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Theorem of the Complement and an Application to Rolle Leaves

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Introduction

Motivation

This diploma thesis deals with o-minimal structures. Those are a creation of model theorists, but they are also useful to prove some interesting facts in differential topology. Thus in tame topology we pay special attention to sets definable in a model-theoretic structure, particulary in an o-minimal structure. For example the sets definable in the ordered field of real numbers $\overline{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 1, 0)$ can be characterized as semi-algebraic sets, i.e. finite unions of subsets of $\mathbb{R}^n$ of the form $\{\overline{\mathrm{x}} \in \mathbb{R}^n \mid f_1(\overline{\mathrm{x}}) = \cdots = f_k(\overline{\mathrm{x}}) = 0 \ \wedge \ g_1(\overline{\mathrm{x}}) > 0 \ \wedge \cdots \wedge \ g_l(\overline{\mathrm{x}}) > 0\}$, where $f_i, g_j \in \mathbb{R}[X]$ are polynomials in the variables $x_1, \ldots, x_n$ and $\overline{\mathrm{x}} = (x_1, \ldots, x_n)$. These sets are the easiest example for an o-minimal structure over the ordered field $\overline{\mathbb{R}} = (\mathbb{R}, <, +, -, \cdot, 1, 0)$. But for talking about o-minimal structures we have to take a look at the definition.

Definition 0.0.1 (Wilkie)**.** A *structure* on $\mathbb{R}$ is a sequence $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$, where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$ such that:

(S1) $\mathcal{S}_n$ is a boolean algebra, i.e. it is closed under union, intersection and complement and contains the whole and the empty set;

(S2) $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

(S3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;

(S4) if $m \geq n, A \in \mathcal{S}_m$, then $\pi[A] \in \mathcal{S}_n$ where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first n coordinates.

$\mathcal{S}$ is called an *o-minimal* structure (over the real field) if

(S5) the boundary of every set in $\mathcal{S}_1$ is finite.

Many of the properties of semi-algebraic sets can be generalized to o-minimal expansions of $\overline{\mathbb{R}}$.

However, semi-algebraic sets are not enough to practize some analysis on the reals, so the question is how we can expand o-minimal structures while keeping the o-minimality.

Another famous application for o-minimal structures is the (sub)analytic topology and geometry. First of all there is an interest in objects defined over the category of subanalytic sets and maps. Now, van den Dries and Miller noticed in [vdDM96] that they can generalize the behaviour of the category of subanalytic sets in analytic geometric categories. They worked out that every analytic geometric category is connected to an o-minimal structure by identifying the analytic manifold $\mathbb{R}^n$ with an open subset of the projective space $\mathbb{P}^n(\mathbb{R})$. This is done via the map $(y_1, \ldots, y_n) \mapsto [1 : y_1 : \cdots : y_n]$ from $\mathbb{R}^n$ to $\mathbb{P}^n(\mathbb{R})$, which allows us to connect the structure $\mathcal{S}$ to a geometrical category $\mathcal{C}$ defined by $\mathcal{S} := \{X \subseteq \mathbb{R}^n \mid X \in \mathcal{C}(\mathbb{P}^n(\mathbb{R}))\}$. Hence, the geometrical category of subanalytic sets corresponds to the o-minimal structure $\mathbb{R}_{an}$. Look for more details in [vdDM96].

Thus, there is interest in o-minimal expansions of the ordered field of real numbers.

That is why we look also at Pfaffian functions, i.e. a finite sequence of functions which is closed under taking derivatives.

Definition 0.0.2. A C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ is called *Pfaffian function* if there exist C^1 functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ with $f = f_k$, such that for each $1 \leq i \leq k$ and $1 \leq j \leq n$, $\frac{\partial f_i}{\partial x_j}$ is expressible as a polynomial in $x_1, \ldots, x_n, f_1, \ldots f_i$.

We can expand structures by means of Pfaffian functions, for example we examine the definable sets in the structure $\mathbb{R}_{\exp}$ or in the structure $\mathbb{R}_{an,\exp}$, where we take the sets definable in $\overline{\mathbb{R}}$ by restricted analytic functions and by the exponential function. Wilkie proved in [Wil99] that the structure generated by $\overline{\mathbb{R}}$ expanded by all Pfaffian functions is again o-minimal.

A more general possibility to build up new structures is the so-called Pfaffian closure which is generated by adding Rolle leaves to an existing structure. Rolle leaves are manifolds on a given set and an 1-form with some nice properties. Speisegger proved in [Spe99] that this construction enables us to expand an o-minimal structure on the reals and preserve the o-minimality of the structure.

In this thesis we will develop a different way to prove this statement. It can be applied to the special case of Pfaffian functions, since the graph of any Pfaffian function is a Rolle leaf and so for example $\mathbb{R}_{\exp}$ is again an o-minimal structure or more general $\overline{\mathbb{R}}$ expanded by any Pfaffian function generates an o-minimal structure.

A powerful method to expand o-minimal structures is Wilkie's theorem of the complement. There we assume that we have a collection of sets which satisfies some properties of an o-minimal structure, such as containing semi-algebraic sets, the cartesian product, the intersection of two sets and the image under a linear bijection. Then we take the Charbonnel closure, which satisfies nearly all properties of an o-minimal structure. The only thing missing is that the Charbonