Lemma 2.9. *Assume λ is Mahlo in M and $\kappa < \lambda$ is a regular cardinal of M. Then if G is $Col(\kappa, < \lambda)$-generic over M, there is no reshaped set $A \subset \kappa^+$ in $M[G]$.*

Proof. Assume $\dot{A}^G \subset \kappa^+$ is an element of $M[G]$. Since G is $Col(\kappa, < \lambda)$-generic/M, for every inaccessible in M cardinal, $G \restriction \xi$ is $Col(\kappa, < \xi)$-generic/M. Additionally, there is a club set of ordinals such that $\dot{A}^{G\restriction\xi} \cap \xi$[3] is defined and equal to $\dot{A}^G \cap \xi$. Since we have stationary many inacessibles below κ^+, there is a ξ such that $\dot{A}^G \cap \xi \in M[G_\xi]$ and at the same time $M[G_\xi]$ is a $Col(\kappa, < \xi)$-generic extension. But then $M[A \cap \xi] \vDash \overline{\overline{\xi}} = \kappa^+$, which means that A^G is not reshaped. □

The same result holds true for coding. Assume that we are in the same situation described in the above lemma. Then if there was a set $B \subset \kappa$ in $M[G]$, such that $M[B] = M[G]$, then the interval $[\kappa, \kappa^+)$ would be trivially reshaped by B -as $M[B] = M[G]$ knows there are no cardinals in (κ, κ^+)-, contradicting what we proved above. Thus even under the existence of a little consistency strength, forcing may be required to carry out coding.

Why reshape?

A direct answer to this question is that reshaping pushes the requirements of coding one cardinal above. To make this concrete, imagine the situation where we want to code a set $A \subset (\kappa, \kappa^+)$ below κ over M. In our first attempt, we managed to perform this coding by assuming that M satisfies weak covering at κ. If this is not true, there are not enough a.d. sets in M to carry out the coding and the only direct way to proceed is to collapse κ^+ to κ.

The other alternative is to reshape A and then code it. Going back to the requirements of the forcing on M, we see that we need $P(\kappa^+) \subset M[A]$. If we are lucky enough and there is a $\lambda \geq \kappa^{++}$ where weak covering holds and we can also find a set $A' \subset \lambda^+$ such that $P(\lambda) \subset M[A']$, then we can collapse λ to κ^+ and get the environment we needed to reshape. In this situation, we were still quite invasive on V, by collapsing a lot of cardinals above κ^+, but κ^+ remained intact, thus the problem of not collapsing is shifted one step above. This procedure is described in [28], section 2, and we analyze it here in chapter 4 for the cases we will be interested in.

[3] Note that here we slightly abuse the notation. The formal way to define $\dot{A}^{G\restriction\xi} \cap \xi$ is $(\dot{A} \restriction_{rec} Col(\kappa, < \xi))^{G\restriction\xi} \cap \xi$, where $\dot{A} \restriction_{rec} Col(\kappa, < \xi)$ is A with all the pairs with first element in $Col(\kappa, < \lambda) \setminus Col(\kappa, < \xi)$ recursively removed.

If one tries to keep shifting the collapses further above, he would have to reshape further above, in the interval (κ^+, κ^{++}), then code below κ^+ and then repeat the procedure mentioned above. By extending this idea to a class forcing, we could potentially code all of V into a real x over M, given that M satisfies enough covering and condensation properties. For the case of L, this is Jensen's "coding the universe" theorem.

Coding the universe

Though the idea we describe above is a reasonable basis to begin coding V over L, the forcing required is actually quite more involved, as one has to simultaneously perform the a.d. codings and the reshapings between all cardinals, as well as take care of the limit ordinals. This forcing is described on the first half of [1]. A guide for reading this book is contained in [5] and shorter version of the proof appears in [7], sections 4.2, 4.3. This coding result has been improved by R. David, who coded V into a Π^1_2-definable real over L.

Theorem 2.10. *Suppose GCH holds in V. Then,*

(a) ([1], 0.1) There is a class forcing $\mathbb{P}$ which codes the universe to a real, i.e. if G is $\mathbb{P}$-generic over V, then $V[G] = L[x]$, $x \subset \omega$. Furthermore, $V[G]$ also satisfies GCH and the large cardinal properties of being Mahlo, weakly compact, indescribable, subtle, ineffable and α-Erdös, $\alpha < \omega_1$ are preserved.

(b) ([3], Thm 4) An elaboration of the above forcing produces a generic G such that $V[G] = L[r]$ and $V[G] \vDash$ "r is a Π^1_2 singleton".

There exist further extensions of Jensen's coding theorem to inner models which contain more large cardinal consistency strength. At [6], one can find a version of the coding over $L[\mu, x]$, where μ is a normal measure on a measurable cardinal and $x \in \mathbb{R}$. For an overview of such generalizations, look at [8]. On the direction of forcing to locally code a set and add a well-ordering of the reals there is [28] where the coding is performed over K under the non-existence of inner models with strong cardinals. In the chapters to follow, we attempt to generalize the latter result at the level below the existence of strong cardinals of a specific order-type.

Chapter 3

Failures of Condensation

Up to now we have considered only coding into an inner model M which almost resembles L, mainly due to its condensation properties. When one decides to switch to some other inner model, such as K, we have already seen that new difficulties arise, due to the fact that condensation might fail for those models. Condensation was crucial in the proof of distributivity of reshaping, where the collapsed structures were of the form $\mathfrak{A}_i = M|\kappa_i[A \cap \alpha_i]$. If we replace M with $K = L[E]$ those structures will be $\mathfrak{A}_i = J_{\kappa_i}[A \cap \alpha_i, \overline{E \restriction \mu}]$. To ensure condensation, we would have to prove that $\overline{E \restriction \mu}$ is an initial segment of E. As we will see in this chapter, if K contains an α-strong cardinal κ, $\alpha \geq \kappa^{++}$, then there exist elementary embeddings into initial segments of K, such that condensation fails. This will be an indication that we have to work with models different than K to carry out coding, or alternatively build A in a more elaborate way. In the subsequent chapters, we will focus on the first approach.

3.1 An embedding into $K||\alpha$, where α is the length of the extender

Assume the existence and fix a total-for-K extender F of the K-sequence with critical point κ and length $\alpha \geq \kappa^{++}$. We will show that after collapsing the cardinals below κ in K, we can find an embedding $j : \mathcal{M} \to K||\alpha$, for a countable mouse $\mathcal{M}$, which fails to satisfy the strong condensation hypothesis we used for coding. In fact, by the homogeneity of the collapse, there will be 2^ω such counterexample embeddings.

Note that in the section 3.2, we will prove that this still holds true if we replace α with any regular cardinal $\theta \geq \alpha$.

Lemma 3.1. $K^{Col(\omega,<\kappa)} \models$ *" there exists an $\overline{\alpha}$ and a set $\mathcal{A}$ of 2^ω many extenders, such that for each $F' \in \mathcal{A}$:*

(a) $F' \notin K$;

(b) $(K|\overline{\alpha}, F')$ is a premouse;

(c) there is an elementary embedding $\pi' : (K|\overline{\alpha}, F') \to K||\alpha$ with critical point $\overline{\kappa}$ such that $\pi'(\overline{\kappa}) = \kappa = \omega_1$."

Proof.

Step 1: There is at least one such F'.

Suppose $\pi : K \to Ult(K; F) = M$ is the embedding induced by F. We are going to extend π to $\tilde{\pi} : K[H] \to M[G]$, where H is $Col(\omega, < \kappa)$-generic over K and G is $Col(\omega, < \pi(\kappa))$-generic over M (see Figure 3.1). We are able to do this because of the next claim.

Claim 3.2. *If G is $M^{Col(\omega,<\pi(\kappa))}$-generic, then $H = \pi^{-1}$"G is $K^{Col(\omega,<\kappa)}$-generic.*

Proof. Let A be a maximal antichain of $Col(\omega, < \kappa)$. The size of A is strictly less than κ since the Levy collapse has the κ-cc property. This implies that $\pi(A) = \pi$"A[1]. But $\pi(A)$ is a maximal antichain of $Col(\omega, < \pi(\kappa))$ because of elementarity, thus there is a $p \in G \cap \pi$"A. This implies that $\pi^{-1}(p) \in H \cap A$. Since H intersects every antichain of $Col(\omega, < \kappa)$, it is generic over K. □

We lift π to $\tilde{\pi} : K[H] \to M[G]$ in the usual way, i.e. $\tilde{\pi}(\dot{a}^H) = \pi(\dot{a})^G$. We only need to check that $\tilde{\pi}$ is elementary. The following argument is enough:

$$K[H] \models \phi(\dot{a}^H) \Leftrightarrow \exists p \in H \ K \models p \Vdash \phi(\dot{a}) \Leftrightarrow M \models \pi(p) \Vdash \phi(\pi(\dot{a}))$$

[1]In fact $\pi(A) = A$ in this case but the way we form the proof it a applies for every forcing satisfying the κ-cc.

$$\Leftrightarrow \exists q \in G \; M \models q \Vdash \phi(\pi(\dot{a})) \Leftrightarrow M[G] \models \phi(\pi(\dot{a})^G).$$

Everything is direct except from the left direction of the first equivalence in the second line, which requires the claim above. We apply the argument for $A = \{p \in Col(\omega, < \kappa) : p \Vdash \phi(\dot{a}) \vee p \Vdash \neg\phi(\dot{a})\}$ which is dense. Since $M \models q \Vdash \phi(\pi(\dot{a}))$ for some $q \in G$ and G must meet $\pi(A)$ at some condition r, $r \Vdash \phi(\pi(\dot{a}))$. The inverse image of r is the condition needed in H.

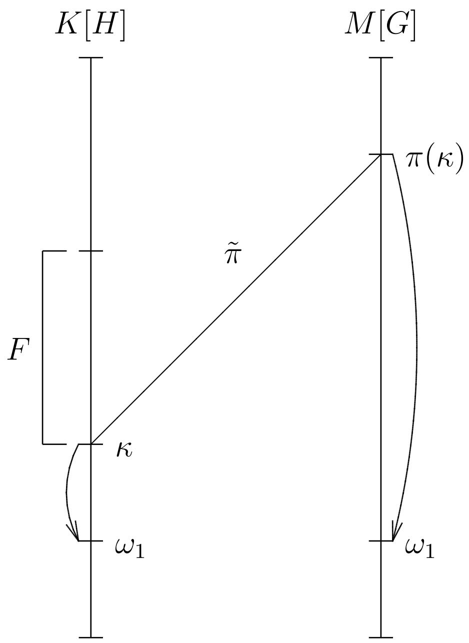

FIGURE 3.1: Lifting π to $\tilde{\pi}$

Now consider the restricted embedding $\pi \restriction (K||\alpha)$ and also pick G so that it is $Col(\omega, < \pi(\kappa))$-generic over both K and M. In the current situation, $\pi \restriction \alpha$ lies in K and $K[G]$ sees additionally that $K||\alpha$ is countable. Our strategy is to pull $\pi \restriction \alpha$ inside $M[G]$ and then using $\tilde{\pi}$ pull it back to $K[H]$. This way, the final embedding will have the desired form. To carry out the first step, we use the sublemma below. This is a very standard and simple way of pulling an embedding inside a structure.

Sublemma 3.3. *Suppose $(M, F), (N, E, E')$ are given structures, where M is countable. Furthermore, the relations F and E are of the same arity. Then there is a tree T of at most countable height looking for an F' and an elementary embedding $\pi : (M, F, F') \to (N, E, E')$.*

Proof. Let $(a_i : i \in \omega)$ be an enumeration of the elements of M and l the arity of E'. We define the tree

$$\begin{aligned} T = \{(\overline{F}, \vec{b}) \in {}^{<\omega}(M^l) \times {}^{<\omega}N :\ & \text{such that for some } n \in \omega, \\ & \text{length}(\vec{b}) = n,\ \overline{F} \subseteq \{a_0, \dots, a_n\}^l \text{ and for each formula } \phi, \\ & (M, F, \overline{F}) \vDash \phi(a_0, \dots, a_n) \Leftrightarrow (N, E, E') \vDash \phi(\vec{b})\}. \end{aligned}$$

This tree is actually building all possible finite predicates $\overline{F} \subset {}^{\omega}M$ and for each one of them, all the finite elementary embeddings from $(M, F, \overline{F})$ to (N, E, E'). Therefore the existance of a predicate F' and a full corresponding embedding is equivalent to the existence of an infinite branch on T. □

Note that this argument is quite flexible and it can be applied at any complicated situation[2] in order to translate the existence of an object to the ill-foundedness of a tree. As long as the tree has countable hight and exists in the structure of interest, one can use absoluteness to find a similar object inside the structure.

Now let $T \in M[G]$ be the tree looking for an F^* together with an embedding $\pi^* : (K|\alpha, F^*) \to \pi(K||\alpha)$. By the existence of F', T is ill-founded in V and by the absoluteness of well-foundedness, it is also ill-founded $M[G]$. Consider a embedding π^* and an extender F^* in $M[G]$ witnessing the corresponding ill-foundedness. This embedding still satisfies $\pi^*(\kappa) = \pi(\kappa)$. By pulling back the statement "$\exists F'\ \exists\overline{\alpha}\ \exists\pi'\ \pi' : (K|\overline{\alpha}, F') \to (K||\pi(\alpha))$" via $\tilde{\pi}$, we get an embedding $\pi' : K||\overline{\alpha} \to K||\alpha$ in $K[H]$ such that $\pi'(\overline{\kappa}) = \kappa = \omega_1^{K[H]}$ and $crit(\pi') = \overline{\kappa}$. Let F' be the top extender of $K||\overline{\alpha}$.

To complete the proof of step 1, we need to check that $F' \notin K$. Suppose not. Since $K|\alpha$ is the ultrapower of $K|\overline{\alpha}$ by F', $\overline{\alpha}$ is a cardinal in $K|\alpha$ thus also in K. But then since $F' \in K$, $\overline{\alpha} \subseteq \{i_{F'}(f)(a) : f : [\overline{\kappa}]^{\overline{\overline{a}}} \to \overline{\kappa}, a \in [\lambda]^{<\omega} \in K\}$, where λ is the supremum of the generators of F'. So $card(\overline{\alpha}) \leq \overline{\kappa}^+ \cdot \lambda < \overline{\alpha}$, which is a contradiction.

[2] E.g. looking for a countable iteration of countable premice.

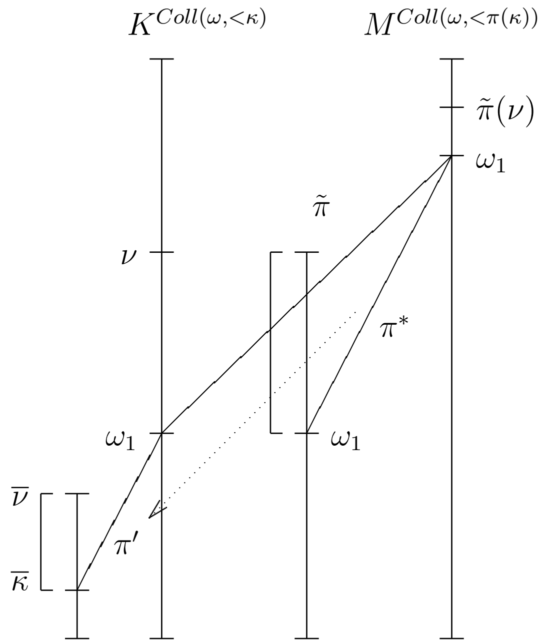

FIGURE 3.2: Pulling π^* back to π'

Step 2: There are 2^ω such extenders.

Since $N = (K|\overline{\alpha}, F')$ is countable, we can code it to into a real and find a nice name $\dot{N}$ for it. $\dot{N}$ has size $\mu < \kappa$ because $Col(\omega, < \kappa)$ satisfies the κ-cc. This means that $\dot{N}$ is already a name of $Col(\omega, \mu)$ for some $\mu < \kappa$. Therefore, we may split $Col(\omega, < \kappa)$ to the forcings $\mathbb{P} = Col(\omega, \mu)$ and $\mathbb{Q} = Col(\mu, < \kappa)$. For those notions we get:

(a) $\overline{\overline{\mathbb{P}}} < \kappa$.

(b) $\dot{N} \in \mathbb{P}$.

(c) $Col(\omega, < \kappa) = \mathbb{P} * \mathbb{Q}$.

Since $\mathbb{P}$ is countable in $K^{Col(\omega,<\kappa)}$, it contains an enumeration $(A_n : n \in \omega)$ of the maximal antichains of $\mathbb{P}$. This enables us to define inductively a binary tree $S = ((p_x : x \in^{<\omega} 2), <)$ of conditions and a sequence $((a_n, X_n) : a_n \in [\overline{\alpha}]^{<\omega}), X_n \subseteq [\overline{\kappa}]^{\overline{\overline{a}}_n}$ with the following properties:

(a) $x \leq_{lex} y \Rightarrow p_x \leq p_y$;

(b) $\overline{\overline{x}} = 2n+1 \wedge x = y^\frown 1 \Rightarrow p_x \Vdash (\check{a}_n, \check{X}_n) \in \dot{F}'$;

$\overline{\overline{x}} = 2n+1 \wedge x = y^\frown 0 \Rightarrow p_x \Vdash (\check{a}_n, \check{X}_n) \notin \dot{F}'$;

(c) $\overline{\overline{x}} = 2n \Rightarrow \exists q \in A_n \; p_x \leq q$.

This is possible because F' is countable so we can always pick a pair (a, X) not decided by the finitely many conditions in hand and each p can be strengthened to hit one of the maximal antichains.

In $K[H]$, T has $2^{\aleph_0}$ many branches which are all generic for $\mathbb{P}$. Each one of the generics defines a different extender F' hence there are $2^{\aleph_0}$ many such extenders in $K[H]$. □

3.2 Embeddings to $K|\theta$, $\theta \geq \alpha$

Here we make the lemma of the previous section a bit more flexible by creating an embedding $K|\theta$, where θ is a regular cardinal above the length of the extender we used. This slight generalization is enough to disprove the condensation for the case we are interested in. The proof is almost identical to the one in the preceding subsection.

Lemma 3.4. *Assume F is a (κ, α) total-for-K extender on the K-sequence and θ some regular cardinal $\geq \alpha$. Then $K^{Col(\omega,<\kappa)} \models$ " there exists an $\overline{\alpha}$ and a set $\mathcal{A}$ of 2^ω many extenders such that for each $F' \in \mathcal{A}$:*

(a) $F' \notin K$;

(b) $(K|\overline{\alpha}, F')$ is a premouse;

(c) there exists a premouse $M \rhd (K|\overline{\alpha}, F')$ and an elementary embedding $\pi' : M \to K|\theta$ with critical point $\overline{\kappa}$, such that $\pi'(\overline{\kappa}) = \kappa = \omega_1$."

Proof.

Suppose $\pi : K \to Ult(K, F) = M$. Just as before, π can be lifted to an embedding $\tilde{\pi} : K[H] \to M[G]$.

Let $\mathcal{N}$ be the transitive collapse of $Hull^{K|\theta}(\alpha + 1)$ and $\sigma : \mathcal{N} \to K|\theta$ the uncollapsing map. The composition $\pi \circ \sigma : \mathcal{N} \to \pi(K|\theta)$ will now play the role of $\pi \restriction K||\alpha$. This composition witnesses the existence of an infinite branch of $T \in K[G]$ searching for $\mathcal{N}, j$ such that:

(a) $\mathcal{N}$ is a premouse.

(b) $\mathcal{N}|\alpha = K|\alpha$ and $E^{\mathcal{N}}_{\alpha} \neq \emptyset$.

(c) $j : \mathcal{N} \to \pi(K|\theta)$ and j extends the ultrapower map of $K||\alpha$ with $E^{\mathcal{N}}_{\alpha}$.

T also has an infinite branch in $M[G]$ and by pulling back via π, we get the desired $\pi' : \overline{\mathcal{N}} \to K|\theta$. To verify that $E^{\overline{\mathcal{N}}}_{\overline{\alpha}} \notin K$, we need to notice that $\overline{\mathcal{N}}|\overline{\alpha} = K|\overline{\alpha}$, π' extends the ultrapower of $K|\overline{\alpha}$ by $E^{\overline{\mathcal{N}}}_{\overline{\alpha}}$ and then work exactly like in the previous section. Also in the same fashion, we may get $2^{\aleph_0}$ many such extenders. □

The above lemma implies that we do not have enough condensation to run the reshaping for K the same way we did in the previous chapter. This is summed up in the following:

Corollary 3.5. *Assume an α-strong cardinal κ exists in the sequence of K and $\kappa^{++} < \omega_2$. Then for every regular cardinal $\theta \geq \omega_2$, there is an elementary embedding $j : \mathcal{P} \to K|\theta$ such that $\mathcal{P} \ntrianglelefteq K$.*

Proof. The embedding $\pi' : \overline{\mathcal{N}} \to K|\theta$ derived from the above lemma is enough, since $\overline{\mathcal{N}}$ contains an extender which is not in K. □

Chapter 4

Preparing V for Coding

Near the end of chapter 2 and while justifying the use of reshaping, we argued that one could potentially code below some cardinal without collapsing it. Now we give a detailed account of the first step towards this direction. This step involves picking an appropriate cardinal δ and collapsing it to ω_1. Once this is done the right way, we should be ready to directly code from the interval (ω_1, ω_2) below ω_1. δ will be a cardinal where weak covering holds in order to enable coding, but at the same time it will be a cutpoint allowing only extenders from strong cardinals to overlap it. Its second property will come in handy in chapter 5, where we will need to reshape the interval (ω, ω_1). The procedure we are following here is a straightforward generalization of the one which appears in section 2 of [28], and it is appropriate for the case where the existence of strong cardinals is not ruled out. Nevertheless, we assume that the strong cardinals of K lie below ω_1 and as always, we are below a Woodin cardinal.

4.1 Stationary preserving forcings

There are several properties of a forcing notion which imply the preservation of ω_1 and in their individual ways express that H_{ω_1} does not change a lot in the generic extension. They begin from the strongest ones such as c.c.c. and ω-closedness and progress to weaker ones like properness and even weaker ones such as semiproperness and stationary preservation. In many ways those properties are connected to large cardinal consistency strength. The usual pattern is that one requires for some proposition to hold for all forcings with a given property. When this property is replaced by a weaker one -thus the class of considered forcings gets larger- more consistency strength is needed to prove the consistency of the proposition holding for those forcings. In our context, where we are interested in adding well orderings of the reals at the level of strong cardinals, the corresponding property seems to be stationary preservation. To get more motivation, one could look at the introduction of [27], especially the table on the 4th page, as well as jump to the 7^{th} chapter where we talk about absoluteness and well-orderings.

Definition 4.1. A forcing notion $\mathbb{P}$ is called *stationary preserving* if every stationary subset of ω_1 remains stationary in the generic extension.

4.2 Picking a starting point

Throughout this chapter and chapter 5, we make the following assumption regarding the strong cardinals of K:

$$K \models \text{“}\kappa \text{ is a strong cardinal} \Rightarrow \kappa < \omega_1\text{”}$$

Given the assumption that there is no inner model with a Woodin cardinal, we know by weak covering that K is close to V at several cardinals. To be more precise, we know that for the stationary set of singular cardinals below Ω, their successors are computed correctly. This is enough to run a first almost disjoint coding. We need to make sure though that some additional properties hold so that we can further reshape and code. We are looking for a δ which satisfies the following:

(a) $\delta^\omega = \delta$;

(b) δ is singular;

(c) δ is a cutpoint of K above ω_1.

(d) δ is such that $K|\delta$ agrees with K on which cardinals are strong.

We give the definition of (c) below.

Definition 4.2. δ is a *cutpoint* of an extender model $M = L[\vec{E}]$ *above some ordinal* $\eta \in OR \cap M$ iff for each extender E_β of M with $crit(E_\beta) \geq \eta$, $\beta \geq \delta \Rightarrow crit(E_\beta) \geq \delta$.

This means that the only extenders from $\vec{E}$ which are allowed to overlap δ, are the ones that have critical point below η (see figure 4.1). Since we assume that every strong cardinal of K is below ω_1, we have the possibility to find such cardinals.

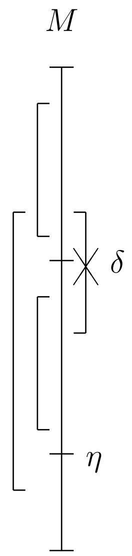

FIGURE 4.1: Cutpoint above η

Property (a) ensures that δ^+ will be preserved while collapsing δ to ω_1. Property (b) implies, by weak covering, that $(\delta^+)^K = \delta^+$, so $[\delta, \delta^+)$ is already reshaped in K. Property (c) renders δ to be a point in the core model where there the extenders do not overlap that much.If there were no strong cardinals, this could actually be a real cut point separating the extenders of K into two disjoint parts. However we will allow strong cardinals to exist below ω_1, thus we have to accept overlaps with their extenders. Later on, this property of δ will give us some form of condensation for some premice and this is why we need it. Finally, property (d) is used to transfer the fact that $K \vDash$"κ is a strong cardinal $\Rightarrow \kappa < \omega_1$" from K to $K|\delta$.

Lemma 4.3. *Let $K \vDash$"κ is a strong cardinal $\Rightarrow \kappa < \omega_1$" hold and assume there is no inner model with a Woodin cardinal. Then there exists a δ satisfying (a)-(d).*

Proof. First we need to verify that the set D of ordinals where (c) holds is a club below Ω. Given an $\alpha < \Omega$, and the fact that there are no strongs above ω_1, we may construct a sequence $(\alpha_i : i < \omega)$ such that $\alpha_0 = max(\alpha, \omega_1)$ and α_{i+1} is the supremum of the lengths of extenders in the K-sequence with critical points in $[\omega_1, \alpha_i)$. The supremum of this sequence satisfies (c), thus D is unbounded. D is also closed as any extender overlapping the supremum of a sequence from D will overlap some element below it.

There is also a club D', where property (d) holds. Let D' be the set of ordinals below Ω, such that $\alpha \in D'$ iff for every $\beta < \alpha$, if $K \vDash$ "β *is not strong*", then α is larger than the supremum of the heights of extenders in K with critical point β. This set is obviously closed and it is also unbounded. This is true since above every ordinal, one can consider the first limit of the closures under the aforementioned supremums, which lies in D'.

Now let $D'' = D \cap D'$, thus a club where (c) and (d) hold. Pick an increasing sequence $(\delta_i : i < \omega_1)$ of its elements, such that $\delta_i^\omega < \delta_{i+1}$ and $\omega_1 < \delta_0$. Its supremum, δ, will satisfy $\delta^\omega = \sum\limits_{i<\omega} \delta_i^\omega = \delta$ and $cf(\delta) = \omega_1 < \delta$. Thus δ satisfies (a)-(d), as needed. □

The first forcing we apply will code H_{δ^+} over J_{δ^+}. In order to proceed this way, we need that $2^\delta = \delta^+$, otherwise any $J_{\delta^+}[B]$, $B \subset \delta^+$ will have smaller cardinality than H_{δ^+}. To make sure this requirement is fulfilled, we apply first the preliminary forcing $Col(\delta^+, 2^\delta)$.

Lemma 4.4. *Assume $K \vDash$"κ is a strong cardinal $\Rightarrow \kappa < \omega_1$" and the non-existence of inner models with Woodin cardinals. Then there is a $B_0 \subset \delta^+$ such that $H_{\delta^+} = J_{\delta^+}[B_0]$.*

Proof. Every $x \in H_{\delta^+}$ can be coded into an $a_x \subset \delta$. This is possible, since there is a bijection $f : \overline{x} \to \delta$ from the transitive closure of x to δ that copies the predicate $\in$ on $\overline{x}$, to the binary relation $\in'$, i.e. $m \in' n \Leftrightarrow f^{-1}(m) \in f^{-1}(n)$. Then $\in'$ can be coded to $a_x \subset \delta$ by Gödel's pairing function. This means that a_x contains all the information needed to reconstruct x, in the sense that for some $\alpha < \delta^+$, $x \in J_\alpha[a_x] \subset J_{\delta^+}[a_x]$. We glue now all those sets together, with the function h. Let $h : \delta^+ \to 2$, $h(\delta\alpha + \gamma) = g(\alpha)(\gamma)$, where $\alpha < \delta^+$, $\gamma < \delta$ and g is a bijection between δ^+ and ${}^{\delta}2$. Then $\breve{h} = B_0 \subset \delta^+$ has the desired property. □

4.3 Collapsing δ to ω_1

After picking the right δ we have an environment just like the one in the a.d. coding section of chapter 2. The only procedure left before beginning to code, is to move this environment at the level we are interested in. This means that we have to collapse δ to ω_1 and verify that we still have the required conditions. K does not change after applying $Col(\omega_1, \delta)$, therefore δ remains a cutpoint above ω_1. Due to the fact that $\delta^\omega = \delta$, $(\delta^+)^K = \omega_2$, so the only property left to verify is that H_{ω_2} is still of the form $J_{\omega_2}[B]$ for some $B \subset \omega_2$.

Lemma 4.5. *Assume* $K \vDash$"*κ is a strong cardinal* $\Rightarrow \kappa < \omega_1$" *and the non-existence of inner models with Woodin cardinals. Then* $H_{\omega_2} = J_{\omega_2}[B_1]$[1], *where* $B_1 = B_0 \oplus G_0$.

Proof. Suppose $x \in H_{\omega_2}$. We can convert x to an $a_x \subset \omega_2$, as in lemma 4.4, and then choose a nice name $\dot{a}_x$ for it. Since P_0 satisfies the δ^+-cc, $\dot{a}_x \in H^V_{\delta^+} = J_{\delta^+}[B] = J_{\omega_2^{V_1}}[B]$ which in turn implies that

$$a_x = \dot{a}_x^G \in J_{\omega_2}[B_0 \oplus G_0] = J_{\omega_2}[B_1].$$

The inverse inclusion is immediate by the definition of H_{ω_2}. □

Note here, that in the above proof we assumed that $\omega_2 = \delta^+$. This still holds true because the forcing has size $\delta^\omega = \delta$, thus it satisfies the δ^+-cc. The same forcing is ω-closed so it preserves stationary subsets of ω_1. The current situation is depicted in figure 4.2.

[1] We write H_{ω_2} for the corresponding model defined in V_1, i.e. for $H^{V_1}_{\omega_2^{V_1}}$, in order to avoid an excessive use of superscripts. This will be done without warning in the next proofs.

In the next chapter we will begin from this nice environment in the interval $[\omega, \omega_2]$ and proceed to code below ω. The reader has to always keep in mind though, that apart from the non-existence of inner models with Woodin cardinals, we made heavy use of the assumption $K \models$"κ is a strong cardinal $\Rightarrow \kappa < \omega_1$". Without it, we could not rule out the existence of strong cardinals in the interval $[\omega_1, \delta]$, which create further complications, as we will see later on.

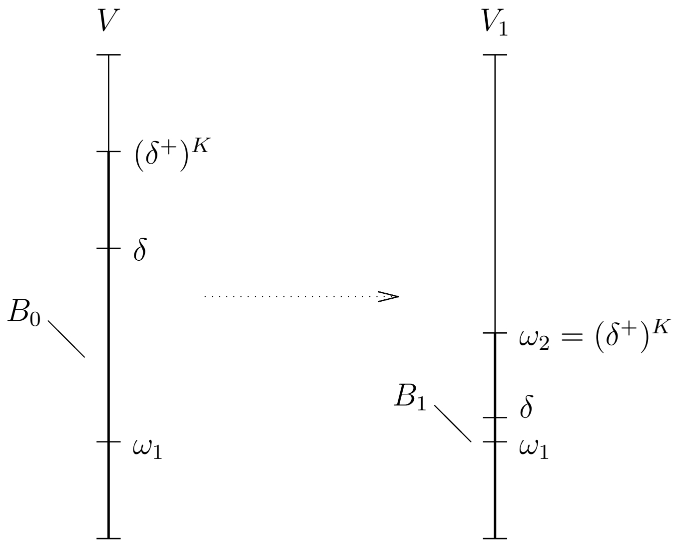

FIGURE 4.2: $\mathbb{P}_0$

Chapter 5

When the Strong Cardinals are below ω_1^V

We prove in this chapter the first of our two coding results. Our task is to code a given set $A \subset \omega_1$ into a real x over some reasonably definable structures, using a stationary preserving forcing. The result of this procedure is producing a generic extension where A is $\Delta^1_{n+3}(x)$ or it belongs to $J_\alpha(x)$. The actual complexity of A in the extension is determined in a straightforward way by the order type of the strong cardinals of K which exist below ω_1. In our setting, we assume that we are below a Woodin cardinal and furthermore that there are no strong cardinals in K above ω_1^V.

5.1 The theorem, a sketch of the proof

We assume by stating in full detail the theorem we will prove and giving a short diagram of the proof. The proof consists of four forcings which produce the final coding. In particular they are a collapse, an a.d. coding, a reshaping and another a.d. coding. The most elaborate part which will occupy most of the chapter, is the reshaping forcing. Another interesting part is also the decoding procedure in the final coding, which also justifies our choice of coding structures. The first forcing produces a nice environment and has already been described in chapter 4 and the others are simple variations of what appeared in chapter 2. We will prove the following:

Theorem 5.1. *Suppose there is no inner model with a Woodin cardinal and* $K \vDash$"κ *is a strong cardinal* $\Rightarrow \kappa < \omega_1$". *Let also* $A \subset \omega_1$ *and* $\tilde{A}$ *be the set of reals coding* A. *Then there is a forcing* $\mathbb{P}$ *such that:*

(a) if there are at most n *strong cardinals in* $J^K_{\omega_1}$*, then* $\tilde{A}$ *is* $\mathbf{\Delta}^1_{n+3}$ *in the generic extension;*

(b) if the order type of the set of strong cardinals in $J^K_{\omega_1}$ *is strictly below* $\omega\theta \leq \omega_1$*, then* $\tilde{A} \in J_{1+\theta}(\mathbb{R})$ *in the generic extension;*

(c) $\mathbb{P}$ *preserves stationary subsets of* ω_1.

We are going to split the proof of the above theorem into four steps, where in each step we force with the notion $\mathbb{P}_i$ extending the universe to $V_{i+1} = V_i[G_i]$ ($V_0 = V$) and coding the new information into the set B_{i+1}, where $0 \leq i \leq 3$. The final part of the coding will be performed using the mice $(\mathcal{M}_\alpha : \alpha < \omega_1)$ instead of K. We will give a detailed description of them later on and also prove that they are produced in a canonical fashion. The forcing notions used in those steps are briefly described below (see also figures 1-4):

(a) $\mathbb{P}_0 = Col(\delta^+, 2^\delta) \star Col(\omega_1, \delta)$, where δ is the appropriate cardinal mentioned in the previous chapter. We have that $H_{\omega_2} = L_{\omega_2}[B_1]$, $B_1 \subset \omega_2$, in V_1.

(b) $\mathbb{P}_1$ codes B_1 into $B_2 \subset \omega_1$ over $K|\omega_2$. In V_2, $H_{\omega_2} = K|\omega_2[B_2]$.

(c) $\mathbb{P}_2$ reshapes ω_1 over the elements of $(\mathcal{M}_\alpha : \alpha < \omega_1)$. B_3 is produced by merging B_2 with the reshaping function and $K|\delta$.

(d) $\mathbb{P}_3$ codes B_3 together with A into a real x over the sequence $(\mathcal{M}_\alpha : \alpha < \omega_1)$.

Throughout our exposition and without further reference, we will assume that there exists no inner model containing a Woodin cardinal and additionally that $K \vDash$ "κ is an strong cardinal $\Rightarrow \kappa < \omega_1$".

5.2 Coding into a subset of ω_1

We continue exactly from the point we reached at the end of chapter 4. Namely, δ is a cutpoint of K above ω_1 and it has been collapsed to ω_1. Furthermore, $H_{\omega_2} = J_{\omega_2}[B_1]$. Now in $K[G_0]$, $2^{\omega_1^V} = 2^\delta = \delta^+ = (\omega_1^+)^{K[G_0]}$, since $\overline{\overline{\delta}} = \omega_1^V$ and $2^\delta = (\delta^+)^K$. Therefore, we can use lemma 2.2 to produce a sequence $(y_i : i < \omega_2) \in K[G_0]$ of a.d. sets. We use this sequence to code down to ω_1 (see figure 5.1).

Let $\mathbb{P}_1$ be the a.d. forcing which codes B_1 to a subset of ω_1 using the a.d. sets from $(y_i : i < \omega_2)$. Let $\bigcup pr_1(G_1) = P$, $\bigcup pr_2(G_1) = P^*$ and set $B_2 = P \oplus G_0$. We check below that $\mathbb{P}_1$ satisfies the usual properties of almost disjoint coding along with (b). We include a shorter version of the proofs appearing in chapter 2 for convenience and since the same argument appears a couple of pages later.

Lemma 5.2. *The following hold true for $\mathbb{P}_1$:*

(a) $B_1 \in K[B_2]$.

(b) $H_{\omega_2} = K|\omega_2[B_2]$.

(c) $\mathbb{P}_1$ *has the* ω_2*-c.c.*

(d) $\mathbb{P}_1$ *is* ω*-closed.*

Proof.

(a) We have that $P^* = B_1$ which implies that $p(i) = 1 \Leftrightarrow \breve{p} \cap y_i = \emptyset$, by a density argument. Therefore, $B_1 \in K[B_2]$.

(b) Since $(y_i : i < \beta) \in K|\omega_2[G_0]$ for every $\beta < \omega_2$, we may refine the above argument to $B_1 \subset K|\omega_2[B_2]$, which implies that $H_{\omega_2}^{V_1} = L_{\omega_2}[B_1] \subset K|\omega_2[B_2]$. We can code each $x \in H_{\omega_2}$ into $a_x \subset \omega_1$ and find a nice name $\dot{a}_x \in H_{\omega_2}^{V_1}$ for it, because of (c). But then $\dot{a}_x \in K|\omega_2[B_2]$ and since $P \in K|\omega_2[B_2]$, $a_x \in K|\omega_2[B_2]$. Hence $H_{\omega_2} \subset K|\omega_2[B_2]$. To verify the inverse inclusion, one has to notice that for every $\alpha < \omega_2$, $K|\beta[B_2]$ is transitive and has cardinality ω_1, thus belongs to H_{ω_2}.

(c) We have to see here that two conditions with the same first coordinates are compatible, thus the size of a maximal antichain can be at most equal to the number of possible first coordinates. But they are $2^{<\omega_1} = \omega_1$ many so ω_2-c.c. holds.

(d) Suppose $((p_n, p_n^*) : n < \omega)$ is a decreasing sequence of conditions in $\mathbb{P}_1$. Then $(\cup_{n\in\omega} p_n, \cup_{n\in\omega} p_n^*)$ is still a condition which extends each p_n, $n < \omega$.

□

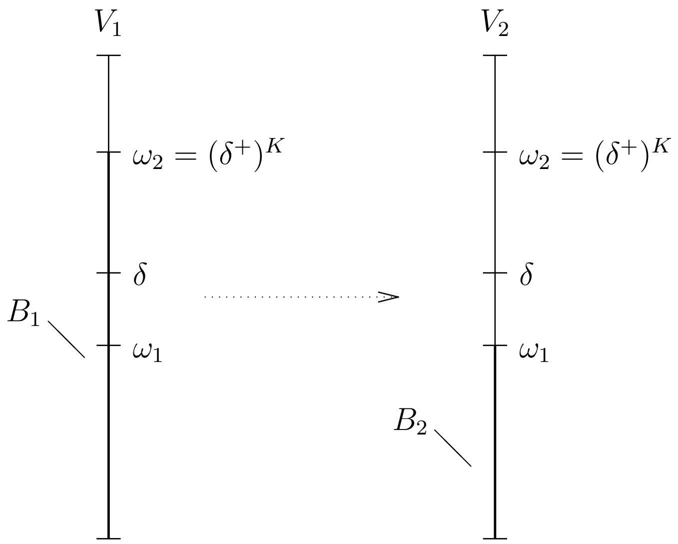

FIGURE 5.1: $\mathbb{P}_1$

Before proceeding further on, we add the information contained in $K|\delta$ to B_2. Let $f : K|\delta \to \omega_1$ be a bijection in $K|\omega_2[B_2]$ and B_2' be $B_2 \oplus a_f$.

5.3 Coding down to a real

What we achieved up to now, is to code H_{ω_2} with a subset of ω_1 over K and at the same time make sure that the extenders of K are as separated as possible in the interval (ω_1, ω_2) at δ. The latter is what enables us to get some kind of condensation which is used in the proof of distributivity and stationary preservation of the reshaping of B_2'. Since K is not a good candidate to get this condensation, according to chapter 3, we change our coding structures.

The idea is to pick directly the structures which might occur in the transitive collapses which appear in the proof of distributivity. More specifically, we code over the sequence $(\mathcal{M}_\alpha : \alpha < \omega_1)$, where $\mathcal{M}_\alpha$ is a large enough mouse produced by a transitive collapse from the distributivity proof, which has critical point α. The latter statement is not completely accurate but contains the main idea and will make more sense once the reader goes through the details of this section. The major step here is to prove that the candidates for $\mathcal{M}_\alpha$ are lined up, which is actually the condensation property we were talking about.

The whole procedure is organized into three sections. On the first one we motivate and derive a definition of the coding structures by looking at specific embeddings. The second one contains the definition of our reshaping as well as a proof of its distributivity and stationary preservation. Finally, the third one is the place where we perform the final coding and also describe the procedure of picking the a.d. sets and simultaneously decoding B_2'.

5.3.1 Choosing the coding structures

We begin by making some observations which allow us to view K as a stack of certain mice. The following lemma will not be used directly, but the idea of constructing K this way will be imitated in defining our sequence of mice. Furthermore, the proof that we have indeed defined a stack will also provide inspiration for the proof that the mice which are candidates for a given place in the sequence will line up.

Before moving further, we state a fact which enables us to use phalanges with first root K, instead of the appropriate, for each case, P witnessing the A_0-soundness of an initial segment of K. The reason for doing that is mostly aesthetic and it happens quite often in the bibliography that $(K, \mathcal{N}, \alpha)$ is to be interpreted as $(P, \mathcal{N}, \alpha)$, where P is the appropriate witness.

Fact 5.3. *Suppose $\mathcal{M}$ is a mouse and P is a witness of the A_0-strongness of $K|\delta = \mathcal{M}|\delta$. Then $(K, \mathcal{M}, \delta)$ is iterable iff $(P, \mathcal{M}, \delta)$ is iterable.*

The lemma below contains the properties a mouse $\mathcal{N}$, containing $K|\delta$, has to satisfy in order to be an initial segment of $K|\omega_2$. Keep in mind that this definition is possible since we are in a very special case. In our context, ω_2 is δ^+ and δ was chosen to be singular and a cutpoint above ω_1. Furthermore we assumed that the strong cardinals of K lie below ω_1. The general idea of stacking mice already appears in some way in the definition of K below a measurable and is further explored in [12].

Lemma 5.4. *Suppose we are in the setting of the generic extension described in section 5.2. Then $\mathcal{N} \lhd K|\omega_2$ iff there is an $\mathcal{N}' \trianglerighteq \mathcal{N}$ such that:*

(a) $K|\delta \trianglelefteq \mathcal{N}'$*;*

(b) $\rho_\omega(\mathcal{N}') \leq \delta$*;*

(c) $\mathcal{N}'$ *is ω-sound;*

(d) $\mathcal{N}' \models$ *"η strong $\Rightarrow \eta < \omega_1^V$";*

(e) $(K, \mathcal{N}', \omega_1)$ *is iterable.*

Proof.

($\Rightarrow$)

Let $\mathcal{N} \lhd K|\omega_2$. Every $\mathcal{N} \cup K|\delta \trianglelefteq \mathcal{N}' \lhd K|\omega_2$ satisfies immediately properties (a) and (c). For (e) just notice that any iteration of $(K, \mathcal{N}', \omega_1)$ can be embedded to an iteration of K, since both $K, \mathcal{N}'$ are initial segments of K.

Now consider a cardinal $\mu \geq \omega_2$ which is such that $K|\mu$ agrees with K on which cardinals are strong[1]. Then the transitive collapse $\mathcal{N}'$ of the Σ_1-hull of $ht(\mathcal{N})$ in $K|\mu$, is an initial segment of K, which projects to δ and also satisfies (d). Therefore $\mathcal{N}'$ satisfies (a)-(e), and finishes the proof of this direction.

[1]This μ can be produce exactly the way we defined δ so that $K|\delta$ agrees with K on strong cardinals.

($\Leftarrow$)

Let $\mathcal{N}, \mathcal{N}'$ satisfy (a)-(e). By the fact 5.3 we may work with witnesses of strong soundness instead of K. We compare P and $(P, \mathcal{N}', \omega_1)$, where P witnesses the A_0-soundness of $K|lh(\mathcal{N}')$. Let $\mathcal{M}_\theta^{\mathfrak{T}}$ and $\mathcal{M}_\theta^{\mathfrak{U}}$ be the last models produced by this comparison. First we fix the root of $(P, \mathcal{N}', \omega_1)$.

Claim 5.5. *The root of* $(P, \mathcal{N}', \omega_1)$ *is* $\mathcal{N}'$.

Proof. Suppose on the contrary that the root is P. We can use the fact that P computes successors correctly on a stationary set of regular cardinals and show that $(P, \mathcal{N}', \omega_1)$ behaves like a universal weasel in the iteration[2]. As P is also a universal weasel, $\mathcal{M}_\theta^{\mathfrak{T}} = \mathcal{M}_\theta^{\mathfrak{U}}$ and none of the sides drops. Suppose α, α' are the critical points of the first extenders $E_\beta^{\mathfrak{T}}, E_\gamma^{\mathfrak{U}}$ appearing on the corresponding cofinal branches -thus also the critical points of the embeddings $i_{0,\theta}^{\mathfrak{T}} : P \to \mathcal{M}_\theta^{\mathfrak{T}}$ and $i_{0,\theta}^{\mathfrak{U}} : P \to \mathcal{M}_\theta^{\mathfrak{U}}$. Then both α, α' are the least ordinal in $\mathcal{M}_\theta^{\mathfrak{T}} = \mathcal{M}_\theta^{\mathfrak{U}}$ not having the definability property, so $\alpha = \alpha'$.

Now $\Phi = \{\xi \in OR : i_{0,\theta}^{\mathfrak{T}}(\xi) = i_{0,\theta}^{\mathfrak{U}}(\xi) = \xi\}$ is thick in P, hence by the hull property at α, for each $X \subset \alpha$ there is a Skolem term τ and a parameter $\bar{c} \in \Phi^{<\omega}$ so that $X = \tau^P[\bar{c}]$. Suppose without loss of generality that $\alpha^* = min\{i_{0,\theta}^{\mathfrak{T}}(\alpha), i_{0,\theta}^{\mathfrak{U}}(\alpha)\} = i_{0,\theta}^{\mathfrak{T}}(\alpha)$. We have then that

$$i_{0,\theta}^{\mathfrak{T}}(X) \cap \alpha^* = i_{0,\theta}^{\mathfrak{T}}(\tau^P[\bar{c}]) \cap \alpha^* = \tau^{\mathcal{M}_\theta^{\mathfrak{T}}}[\bar{c}] \cap \alpha^* =$$

$$= i_{0,\theta}^{\mathfrak{U}}(\tau^P[\bar{c}]) \cap \alpha^* = i_{0,\theta}^{\mathfrak{U}}(X) \cap \alpha^*$$

The latter implies that $E_\beta^{\mathfrak{T}} = (E_\gamma^{\mathfrak{U}} \restriction \nu)^*$ for some $\nu < \alpha^*$. But this means that the two extenders are compatible, which is impossible since they were used in a comparison. □

P is universal, so it wins the comparison and the $(P, \mathcal{N}', \omega_1)$ side has no drops. But the first extender used on the cofinal branch of that side has critical point above ω_1, according to the previous claim. Moreover, its index is above δ since $\mathcal{N}'$ and P agree up to that level and given that δ is a cutpoint above ω_1, its critical point is actually above δ. But $\mathcal{N}'$ projects below δ, hence this extender can not be used without creating a

[2]This runs essentially the same way as the proof of the fact that every weasel computing successors correctly on a stationary set of regular cardinals, is universal.

non-sound model in the iteration. This means that $\mathcal{M}_\theta^{\mathfrak{U}} = \mathcal{N}'$, i.e. the $(P, \mathcal{N}', \omega_1)$ side does not move.

Now we look at the P side. The length of the first extender used in the cofinal branch of the iteration is a cardinal in $\mathcal{M}_{0,\theta}^{\mathfrak{T}}$ so it cannot be in $(\delta, ht(\mathcal{N}')]$, as $\mathcal{N}'$ is sound and projects below δ. It is also above δ, because P and $\mathcal{N}'$ agree up to that point, hence it is also above $ht(\mathcal{N}')$, which is a contradiction. Thus the P side also doesn't move implying that $\mathcal{N}' \lhd P \Rightarrow \mathcal{N}' \lhd K$. □

The rough idea behind the proof of reshaping ω_1, is to collapse a countable hull of $K|\omega_2[B_2']$ along with some sequence $(D_i : i < \omega)$ of dense sets and a condition p_0 to some model $\overline{K}$. Then ideally we would be able to construct a sequence $(p_i : i < \omega)$ in $\overline{K}$ which hits the dense sets $(\overline{D}_i : i < \omega)$. Then it is possible to see definably over $\overline{K}$ that this sequence of conditions is countable.

At this point we would like to be able to pull this fact back to K but it is not generally true that $\overline{K}$ is contained in there. Therefore, we choose to reshape directly over the structure $\overline{K}$ instead of reshaping over K. In order to get nice definability properties when we decode, we use a version of lemma 5.4 that describes the structures $\overline{K}$ in a similar way. This is actually the place where the assumption that the strong cardinals are below ω_1 is used.

Lemma 5.6. *Suppose* $j : \overline{K}[B_2' \cap \alpha] \to K|\omega_2[B_2']$ *is an elementary embedding with critical point* α*, thus* $j(\alpha) = \omega_1$*. Let* $\overline{\delta} = j^{-1}(\delta)$ *and* $\mathcal{N}_\alpha = \overline{K}|\overline{\delta}$ *be the structure coded by* $B_2' \cap \alpha$*. Then* $\overline{K}$ *has the following properties:*

(a) $\overline{\delta}$ *is the largest cardinal of* $\overline{K}$*;*

(b) $\overline{K} \vDash$"η *is strong* $\Rightarrow \eta < \alpha$"*;*

(c) $(K, \overline{K}, \alpha)$ *is iterable.*

Proof. Everything is immediate from elementarity except from (c). Since $j \restriction \overline{K} : \overline{K} \to K|\omega_2$ is elementary, we can lift any putative iteration of $(K, \overline{K}, \alpha)$ to an iteration of $(K, K|\omega_2, \alpha)$, therefore $(K, \overline{K}, \alpha)$ is iterable. □

Definition 5.7. We use C^* to denote the the set of all $\alpha < \omega_1$ which are critical points of embeddings of the form described above and $\mathcal{N}_\alpha$ to denote the structure coded by $B_2' \cap \alpha$ with the technique of lemma 4.4. This set contains a club subset of ω_1.

Note that $\mathcal{N}_\alpha$ is the Mostowski collapse of the part of $K|\delta$ coded by $B_2' \cap \alpha$, hence it is the same for any $\overline{K}$ and j with critical point α. We define now the property $(*)_\alpha$ which restricts us to mice over $\mathcal{N}_\alpha$ that line up and also contain any $\overline{K}$ derived from a j as in lemma 5.6.

Definition 5.8. $\mathcal{N}$ has the $(*)_\alpha$ property iff:

(a) $\mathcal{N}_\alpha \trianglelefteq \mathcal{N}$;

(b) $\mathcal{N}$ is ω-sound;

(c) $\overline{\delta}$ is the largest cardinal of $\mathcal{N}$, where $\overline{\delta} = \mathcal{N}_\alpha \cap OR$;

(d) $\mathcal{N} \models$ "η *strong* $\Rightarrow \eta < \alpha$";

(e) $(K, \mathcal{N}, \alpha)$ is iterable.

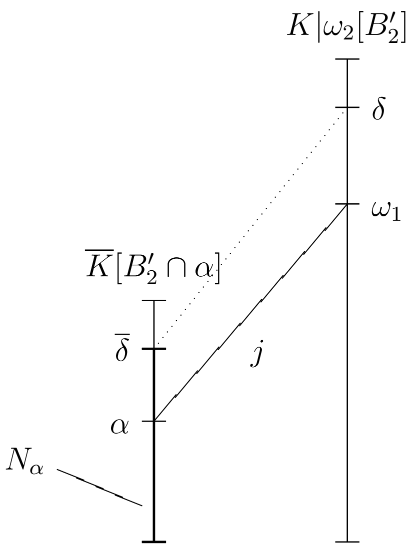

FIGURE 5.2: The structures $\mathcal{N}_\alpha$

Lemma 5.9. *Let $\mathcal{N}, \mathcal{N}'$ satisfy $(*)_\alpha$. Then $\mathcal{N} \trianglelefteq \mathcal{N}'$ or $\mathcal{N}' \trianglelefteq \mathcal{N}$.*

Proof. Suppose P is a universal weasel witnessing the A_0-strongness of $K|\alpha$. By α-goodness and the above fact, we can compare $(P, \mathcal{N}, \alpha)$ with $(P, \mathcal{N}', \alpha)$. Let $\mathcal{M}_\theta^{\mathfrak{T}}$ and $\mathcal{M}_\theta^{\mathfrak{U}}$ be the last models produced by this comparison.

Claim 5.10. *It cannot be the case that the root of both* $(P, \mathcal{N}, \alpha)$ *and* $(P, \mathcal{N}', \alpha)$ *is* P.

Proof. Suppose on the contrary that both phalanges root on P. We can use the fact that P computes successors correctly on a stationary set of regular cardinals and show that $(P, \mathcal{N}', \omega_1)$ behaves like a universal weasel in the iteration[3]. As the same holds for $(P, \mathcal{N}, \alpha)$, $\mathcal{M}_\theta^{\mathfrak{T}} = \mathcal{M}_\theta^{\mathfrak{U}}$ and none of the sides drops. Suppose $\alpha, \alpha' < \omega_1$ are the critical points of the first extenders $E_\beta^{\mathfrak{T}}, E_\gamma^{\mathfrak{U}}$ appearing on the corresponding cofinal branches -thus also the critical points of the embeddings $i_{0,\theta}^{\mathfrak{T}} : P \to \mathcal{M}_\theta^{\mathfrak{T}}$ and $i_{0,\theta}^{\mathfrak{U}} : P \to \mathcal{M}_\theta^{\mathfrak{U}}$. Then both α, α' are the least ordinal in $\mathcal{M}_\theta^{\mathfrak{T}} = \mathcal{M}_\theta^{\mathfrak{U}}$ not having the definability property, so $\alpha = \alpha'$.

Now $\Phi = \{\xi \in OR : i_{0,\theta}^{\mathfrak{T}}(\xi) = i_{0,\theta}^{\mathfrak{U}}(\xi) = \xi\}$ is thick in P, hence by the hull property at α, for each $X \subset \alpha$ there is a Skolem term τ and a parameter $\overline{c} \in \Phi^{<\omega}$ so that $X = \tau^P[\overline{c}]$. Suppose without loss of generality that $\alpha^* = min\{i_{0,\theta}^{\mathfrak{T}}(\alpha), i_{0,\theta}^{\mathfrak{U}}(\alpha)\} = i_{0,\theta}^{\mathfrak{T}}(\alpha)$. We have then that

$$i_{0,\theta}^{\mathfrak{T}}(X) \cap \alpha^* = i_{0,\theta}^{\mathfrak{T}}(\tau^P[\overline{c}]) \cap \alpha^* = \tau^{\mathcal{M}_\theta^{\mathfrak{T}}}[\overline{c}] \cap \alpha^* =$$

$$= i_{0,\theta}^{\mathfrak{U}}(\tau^P[\overline{c}]) \cap \alpha^* = i_{0,\theta}^{\mathfrak{U}}(X) \cap \alpha^*$$

The latter implies that $E_\beta^{\mathfrak{T}} = (E_\gamma^{\mathfrak{U}} \upharpoonright \nu)^*$ for some $\nu < \alpha^*$. But this means that the two extenders are compatible, which is impossible since they were used in a comparison. □

Assume now without loss of generality that the root of $(P, \mathcal{N}, \alpha)$ is $\mathcal{N}$. We will use the following claim to deal with the possibility that the other root is P and also to complete the proof of the lemma.

Claim 5.11. *Suppose that one of the phalanges, assume* $(P, \mathcal{N}, \alpha)$, *is rooted on* $\mathcal{N}$. *Then this side of the iteration does not move.*

[3]This runs essentially the same way as the proof of the fact that every weasel computing successors correctly on a stationary set of regular cardinals, is universal.

Proof. The first extender E used on the $(P, \mathcal{N}, \alpha)$ side must have height at least δ, given that the structures agree below this ordinal. By (c) of the definition of $\mathcal{N}$, the critical point of this extender has to be below α or above $\overline{\delta}$. The first case is impossible because $\mathcal{N}$ is the root of the phallanx, thus no extenders with critical point below α are used in the comparison. Given that $\overline{\delta}$ is the largest cardinal of $\mathcal{N}$, the critical point can also not be above $\overline{\delta}$, therefore the $(P, \mathcal{N}, \alpha)$-side does not move. □

Now assuming that the second phalanx has root P, we get by the above claim that $\mathcal{N}$ is an initial segment of an iterate of P. Since $\mathcal{N}, \mathcal{N}'$ agree up to $\overline{\delta}$, the index of the first extender used should be strictly above $\overline{\delta}$. But then this means that some cardinal would exist in the interval $(\overline{\delta}, ht(\mathcal{N}))$ in the iterate of P thus also in $\mathcal{N}$, contradicting the maximality of $\overline{\delta}$.

Therefore the only possibility is that the roots of the phalanges are $\mathcal{N}, \mathcal{N}'$ which proves the lemma by claim 5.11. □

5.3.2 Reshaping

We proceed by reshaping below ω_1. The first observation here is that we only need to reshape on a club.

Lemma 5.12. *Suppose C is a club below ω_1 which is reshaped over some inner model M, i.e. for every $\alpha \in C$, $M[C \cap \alpha] \models \overline{\overline{\alpha}} = \omega$. Then there is a $g : \omega_1 \to 2$ which reshapes ω_1 over M.*

Proof. First assume that C has been reduced to a club of limit ordinals. Let f be such that for every $\alpha \in C$, $f \restriction [\alpha, \alpha + \omega)$ codes an $h : \omega \to \alpha'$ collapsing α', where $\alpha' = min(C \setminus (\alpha + \omega))$. If $f' : \omega_1 \to 2$ is the function reshaping C over M set $g = f \oplus f'$. Suppose now that $\beta \notin C$. Then $C \cap \beta$ has a maximum β_0 and the successor β_0' of β_0 in C is bigger than β. Hence if $\beta \in (\beta_0, \beta_0 + \omega)$, then it is reshaped due to the fact that β_0 is already reshaped, else $f \restriction [\beta_0, \beta_0 + \omega) \subset g \restriction \beta$ so again $M[g \restriction \beta] \models \overline{\overline{\beta}} = \omega$. □

The set we are going to reshape is C^*, which contains a club (see definition 5.7). In the definition below we define simultaneously what a condition is as well as a sequence of structures which we reshape and a canonical sequence of a.d. sets. Notice that though the structures $\mathcal{N}_\xi$ are chosen from the lined up collection of mice satisfying $(*)_\xi$, they might have different heights for different conditions. The same holds for the a.d. sets.

Definition 5.13. $p \in \mathbb{P}_2$ iff:

(a) $p : \alpha \to 2,\ \alpha < \omega_1$;

(b) there exist sequences $(\mathcal{N}^\xi : \xi \in (\alpha+1) \cap C^*)$ and $(z_\xi : \xi \in \alpha \cap C^*)$ such that $\mathcal{N}^\xi$ is the least in height model $\mathcal{N}$ satisfying:

1. $\mathcal{N}[B_2' \cap \xi, p \restriction \xi] \vDash \overline{\overline{\xi}} = \omega$ where $\mathcal{N}$.
2. $\mathcal{N}$ has the $(*)_\xi$ property;
3. $\{z_i : i < \xi\} \subset \mathcal{N}[B_2' \cap \xi, p \restriction \xi]$, where $z_i \subset \omega$, $i < \xi$, is the $\mathcal{N}^i[B_2' \cap i, p \restriction i]$-least set almost disjoint from all elements of $\{z_j : j < i\}$.[4]

$p \leq q$ iff $p \supseteq q$.

We set $B_3 = B_2' \oplus G_3$. The next step is to prove that $\mathbb{P}_2$ is ω-distributive, thus it preserves ω_1. Using this fact we will show then that it is actually stationary preserving.

Lemma 5.14. $\mathbb{P}_2$ *is* ω*-distributive.*

Proof. Let $p \in \mathbb{P}_2$ and $(D_i : i < \omega)$ be a countable sequence of dense sets. We must find some $q \leq p$ which hits every D_i, $i < \omega$. This will happen at some countable elementary substructure containing enough information to construct q. Pick a countable $X \prec (K|\omega_2[B_2'], (D_i : i < \omega), \mathbb{P}, p, \vec{E} \restriction \omega_2)$, where $\vec{E}$ is the extender sequence of K. Let $\mathcal{M} = (\overline{K}[B_2' \cap \alpha], (\overline{D}_i : i < \omega), \overline{\mathbb{P}}, p, \overline{\vec{E}} \restriction \omega_2)$ be its transitive collapse and j be the collapsing map. Clearly $\alpha \in C^*$, $\overline{K}$ satisfies $(*)_\alpha$ by lemma 5.6.

[4]This definition makes sense because the mice satisfying $(*)_i$ are lined up, therefore they define the same z_i. Namely, if $k \leq i, j$, then $\mathcal{N}^i$ and $\mathcal{N}^j$ define the same minimal sequence of a.d. sets $\{z_l : l < k\}$.

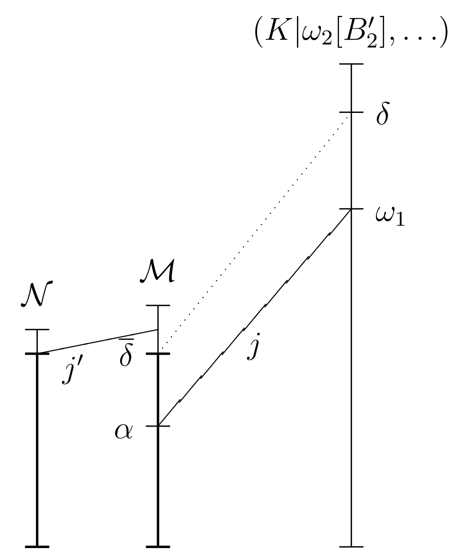

FIGURE 5.3: The collapsed structures

Now, inside $\mathcal{M}$, we consider a Σ_n elementary subset X' for some n large enough to preserve the needed properties of the predicates we considered. Let $\mathcal{N}$ be its transitive collapse and j' the corresponding map (see figure 5.3). The reason for considering the second hull, is that now $\mathcal{M} \vDash \overline{\overline{\mathcal{N}}} = \alpha$. Let $(E_k : k < \alpha) \in \mathcal{M}$ be an enumeration of the club subsets of α in $\mathcal{N}$. Then $\mathcal{M}$ contains $E = \triangle_{i<\alpha} E_i$ which is a fast club for α in $\mathcal{N}$. The idea now is to define a decreasing sequence of conditions $(p_i : i < \omega)$ which extend $p = p_0$ and hit the dense sets $(\overline{D}_i' : i < \omega)$ in such a way that the lengths of the conditions do not belong to any of the elements of some sequence $(E_{k_i} : i < \omega)$ of clubs. This will enable us to code information in E which guarantees that $q = \cup_{i\in\omega} p_i$ is actually a condition of $\mathbb{P}_2$.

If $r \in \overline{\mathbb{P}}'$, $\gamma < \alpha$, $|r| \leq \xi$ and $\xi + \omega < \gamma$ then we define:

(a) $r^{\xi,\gamma,1} = r^\frown x^\frown 0^{otp(\xi\setminus|r|)\frown} 1^\frown 0^{\gamma\setminus(\xi+\omega)}$, where x is a real collapsing the length of $r^{\xi,\gamma,1}$ and coding the almost disjoint sets $(z_i : i < |r|)$ associated with r;

(b) $r^{\xi,\gamma,i,2}$ is the $<_{\mathcal{N}}$-least element of $\overline{D}_i'$ extending $r^{\xi,\gamma,1}$;

(c) $f_{r,i} : [|r|, \alpha) \to [|r|, \alpha)$ is such that $f_{r,i}(\gamma) = sup_{\xi<\gamma}(|r^{\xi,\gamma,i,2}| + 1)$;

(d) $E_{r,i} = \{f^{\beta}_{r,i}(|r|+\omega) : \beta < \alpha\} \cap E$, where f^{β} is the continuous iteration of f, β many times[5].

The notation above seems to be a little complicated, but the situation it describes is fairly simple. We are given r and we wish to extend it by marking one ordinal. This is simple, but if we wish to further extend it and hit $\overline{D}'$, we have no control over the length of the new condition. Hence for each $\gamma < \alpha$ we consider all the possible ways of extending r by adding x, a string of zeros and a 1 on the ξ^{th} position. Then $f_{r,i}$ maps each γ to the supremum of the lengths of such extensions. Finally we form $E_{r,i}$ by applying $f_{r,i}$ to $|p| + \omega$ iteratively and keeping the images. We also intersect this set with E in order to be in this fast club which is common for all the clubs that might come up this way. Note that $E_{r,i}$ contains a final segment of E. The situation now, is such that whenever we extend some condition r to $r^{\xi,\xi+\omega,i,2}$, for some $\xi \in E_{r,i}$, then no element of E apart from ξ is changed.

Being able to add information on the conditions and recover it from E, we begin defining the p_i's. Fix an $E^* \subset E$ cofinal and of order-type ω.[6] We set then $p_{i+1} = p_i^{\xi_i,\xi_i+\omega,i,2}$, where ξ_i is the least element of E^* in the interval $[|p_i|, \alpha)$. Then the condition $q = \cup_{i<\omega} p_i$ has value 1 exactly at those elements of E we marked while extending p. Since $E \in \mathcal{M}$, $\overline{K}[B_2', q]$ contains a cofinal subsequence of E^* which collapses α. This means that q satisfies property (ii)(a) of being a condition of $\mathbb{P}$ and by elementarily and the fact it is the limit of the conditions p_i, it also satisfies (ii)(b). So we have that q is a condition which hits all the dense sets, therefore we are done. □

Lemma 5.15. $\mathbb{P}_2$ *preserves the stationary subsets of* ω_1.

Proof. Suppose $S \subset \omega_1$ is a stationary set and $p \Vdash$"$\dot{C}$ is a club". Working as in lemma 4.5, we may assume $\dot{C} \in H_{\omega_2}$. We fix an $\alpha \in S \cap C^*$ which is also the critical point of some elementary embedding $j : \mathcal{A} \to \mathcal{B}$, where
$\mathcal{A} = (\overline{K}[B_2' \cap \alpha], \overline{\dot{C}}, \overline{P}, p_0, \overline{\vec{E} \restriction \omega_2})$ and
$\mathcal{B} = (K|\omega_2[B_2'], \dot{C}, \mathbb{P}, p_0, \vec{E} \restriction \omega_2)$.

[5] I.e. $f^0 = f$, $f^{\beta+1} = f \circ f^{\beta}$ and for limit ordinals β, $f^{\beta} = sup_{i<\beta} f^i$.
[6] E^* will probably not even be in $\mathcal{M}$, but part of it will be recovered by our final condition and $E \in \mathcal{M}$.

Pick a cofinal sequence $(\alpha_i : i \in \omega)$ which converges to α and define the sets $D_i = \{r : dom(r) \geq \alpha_i \wedge \exists \beta \geq \alpha_i \ r \Vdash \check{\beta} \in \dot{C}\}$ which are open dense, since by the previous lemma, $\Vdash \dot{\omega}_1 = \check{\omega}_1$. It is clear that if we construct a decreasing sequence $(p_i : i < \omega)$ such that $p_0 = p$ and $p_i \in D_i$, then $q = \bigcup_{i<\omega} p_i$ will force $\check{\alpha}$ to be in $\dot{C}$ as long as q is a condition. Working exactly the same way as in the above lemma, with the aforementioned dense sets, we get this condition q. But then this means that $\Vdash$ "$\check{a} \in \dot{C} \Rightarrow \check{S} \cap \dot{C} \neq \emptyset$". □

5.3.3 The final coding and complexity analysis

Now that we have reshaped on the ordinals of ω_1 over a sequence of structures, we can inductively define a sequence of almost disjoint sets needed for the final coding. This sequence is defined using the reshaped structures which are associated to the conditions contained in the generic.

Definition 5.16.

(a) **reshaped structures:** For every $\alpha \in C^*$, we call a structure $\mathcal{M}_\alpha$ $\alpha - reshaped$, if it is equal to $\mathcal{N}^\alpha$ relative to the condition $G_3 \cap \alpha$, in the sense of definition (b), 1. of 5.13.

(b) **coding sets:** We define the sequence $(z_\alpha : \alpha \in \omega_1)$ exactly as in (b), 3. of definition 5.13 and relative to the conditions $G_3 \cap \alpha$, $\alpha < \omega_1$.

The final forcing we apply, is the almost disjoint coding of $B_3 \oplus A = B_3'$ to a real, where $A \subset \omega_1$. The almost disjoint sets come from the sequence $(z_\alpha : \alpha \in \omega_1)$, as defined above. Note that every notion is defined according to definition 5.13 and using the generic G_3 to generate the conditions required.

Definition 5.17. $(p, p^*) \in \mathbb{P}_3$ iff:

(a) $p : \alpha \to 2$, $\alpha < \omega$;

(b) $p^* \subset B_3'$, $\overline{\overline{p^*}} < \omega$.

$(p, p^*) \leq (q, q^*)$ iff:

(a) $p \supseteq q$, $p^* \supseteq q^*$;

(b) $\forall i \in q^*$ $(\breve{p} \setminus dom(q)) \cap z_i = \emptyset$

In the same fashion as in the last almost disjoint coding, we have that $\mathbb{P}_3$ satisfies the c.c.c. and we let r be the real added by this forcing. We are now ready to prove the theorems mentioned in the first section.

Proof. (of **Theorem 5.1**) At this point everything is set and we only need to check (a)-(c).

(a): First we show that $(*)_\alpha$ is $\mathbf{\Pi}^1_{n+2}$. The most complicated part of the definition of $(*)_\alpha$ is the iterability of $(K, \mathcal{M}, \alpha)$ which is actually $\mathbf{\Pi}^1_{n+2}$. This is by [10], proof of corollary 2.18, and using the fact that there are have at most n strong cardinals below ω_1^V. Then "$\mathcal{M}_\alpha$ is an α-reshaped structure" is $\mathbf{\Pi}^1_{n+2}$ (we code α by a real).

Let:

- $\phi(b, x) =$"x is a β-reshaped structure, where β is the ordinal coded by $b \subset \omega$";
- $\psi(x, y, z) =$ "x is the $<_y$-least set almost disjoint from the elements of the sequence z".

Then $a \in \tilde{A}$ is defined by

$$\exists(\mathcal{M}_\beta : \beta \leq \alpha)\ \exists(z'_\beta : \beta \leq \alpha)\ \forall \beta \leq \alpha$$

$$[\phi(\beta, \mathcal{M}_\beta) \wedge \psi(z'_\beta, \mathcal{M}_\beta[B'_3 \cap \beta], (z'_\gamma : \gamma < \beta)) \wedge \overline{\overline{r \cap z'_\alpha}} < \omega].$$

The above sentence is $\mathbf{\Sigma}^1_{n+3}$ as ϕ is $\mathbf{\Pi}^1_{n+2}$. Here $(z'_\beta : \beta \leq \alpha) = (z_\beta : \beta \leq \alpha)$, since the β-reshaped structures, for $\beta \leq \alpha$, are lined up. This way we are able to use r together with the a.d. sets to decode.

Since the sequence $(\mathcal{M}_\beta : \beta \leq \alpha)$ is uniquely determined, we may switch the quantifier in the above formula to a universal one and thus get a $\mathbf{\Pi}^1_{n+3}$ definition of $\tilde{A}$. Therefore, there is a $\mathbf{\Delta}^1_{n+3}$ definition of it.

(b): This works using the definition mentioned above and deriving the appropriate definition of being having the $(*)_\alpha$ property, along with lemma

2.3 of [26]. This way we get that "$\mathcal{M}_\alpha$ is an α-reshaped structure" is in $J_{1+\theta}(\mathbb{R})$.

(c): Our forcing construction is the iteration of the four forcings we introduced and since each one of them is stationary preserving, their iteration is also stationary preserving. $\square$

Chapter 6

Coding higher above

On chapter 5 we have focused on coding a subset of ω_1 to a real. Now we have a look at the possibilities of running a similar coding on larger cardinals. Namely, we wish to code a subset of $\kappa = \iota^+$, where ι is a regular cardinal greater than ω_1, to a subset of ι. We follow the same strategy as before, i.e. we collapse an appropriate cardinal δ to κ and afterwards code in two steps. Once more, most of the difficulties arise at the proof of distributivity of the reshaping, which is needed to run the second coding.

Remember that proving distributivity involved constructing a countable decreasing sequence of conditions. Then using the "fast club" trick from [31] we were able make sure that their union was also a condition. Additionally, our choice of coding structures provided us with enough condensation for the latter argument. In the current situation, in which we want to code above ω_1, we have some extra tools in our disposal, but at the same time new difficulties arise. The main advantage comes from the fact that we can get enough condensation by simply using the argument form the proof of weak covering in [18]. The new difficulty that arises, is the need to build a decreasing sequence of conditions of uncountable length for the proof of distributivity. This creates complications as we are not able to apply the "fast club" trick since we have to deal with several limit points of our sequence. Furthermore, the condensation we derive can only be applied at stages of uncountable cofinality.

Due to the above complications, we choose to code over the model $K[\vec{A}]$ instead of K. $\vec{A}$ is a sequence of countable cofinal subsets of the ordinals below κ with cofinality ω. This sequence enables K to see countable cofinalities and thus deal with the problematic limit points. The mixed model $K[\vec{A}]$ looks like the Chang model, in the sense that it is enchanted with specific countable sequences but it satisfies AC due to its construction.

6.1 Massaging the universe

We choose δ the way we did in chapter 4, though this time we don't need to require that it is any sort of cutpoint. This is because we will get condensation by covering and the $\vec{A}$ sequence instead of using the collapsed structures $\mathcal{M}_\alpha$, whose canonicity depended on the lack of extender overlap. Thus δ satisfies,

(a) $\delta^\iota = \delta$.

(b) δ is singular.

We force with $Col(\delta^+, 2^\delta) \star Col(\kappa, \delta)$ in order to ensure that the subsets of δ appear below δ^+, which is now equal to κ^+. In the produced extension we have that $H_{\kappa^+} = J_{\kappa^+}[B']$ for some $B' \subset \kappa^+$. Then it is enough to apply an a.d. coding to code B' to a subset B of κ, in order to end up in the following setting:

(a) There is no inner model with a Woodin cardinal.

(b) $\kappa = \iota^+$, where $\iota \geq \omega_1$. Set $\kappa^+ = \lambda$.

(c) $K|\lambda[B] = H_\lambda$, for some $B \subset \kappa$.

(d) There is an ordinal $\delta \in (\kappa, \kappa^+)$ such that $(\delta^+)^K = \lambda$.

(e) B_{odd} codes $K|\delta$ in a trivial way over L.

(f) We fix a sequence $\vec{A} = (a_i \subset i : i < \kappa, cf(i) = \omega)$, where a_i is cofinal below i and $otp(a_i) = \omega$. Those sets are witnesses of the true cofinality of the ordinals which have countable cofinality.

Assuming we are in the above configuration, we are going to prove the coding theorem below, which is the main theorem of this chapter.

Theorem 6.1. *Assume we are in the situation described by (a)-(f) above. Then there is a forcing notion such that in the generic extension, $H_\lambda = K|\lambda[\vec{A}, C]$, for some $C \subset \iota$. Moreover, this extension preserves cardinals and cofinalities.*

The critical step, as before, is reshaping the interval $[\iota, \kappa)$ so that we can further code to a subset of ι. Before performing the reshaping though, we must go once more through an analysis of the structures which are produced from the collapses and of how they condense.

6.2 Analyzing the collapsed structures

Before defining the reshaping forcing, we introduce the structures $\mathcal{N}_\gamma$ and their extensions $\mathcal{M}_\gamma$, which will play a central role later on in the coding procedure. Note that the analysis of those structures is very close to the one appearing in the proof of the weak covering lemma (see [18], [17]). For this reason we try to be as consistent as possible with the notation appearing in those papers and we redirect the reader there for the proofs of certain arguments.

Definition 6.2. Suppose Ω is a measurable cardinal and W is a universal weasel witnessing the strong soundness of $K|\lambda$. Let $\pi : \overline{H} \to H_{\Omega+1}$ be the transitive collapsing map of $Y \lhd H_{\Omega+1}$, with $\iota, \delta, \Omega \in Y$, $\overline{\overline{Y}} = \iota$ and such that $crit(\pi) = \gamma$. As usual, we denote the collapsed images of objects by π with bars, e.g. $\overline{\delta} = \pi^{-1}(\delta)$, $\overline{W} = \pi^{-1}(W)$. Let also $\mathcal{N}_\gamma$ be the model coded by $(B \cap \gamma)_{odd}$ over $L|\overline{\lambda}$. Finally, we denote by C^* the set of all γ which are critical points of such maps. One can observe that C^* contains a club below κ.

As we have already seen in previous cases of coding, one would like to use $A \cap \gamma$ in order to decode one more level of A. The generic which will carry out the coding will eventually provide us this information, but we also need to decode some part of $\overline{W}$ in order to run this decoding. This part will come from the comparison of $\mathcal{N}_\gamma$ with K, which up to some level is essentially identical with the comparison of $\overline{W}$ with W. As we will see, in some specific cases, the last model of the W-side will contain $\overline{W}|\overline{\lambda}$ as an initial segment. Before moving to this step though, we use the proof of the covering lemma to show that we can create situations where the last model of the W-side of the $\overline{W}, W$ comparison contains $\overline{W}|\overline{\lambda}$ as an initial segment. In such cases, the elementary submodel which is collapsed will be called "good".

Definition 6.3. Let $Y, W, \overline{W}$ be as in definition 6.2. Suppose also that W_θ is the last model of the W-side of the comparison of $\overline{W}, W$. Then we will call Y *good*, iff $\overline{W}|\overline{\lambda} \lhd W_\theta$.

Lemma 6.4. *[a combination of [18], [17] and [29], 11.55] Suppose Y is as in definition 6.2. Then there is a good $X \supset Y$ such that $\overline{\overline{X}} = \iota$.*

Proof. As mentioned in the comments, this proof is nearly identical with the one of weak covering in [18]. The only difference here is that we remove the assumption that $\overline{H}$ is countably closed using the technique appearing in [29] and the pull-back argument from [17].

The idea is to define $\overline{H}$ as the transitive collapse of $X = \bigcup_{i<\iota} X_i$, $X_0 = Y$, which is a union of structures elementary embedded into $H_{\Omega+1}$. In [18] there is a list of inductive steps $(1)_\alpha - (6)_\alpha$ which are eventually satisfied. When passing from X_i to X_{i+1}, we add a witness for every possible ill-foundedness or badness of iteration trees that might occur in one of $(1)_\alpha - (6)_\alpha$ for the reason that X_i is not omega closed. Then using a reflection argument we show that the induction can be carried out successfully for $\overline{H}$. We avoid repeating the whole proof here. Instead, we mention the definitions of the main objects and provide an exposition of the proofs which need to be changed.

Suppose that $\overline{W}_\eta$ and W_η, $\eta \leq \theta$, are the models produced in the comparison of $\overline{W}$ and W. We define the following:

(a) $\boldsymbol{\kappa} = (\kappa_\alpha : \alpha \leq \varepsilon)$ is an enumeration of the cardinals of $\overline{W}_\theta$ below $\overline{\lambda}$.

(b) $\boldsymbol{\lambda} = (\lambda_\alpha : \alpha < \varepsilon)$, where $\lambda_\alpha = \kappa_{\alpha+1}$.

(c) Let $\eta(\alpha)$ be the least ordinal $\eta \leq \theta$ such that $\kappa_\alpha < \nu_\eta$ if one exists, otherwise $\eta(\alpha) = \theta$. P_α is the least initial segment of $W_{\eta(\alpha)}$ which projects below κ_α, if it exists, else $P_\alpha = W_\alpha$.

(d) $R_\alpha = ult(P_\alpha, E_\pi \restriction \pi(\kappa_\alpha))$, π_α is the ultrapower map, if this ultrapower is well-founded. $m_\alpha = n(P_\alpha, \kappa_\alpha)$ is the largest integer m such that $\rho_m^{P_\alpha} > \kappa_\alpha$, if one exists, else $m = \omega$.

(e) $\mathbf{\Lambda} = (\Lambda_\alpha : \alpha < \varepsilon)$, where $\Lambda_\alpha = sup(\pi_\alpha{}''\lambda_\alpha) = (\pi_\alpha(\kappa_\alpha)^+)^{R_\alpha}$.

(f) S_β is defined by induction. If R_β is a premouse, then $S_\beta = R_\beta$. Otherwise, if R_β is just a protomouse, there is an $\alpha < \beta$ such that $crit(F^{R_\beta}) = \pi_\alpha(\kappa_\alpha)$ and $S_\beta = ult(S_\alpha, F^{R_\beta})$.

(g) Q_β is also defined by induction. If R_β is a premouse, then $Q_\beta = P_\beta$. Otherwise, there is a $\alpha < \beta$ such that $crit(F^{P_\beta}) = \kappa_\alpha$ and $Q_\beta = ult(Q_\alpha, F^\beta)$.

Comments 1. The proof consists of an induction on α showing at each step that $\overline{W}$ agrees with W_θ up to λ_α. One can imagine P_α as the least-/earliest place of the W-side iteration where the models stabilize below λ_α. By the agreement we are opting for, and the the fact that W is the winning side, we may think of P_α as a local approximation of $\overline{W}$ up to some point. Then π_α locally simulates π and R_α accordingly approximates W.

The reason why S_α and Q_α are defined, is to replace R_α with a premouse in case it fails to be one and then compute the corresponding version of P_α. It is easy to check that $S_\alpha = ult(Q_\alpha, E_\pi \restriction \pi(\kappa_\alpha))$ (see [18], lemma 3.13).

In order to ensure the consistency of the definitions of (d)-(g), we must make sure that certain ultrapowers are well founded and that certain phalanges are special (see below). This is taken care of in several lemmas appearing later on.

Definition 6.5. A phalanx $(\boldsymbol{P}; \boldsymbol{\lambda})$ of length ε is *special* iff there is a sequence $\boldsymbol{\kappa}$ such that:

(a) If $\alpha \le \beta < \varepsilon$, then κ_α is a cardinal of P_β and $\lambda_\beta = (\kappa_\beta^+)^{P_\beta}$.

(b) P_α is κ_α-sound, if it is a set protomouse.

(c) P_α satisfies $\kappa(P_\alpha) \le \kappa_\alpha$ in case it is a weasel.

(d) If P_β is not a premouse, then there is a unique $\alpha < \beta$ such that F^{P_β} is a $(\kappa_\alpha, OR^{P_\beta})$ extender over P_α.

$(\boldsymbol{P}; \boldsymbol{\lambda})$ is *very special* if additionally:

(e) When P_α is a weasel, $c(P_\alpha) \setminus \kappa_\alpha = \emptyset$.

(f) If P_α is not a premouse, then it is an active protomouse of type I or II and $n(P_\alpha, \kappa_\alpha) = 0$.

The induction steps used in the proof of the lemma are listed below:

$(1)_\alpha$ If $\overline{E}_\eta \neq \emptyset$, then $lh(\overline{E}_\eta) > \lambda_\alpha$.

$(2)_\alpha$ $(W, S_\alpha; \pi(\kappa_\alpha))$ is an iterable phalanx of premice.

$(3)_\alpha$ $(\overline{W}, Q_\alpha; \kappa_\alpha)$ is an iterable phalanx of premice.

$(4)_\alpha$ $(\boldsymbol{P} \restriction \alpha, \overline{W}; \boldsymbol{\lambda} \restriction \alpha)$ is an iterable, very special phalanx.

$(5)_\alpha$ $(\boldsymbol{R} \restriction \alpha, W; \boldsymbol{\Lambda} \restriction \alpha)$ is an iterable with respect to special trees, very special phalanx of protomice.

$(6)_\alpha$ $(\boldsymbol{S} \restriction \alpha, W; \boldsymbol{\Lambda} \restriction \alpha)$ is an iterable with respect to special trees, very special phalanx.

Assuming that $(1)_{<\alpha}$ holds, the induction step is split into two paths:

$(4)_{<\alpha} \Rightarrow (3)_\alpha \Rightarrow (2)_\alpha \Rightarrow$ next step,
$(2)_{<\alpha} \Rightarrow (6)_\alpha \Rightarrow (5)_\alpha \Rightarrow (4)_\alpha \Rightarrow (1)_\alpha \Rightarrow$ next step

The only one of the above implications which really depends on the countable closure of κ, in [18], is $(3)_\alpha \Rightarrow (2)_\alpha$. In order to surpass this obsticle we will construct X as the union of elementary substructures of $H_{\Omega+1}$, as mentioned before. At each level of this construction we will throw in a witness of a possible failure of $(3)_\alpha \Rightarrow (4)_\alpha$ at the previous step. We begin by defining this tower of elementary structures. For each one of them, the objects $P_\alpha, R_\alpha, \ldots$ can be defined as before and we distinguish them by a power, e.g. $P^i_\alpha, R^i_\alpha \ldots$ correspond to the structure X_i. As for the final structures, $X_\iota, P^\iota, R^\iota, \ldots$ we omit the ι.

Definition 6.6. We define inductively for every $i \leq \iota$ the following embeddings:

(a) $X_0 = Y$. $\pi_0 : \overline{H}_0 \to H_{\Omega+1}$ is the transitive collapsing map of X_0.

(b) X_{i+1} is the least elementary substructure of $H_{\Omega+1}$ such that:

- $X_{i+1} \supset X_i \cup \{X_i\}$.
- If α' is the least $\alpha < \varepsilon^i$ such that the phallanx $(W^i, S^i_\alpha; \pi(\kappa^i_\alpha))$ is not iterable, then the $<$-least witness of this non-iterability is contained in X_{i+1}. This witness will consist of a well-founded relation $\overline{R}$, rudimentary over W^i, an ill-founded relation R, rudimentary over some iterate of W^i with the same definition as $\overline{R}$, and an infinite decreasing sequence $([a_n, f_n] : n < \omega)$ witnessing its ill-foundedness.

(c) $X_i = \bigcup_{j<i} X_i$, for limit ordinals $i \leq \iota$.

Lemma 6.7 ("3.1"). *For every $\alpha \leq \varepsilon$, $(W, S_\alpha, \pi(\kappa_\alpha))$ is iterable.*

Proof. Look at sections 2 and 3 of [17]. The procedure is the same, with the difference that we replace X_j with X. Again some X_i can be picked so that $X_i \cap ran(\psi) \subset X_i$ and the rest of the proof is essentially the same. □

Now by the fact that $(1)_\alpha$ holds for all $\alpha < \varepsilon$, we get that $\overline{W}|\overline{\lambda} \lhd W_\theta$.

□

Lemma 6.8. *Assume W, $\overline{W}$ and $\mathcal{N}_\gamma$ are as in definition 6.2. Let also $W_{\theta'}$ be the last model of the W-side of the comparison between W and $\mathcal{N}_\gamma$. Then $\overline{W}|\lambda \trianglelefteq W_{\theta'}$.*

Proof. Assume that W_θ is the last model of the W-side of the comparison between $\overline{W}$ and W. The comparison between $\mathcal{N}_\gamma, W$ is contained in the comparison between $\overline{W}, W$, thus from lemma 6.4, $\mathcal{N}_\gamma$ does not move in the comparison. But then the extenders that are used in the W-side below the height of $\mathcal{N}_\gamma$, which is above $\overline{\delta}$, are the same in both iterations. But the comparison between $\mathcal{N}_\gamma, W$ ends at the height of $\mathcal{N}_\gamma$ and the only differences above that point will be due to extenders used in the comparison of $\overline{W}, W$. But from lemma 6.4, $\overline{W}|\overline{\lambda} \trianglelefteq W$, therefore there are no cardinals in W_θ, in the interval $(\overline{\delta}, \overline{\lambda})$. This in turn implies that no extenders with indices in this interval were used, thus $W_{\theta'}|\overline{\lambda} = W_\theta|\overline{\lambda} = \overline{W}|\overline{\lambda}$, which is what we wanted.

□

6.3 Reshaping

Now we turn to the definition of the reshaping forcing. This is exactly like the one appearing in chapter 2, with the only difference that it is applied on $K[\vec{A}]$:

Definition 6.9. $p \in \mathbb{P}$ iff:

1. $p : \gamma \to 2$, $\gamma \in C^*$;
2. for every $\gamma' \in C^* \cap (\gamma + 1)$ such that $cf(\gamma') > \omega$,

$$K[\vec{A}, B \cap \gamma', p \restriction \gamma'] \vDash \overline{\overline{\gamma'}} = \iota.$$

3. $p \leq q$ iff $q \subseteq p$.

Since we require that κ must not be collapsed after forcing, we need to make sure that $\mathbb{P}$ preserves it. This can be derived, like before, from distributivity.

Lemma 6.10. $\mathbb{P}$ *is* ι*-distributive.*

Proof. Let $(D_i : i \in \iota)$ be a sequence of open dense sets and $p \in \mathbb{P}$. We use the following tower of structures to construct a condition $q \leq p$ that meets all the elements of $(D_i : i < \iota)$.

(a) $Y_0 = Hull^{H_\lambda}((\iota+1) \cup \{\mathbb{P}, (D_i : i < \iota), p, <\})$;

(b) Y_{i+1} is the $<$-least good Y such that $Y_i \cup \{Y_i\} \subset Y$ and $Y \prec H_\lambda$.

(c) $Y_\eta = \bigcup_{i \in \eta} Y_i$, for limit ordinals $\eta \leq \iota$.

Let $\tilde{\pi}_i : \overline{H}_i = \overline{K}_i[B \cap \gamma_i] \to H_\lambda$ be the uncollapsing map for Y_i -notice that $crit(\tilde{\pi}_i) = \gamma_i$. Before proceeding, we need to check that the structures $\overline{H}_i$ are also good at limit steps. It is enough to check this for ordinals of uncountable cofinality, since $\vec{A}$ provides enough information at countable cofinality levels.

Claim 6.11. *Assume η is a limit ordinal $\leq \iota$, such that $cf(\eta) > \omega$. Then $\overline{H}_\eta$ is good.*

Proof. Fix an η with the properties of the above statement. We begin by noticing that we are high enough to use the covering lemma. Therefore, if $\aleph_2^V \leq \kappa \leq \mu$, then $cf((\mu^+)^K) \geq \overline{\overline{\mu}} \geq \kappa$. This implies that for all such ordinals μ, the set $Y_\eta \cap (\mu^+)^K$ is bounded below $(\mu^+)^K$. As a result of that, if $\mu \in Y_\eta$, then $cf(Y_\eta \cap (\mu^+)^K) = cf(\eta) > \omega$.

In order to prove that Y_η is good, we use the above observation and repeat the argument of the covering lemma in order to get enough agreement between $\overline{K}_\eta$ and the corresponding W. For this, it is enough to prove that the phalanx $(W^\eta, S^\eta_\alpha; \tilde{\pi}_\eta(\kappa^\eta_\alpha))$ $[(3)_\alpha]$ is iterable, for every $\alpha < \varepsilon_\eta$, as the rest of the implications hold true automatically from the proof in [18]. Until the end of the proof of this claim, for aesthetically reasons, we omit the superscript η.

Assume that the above is not true and let i be the least α such that $(W, S_\alpha, \tilde{\pi}_\eta(\kappa_\alpha))$ is not iterable. Then by [9], lemma 3.3, S_i is also not iterable. We split into two cases depending on the cofinality of κ_{i+1} and prove in both of them that S_i is iterable, thus a contradiction.

- $cf(\kappa_{i+1}) > \omega$:

  Assume $\sigma : \overline{H} \to H_\theta$ is an elementary embedding respecting a well-ordering $<$ of H_θ and that θ is large enough so that H_θ contains Q_i, S_i as well as all the corresponding embeddings and phalanxes that appear up to that level in the proof of covering. Let also $\overline{Q}_i, \overline{S}_i$ be the inverse images of Q_i, S_i via σ and $\tilde{Q}_i = Ult_0(\overline{Q}_i, \sigma \restriction \overline{\kappa}_{i+1})$, $\tilde{S}_i = Ult_0(\overline{S}_i, \sigma \restriction \overline{\kappa}_{i+1})$. Then, if $k : \tilde{Q} \to Q_i$ is the map factoring the ultrapower map, $\tilde{Q}_i \trianglelefteq Q_i$ by condensation, thus $\tilde{Q}_i \trianglelefteq K_\eta$. But then $\tilde{S}_i = Ult_0(\tilde{Q}_i, \tilde{\pi}_\eta \restriction (\kappa_i^+)^{\tilde{Q}_i}) \to \tilde{\pi}(\tilde{Q}_i) \trianglelefteq W$. Therefore S_i is iterable.

- $cf(\kappa_{i+1}) = \omega$:

  Fist notice that κ_{i+1} is not a cardinal of $\overline{K}_\eta$. If it were, then the embedding $\tilde{\pi}_\eta$ would map it to $\tilde{\pi}_\eta(\kappa_{i+1})$ which would in turn be a cardinal of the corresponding W. But then $cf(\tilde{\pi}_\eta(\kappa_{i+1}) \geq \kappa$. But this in turn implies that $cf(\kappa_{i+1}) = cf(Y_\eta \cap \tilde{\pi}_\eta(\kappa_{i+1})) = cf(\eta) \neq \omega$, contradiction.

  As a result, there exists some $\zeta \geq \kappa_{i+1}$ such that $\rho_\omega(\overline{K}_\eta||\zeta) \leq \kappa$. But the phalanx $(\mathbf{Q} \restriction i, \overline{K}_\eta||\zeta, \kappa)$ is iterable as every element of $\mathbf{Q} \restriction i$ is embedded to some S_j, $j < i$, which in turn is embedded to W and $\overline{K}_\eta||\zeta$ is embedded into W. Thus comparing $(\mathbf{Q} \restriction i, \overline{K}_\eta||\zeta, \kappa)$ and $\mathbf{Q} \restriction i + 1$, we get that $Q_i = \overline{K}_\eta||\zeta$. Therefore S_i is iterable as it is produced by a segment of the map π from an initial segment of $\overline{W}$.

□

One can notice from the proofs above that we are actually making constantly use of the arguments from [18], except from the successor cases, where an argument alike to the one in [17] is used to produce good structures, and also excluding the countable cofinality limit cases, where $\vec{A}$ is used.

Now that enough condensation is available, we use the collapsed structures $\overline{H}_i$ to define a decreasing chain of conditions p_i such that

(a) $p_0 = p$;

(b) p_{i+1} = "the constructibly least $r \in \overline{K}_{i+1}[B \cap \gamma_{i+1}]$ such that $r \leq p_i$, $\overline{\overline{r}} \geq \gamma_i$ and $r \in D_i \cap Y_{i+1}$";

(c) $p_\beta = \bigcup\limits_{i \in \beta} p_i$.

Claim 6.12. $K[\vec{A}, B \cap \gamma_i] \vDash \overline{\overline{p}}_i \leq \iota$*, for every* $i \leq \iota$.

Proof. We prove this by induction on $i \leq \kappa$. We consider the following two possibilities (for successor ordinals there is nothing to be done), depending on the cofinality of γ_i:

- $cf(\gamma_i) > \omega$:

We will prove in the next sublemma that $K[\vec{A}, B \cap \gamma_i]$ can see all the structures $(Y_j : j < i)$ and their transitive collapses, $(\overline{K}_j : j < i)$, just the way they are defined in $K[B]$. Then by lemma 6.4, if $j < i$, then $\overline{K}_i \in K[\vec{A}, B \cap \gamma_i]$ since the cofinality of i is uncountable. Therefore we may reconstruct the sequence $(p_j : j < i)$ in $K[\vec{A}, B \cap \gamma_i]$, which witnesses that $\overline{\overline{p}}_i = \overline{\overline{\gamma}}_i = \overline{\overline{i}} \cdot \iota = \iota$. The aforementioned argument is based on the sublemma stated below. Notice that in order to distinguish between the corresponding objects defined in $K[\vec{A}, B \cap \gamma_i]$ and $K[B]$, we use parentheses and a superscript i for the objects of $K[\vec{A}, B \cap \gamma_i]$.

Subclaim 6.13. *Suppose $((X_l)^i : l < \iota)$ is the sequence witnessing the goodness of $(Y_j)^i$, for some $j < i$, as it is built in $K[\vec{A}, B \cap \gamma_i]$. Let $(X_l : l < \iota)$ be the same sequence, but built in $K[B]$. Then for every $l < \iota$, $(X_l)^i$ and X_l are isomorphic and therefore they collapse to the same structures.*

Proof. It is enough to check this for the successor steps, as at limits we consider unions. If the induction is true up to some $l < \iota$, then the process of passing from $(X_l)^i$ to $(X_{l+1})^i$ or from X_l to X_{l+1}, consists of adding a witness of the failure of $(2)_\alpha$ for the least α where it fails. Therefore, it is in enough to prove that this will happen for the same α for both $(X_l)^i$ and X_l. To get this we will show that $(W^j, S^j_\alpha, \Lambda^j_\alpha)^i$ is iterable iff $(W^j, S^j_\alpha, \Lambda^j_\alpha)$ is iterable. Like before, we fix j and omit it from the superscript for the sake of simplicity.

Assuming the induction holds up to α, the iterability of the second phalanx implies the iterability of the first one. This is because $\overline{K}_i[\vec{A}, B \cap \gamma_i]$ is embedded into $K[\vec{A}, B]$ via $\tilde{\pi}_i$, thus $(S)^i_\alpha$ can also be embedded to S_α, therefore, any iteration of $(W, S_\alpha, \Lambda_\alpha)^i$ is embeddable to an iteration of $(W, S_\alpha, \Lambda_\alpha)$.

For the other direction, assume $(W, S_\alpha, \Lambda_\alpha)^i$ is iterable and let's consider first the case where $(S)^i_\alpha$ is a set mouse. By factoring the map $\tilde{\pi}_j$ we get $Ult((Q)^i_\alpha, \tilde{\pi}_i \circ (\tilde{\pi}_j)^i \upharpoonright (\lambda_\alpha)^i) = Ult(Ult((Q)^i_\alpha, (\tilde{\pi}_j)^i \upharpoonright (\lambda_\alpha)^i), \tilde{\pi}_i \upharpoonright (\Lambda_\alpha)^i)$. But

$(S_\alpha)^i = Ult((Q^\alpha)^i, \tilde{\pi}_i \upharpoonright (\lambda_\alpha)^i) \lhd W_i$ and $Ult((S_\alpha)^i, \pi \upharpoonright ((\Lambda_\alpha)^i))$ embeds to $\pi((S_\alpha)^i)$. Since $Ult(S^i_\alpha, \pi \upharpoonright ((\Lambda_\alpha)^i))$ is an initial segment of S_α up to Λ_α and $(W, \pi((S_\alpha)^i), \Lambda_\alpha))$ is iterable, $(W, S_\alpha, \Lambda_\alpha)$ is also iterable.

To deal with the case where $(S_\alpha)^i$ is a proper class we use the fact that $(S_\alpha)^i = Ult(W_i, E^{W_i}_{(\Lambda_\alpha)^i})$. Like before we get $Ult(W_i, \tilde{\pi}_i \circ \pi_{E^{W_i}_{(\Lambda_\alpha)^i}}) = Ult(Ult(W_i, E^{W_i}_{(\Lambda_\alpha)^i}), \tilde{\pi}_i) = Ult((S_\alpha)^i, \tilde{\pi}_i)$. Once more the iterability of $(W, \pi((S_\alpha)^i), \Lambda_\alpha))$ and the agreement of $Ult(S^i_\alpha, \pi \upharpoonright ((\Lambda_\alpha)^i))$ with $(S_\alpha)^i$ guarantee the iterability of $(W, S_\alpha, \Lambda_\alpha)$. □

Using the above subclaim, we get that the sequence $(K_j : j < i)$ can be constructed in $K[\vec{A}, B \cap \gamma_i]$ as exactly as in $K[B]$, thus $(p_j : j < i)$ is also contained in $K[\vec{A}, B \cap \gamma_i]$. Therefore $K[\vec{A}, B \cap \gamma_i] \vDash \overline{\overline{p_i}} = \iota$.

- $cf(\gamma_i) = \omega$:

By the induction hypothesis, for every $j < i$, $K[\vec{A} \cap (\gamma_i + 1), B \cap \gamma_i] \vDash \overline{\overline{p}}_j \leq \iota$. On the other hand, $K[\vec{A}, B \cap \gamma_i] \vDash cf\gamma_i = \omega$, which resolves the case as $p_i = \bigcup_{j<i} p_j$.

□

Following the above, $K[\vec{A} \cap (\gamma_\iota + 1), B \cap \gamma_\iota] \vDash p_\iota \in \mathbb{P}$ and this finishes the lemma because p_ι meets every set of $(D_i : i < \iota)$. □

Remark 6.14. Suppose G is the generic added by $\mathbb{P}$ and set $\tilde{B} = B \oplus \cup G$. $\tilde{B}$ is the reshaped version of B.

Having reshaped the interval $[\iota, \kappa)$, we can proceed to the almost disjoint coding which will eventually code $\tilde{B}$ into C. This coding will be done using a collection of almost disjoint sets $(B_i : i < \kappa)$ which will be inductively generated in $K[\vec{A}]$ along with the initial segments of $\tilde{B}$. In the final extension we want $\tilde{B}$ to be coded by $i \in \tilde{B} \Leftrightarrow$ "$B_i \cap C$ is bounded".

Definition 6.15. For a limit $i < \kappa$, let B_{γ_i} be the constructibly least set in $K[\vec{A}, \tilde{B} \cap \gamma_i]$ which is almost disjoint to the elements of the sequence $(B_j : j < i)$. For $i = j + 1 < \kappa$, let $(B_k : k \in (\gamma_j, \gamma_i])$ be the constructibly least in $K[\vec{A}, \tilde{B} \cap \gamma_i]$ sequence extending $(B_k : k \leq \gamma_j)$ to a sequence of pairwise almost disjoint sets . It is easy to notice that $(B_j : i < \kappa)$ is definable because we have performed the reshaping.

Now we apply the a.d. coding $\mathbb{Q}$ which is using the sets defined above. Let C be the union of the first coordinates of the generic added by $\mathbb{Q}$. Then the usual properties hold true:

Lemma 6.16.

(a) $\mathbb{Q}$ *is* $< \iota$*-closed.*

(b) $\mathbb{Q}$ *satisfies the* ι*-cc.*

(c) $\forall i \in \kappa$ $(i \in \tilde{B} \Leftrightarrow B_i \cap C$ *is bounded).*

Proof. They are all basic properties of the a.d. forcing. □

Having performed all the codings required we can derive theorem 6.1 once we show that we can effectively decode.

Proof. *(of Theorem 6.1)*

We prove by induction on $i \leq \iota$ that $(\tilde{B} \cap \gamma_j : j < i), (B_j : j < \gamma_i) \in K[\vec{A}, C]$. Suppose this is true for every $j < i$. Then $\tilde{B} \cap \gamma_i \in K[\vec{A}, C]$ which implies that $B_{\gamma_i} \in K[\vec{A}, C] \supset K[\vec{A}, B \cap \gamma_i, (B_j : j < \gamma_i)]$. But now $\gamma_i \in \tilde{B}$ iff $C \cap B_{\gamma_i}$ is bounded. Therefore $\tilde{B} \cap \gamma_i \in K[\vec{A}, C]$ thus $(\tilde{B} \cap \gamma_j : j \leq i)$ is also contained in the same set. Here we are using the fact that every almost disjoint set is defined in a unique way using the constructibility structure of $K[\vec{A}, C]$.

Finally $B \cap \sup_{i<\kappa} \gamma_i = B \in K[\vec{A}, C]$ which is what we wanted to prove. Both of the forcings which were mentioned preserve cofinalities, thus the same holds for their two steps iteration that carries out the coding.

□

Chapter 7

Connections with Absoluteness

One long series of questions in set theory concerns the potential of forcing well-orderings of the reals which are definable in a simple way. On the other hand, for every such result there exists a -somehow complementary-one, about forcing absoluteness of certain categories of forcings and for formulas of a certain complexity (for our purposes let's say in $L(\mathbb{R})$).

The factor that determines the category of forcings for which $L(\mathbb{R})$ is absolute, is the large cardinal consistency strength of the universe. For example, a weakly compact cardinal is needed to get absoluteness for c.c.c. forcings, a remarkable cardinal for proper forcings and the existence and full iterability of $M_\omega^\sharp$ for all forcings. Correspondingly, if there is no inner model with a weakly compact cardinal we may add a well-ordering of $\mathbb{R}$ with a c.c.c. forcing. If there is no inner model with a remarkable cardinal we may add this well-ordering with a proper forcing and if there is no inner model with ω Woodin cardinals and an extender above them, then we can only say that there is some forcing that adds such a well-ordering[1].

The case we are interested in here, is the one related to the collection of stationary preserving forcings. The consistency strength of those problems lies at the level of strong cardinals. It is already known by [28] that below a strong cardinal one can force a well-ordering which is $\mathbf{\Delta}^1_3$ using a stationary preserving notion. Here we extend this to the case where strong cardinals are allowed to exist in K, but every ω_2-strong must lie

[1]For a more detailed description of this situation, look at the introduction of [27].

below ω_1^V. We will use the coding theorem we produced in chapter 5 in order to get those results.

7.1 Definable well-orderings via coding

Using the coding theorem (5.1), we may force the existence of nicely definable well-orderings of $\mathbb{R}$. Just like before, the complexity of those orderings directly depends on the number of strong cardinals in $K|\omega_1$.

Theorem 7.1. *Suppose there is no inner model with a Woodin cardinal and $K \models$"κ is a strong cardinal $\Rightarrow \kappa < \omega_1$". Then there is a forcing $\mathbb{P}$ such that:*

(a) if there are n strong cardinals in K below ω_1, then there is a $\boldsymbol{\Delta}^1_{n+3}$ well-ordering of the reals in the generic extension;

(b) if the ordertype of the set of strong cardinals in $J^K_{\omega_1}$ is strictly below $\omega\theta \leq \omega_1$, then there is a well-ordering of the reals in $J_{1+\theta}(\mathbb{R})$ in the generic extension;

(c) $\mathbb{P}$ preserves stationary subsets of ω_1.

Proof.

(a) We repeat the proof of theorem 5.1 with a few modifications. In the beginning we choose a δ which is additionally above 2^ω. This way $\overline{\overline{\mathbb{R}}} = \omega_1$ in all the generic extensions after $\mathbb{P}_0$. Now let $<$ be a well-ordering of $\mathbb{R}$. Using a bijection $f : \omega_1 \to \mathbb{R}$, we may view $<$ as a subset of $\omega_1 \times \omega_1$ which is easily coded to a subset A_0 of ω_1 using Gödel's pairing function. Using again standard techniques, we may also code f to an $A_1 \subset \omega_1$[2].

If A_2 is the set derived from $\mathbb{P}_1$, which codes H_{ω_2} over $K|\omega_2$, then we set $A = A_0 \oplus A_1 \oplus A_2$. We continue by coding A down to a real and getting a $\boldsymbol{\Delta}^1_{n+3}$ definition for it. Since the procedure of retrieving the information of $<$ does not increse this complexity, it is still $\boldsymbol{\Delta}^1_{n+3}$ when viewed as a set of reals.

[2] See the proof of lemma 4.4.

(b) We use the same procedure as in (a). Namely we run the proof of theorem 5.1 by shifting δ up if needed and adding the information of $<$ to the set to be coded. The final comlplexity is formed by the order-type of the strongs in $K|\omega_1$.

(c) Since in both (a) and (b) we used essentially the forcing of theorem 5.1, the proof of this property is the same.

□

Remark 7.2. This is one of the main ways this kind of coding theorems can be applied. One transforms some structure M of size ω_1 to a subset of ω_1, then adjoins it to the set coding H_{ω_2} and codes it down to a real. The low complexity of the coding structures will then guarantee that M also has a low complexity, as it can be decoded using only them and a real.

7.2 Definable well-orderings and absoluteness

The ability to force well-orderings of the reals blocks generic absoluteness, even for categories of forcings so small as the c.c.c. forcings. The absoluteness which is blocked, is that of formulas which have complexity one real quantifier above the complexity of the well-ordering we can add. Furthermore, the weaker form of 2-step absoluteness also fails in the same manner. Before proceeding we define the notions of absoluteness we will need.

Definition 7.3. Suppose $\mathcal{P}$ is a category of forcings. Then,

(a) $\boldsymbol{\Sigma}^1_n$ formulas *are absolute for* $\mathcal{P}$ *forcings* iff for every $\mathbb{P} \in \mathcal{P}$, every Σ^1_n formula ϕ and every vector $\vec{x}$ of reals, $\phi(\vec{x}) \leftrightarrow \Vdash_{\mathbb{P}} \phi(\check{\vec{x}})$.

(b) $J_\alpha(\mathbb{R})$ *is absolute for* $\mathcal{P}$ *forcings* iff for every $\mathbb{P} \in \mathcal{P}$, every formula ϕ and every vector $\vec{x}$ of reals, $J_\alpha(\mathbb{R}) \vDash \phi(\vec{x}) \leftrightarrow \Vdash_{\mathbb{P}} [J_\alpha(\mathbb{R}) \vDash \phi(\check{\vec{x}})]$.

(c) $\boldsymbol{\Sigma}^1_n$ formulas *are 2-step absolute for* $\mathcal{P}$ *forcings* iff for every $\mathbb{P} \in \mathcal{P}$, $\Vdash_{\mathbb{P}}$ "$\boldsymbol{\Sigma}^1_n$ formulas are absolute for $\mathcal{P}$ forcings".

(d) $J_\alpha(\mathbb{R})$ *is 2-step absolute for $\mathcal{P}$ forcings* iff for every $\mathbb{P} \in \mathcal{P}$, $\Vdash_{\mathbb{P}}$ "$J_\alpha(\mathbb{R})$ is absolute for $\mathcal{P}$ forcings".

Below we describe the cases of interest were well-orderings block absoluteness. The proofs are straightforward and they can be applied the same way in different settings.

Lemma 7.4. *Suppose a well-ordering of $\mathbb{R}$ which is $\mathbf{\Pi}^1_{n+3}$ can be added using a forcing of some category $\mathcal{P}$ extending the c.c.c forcings category. Then there is no forcing extension where $\mathbf{\Pi}^1_{n+4}$ formulas are 2-step absolute for forcings of type $\mathcal{P}$.*

Proof. Assume towards a contradiction that $\mathbf{\Pi}^1_{n+4}$ formulas become absolute for $\mathcal{P}$ forcings after applying any $\mathcal{P}$ notion. First use the forcing which adds a $\mathbf{\Pi}^1_{n+3}$ well-ordering of the reals defined by $\psi(x)$. The existence of a $\mathbf{\Pi}^1_{n+3}$ well-ordering defined by ϕ can be expressed by a $\mathbf{\Pi}^1_{n+4}$ formula. The latter states that for every real, coding in a simple way a countable sequence of reals, ϕ is a linear ordering (here we need just the first two reals of the sequence) and the sequnce is not strictly decreasing realtive to ϕ.

Therefore, by our assumption, after further using a $\mathcal{P}$ notion, $\psi(\check{x})$ should still define a well-ordering of $\mathbb{R}$. Now add ω many Cohen reals with a c.c.c. forcing. By Cohen's argument for the independence of AC[3], there is no well-ordering of the reals definable with real parameters. But this forcing is in $\mathcal{P}$ and $\psi(\check{x})$ defines still a well-ordering, contradiction.

□

Lemma 7.5. *Suppose a well-ordering of $\mathbb{R}$ which is in $J_\alpha(\mathbb{R})$ can be added using a forcing of some category $\mathcal{P}$ containing the c.c.c. notions. Then there is no forcing extension where $J_\alpha(\mathbb{R})$ is 2-step absolute for forcings of type $\mathcal{P}$.*

Proof. It is the same idea as above. We add a well-ordering in $J_\alpha(\mathbb{R})$ with a $\mathcal{P}$ forcing. Since $J_\alpha(\mathbb{R})$ is closed under real quantifiers, $J_\alpha(\mathbb{R})$ absoluteness is enough to guarantee that this well-ordering will remain in further $\mathcal{P}$-extensions. But then, adding ω cohen reals, we destroy it getting a contradiction. □

[3]Look e.g. [11], 14.36.

7.3 Limitations on forcing stationary preserving absoluteness

Following the aforementioned connections of coding with absoluteness the reader should not be surprized that we will apply our coding theorem get information about absoluteness. Ideally we would like to directly extend the coding result of [28] and get lower consistency strength bounds. Unfortunately, our assumption of the non-existence of strong cardinals in $K|\omega_2$ above ω_1 does not allow us to proceed this way.

Nevertheless, we can prove that the standard methods of producing absoluteness by collapsing cardinals above strongs cannot be used to get more absoluteness than the one allowed by the ordertype of the strong cardinals of $K|\omega_1$. The main result of this form is the following:

Theorem 7.6 (H. Woodin)**.** *Suppose κ is a strong cardinal. Then 2-step generic absoluteness for $\mathbf{\Sigma}^1_4$ formulas holds in the extension by $Col(\omega, 2^{2^\kappa})$.*

The above is easily extended to the case where more strongs exist below κ. Note though that one cannot reach only absoluteness below $J_{\omega_1}(\mathbb{R})$ since all the strongs are collapsed to ω. For more absoluteness one needs more consistency strength than the mere existence of strong cardinals. This theorem has recently been refined by Trevor Wilson who reduced 2^{2^κ} to κ^+ in [37].

Theorem 7.7 (T. Wilson)**.** *Suppose κ is a strong cardinal. Then 2-step generic absoluteness for $\mathbf{\Sigma}^1_4$ formulas holds in the extension by $Col(\omega, \kappa^+)$.*

The following two theorems provide a limit to the complexity of the sentences which become 2-step absolute by the above collapsing forcings. Furthermore, one only needs to consider stationary preserving forcings to get the limitations. The corresponding complexities depend on the number of strong cardinals of K.

Theorem 7.8. *Suppose λ is a cardinal of K which is not strong and such that there exist n many strong cardinals below it. Then 2-step $\mathbf{\Sigma}^1_{n+4}$ absoluteness for stationary preserving forcings can not be forced by collapsing λ to ω over K.*

Proof. After applying $Col(\omega, \lambda)$, all the strong cardinals of $K|\omega_2$ are below ω_1. Thus we may apply 5.1 to get an extension containing a $\mathbf{\Delta}^1_{n+3}$ well-ordering of the reals using a stationary preserving notion. By 7.4, this makes 2-step $\mathbf{\Pi}^1_{n+4}$ absoluteness false. □

Theorem 7.9. *Suppose λ is a cardinal of K which is not strong and such that there exist $< \omega\theta$ many strong cardinals below it. Then 2-step $J_{1+\theta}[\mathbb{R}]$ absoluteness for stationary preserving forcings can not be forced by collapsing λ to ω over K.*

Proof. Exactly as above, by using 7.5 instead of 7.4. □

One may hope that by Levy collapsing a strong to ω_1, the desired absoluteness might be produced. Nevertheless, this is also not the case as we will see below. In general every forcing which leaves the interval $[\omega_1, \omega_2]$ clean of strong cardinals will not be able to produce the desired absoluteness. We leave it to the imagination of the reader to produce results of the same flavor.

Theorem 7.10. *Let κ be a strong cardinal of K such that there are n or $< \omega\theta$ strong cardinals below it. Then $Col(\omega, < \kappa)$ over K does not produce 2-step $\mathbf{\Pi}^1_{n+4}$ or respectively $J_\theta[\mathbb{R}]$ absoluteness for stationary preserving forcings.*

Proof. Let G be a generic for the $Col(\omega, < \kappa)$ over K. We have that $\kappa = \omega_1^{K[G]}$ and $(\kappa^+)^K = \omega_2^{K[G]}$. We run exactly the argument of chapter 5 to reshape over the collapsed structures and then code down to a real. Everything functions the same way, except from the proof that the collapsed structures are lined up. This actually gets simpler in this case, as there are no cardinals of K in (ω_1, ω_2), therefore both sides in the comparison of the phalanges trivially don't move.

This way we get a version of theorem 5.1 with the current assumptions. Then using 7.4 and 7.5 we get the 2-step-non-absoluteness. □

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Notation

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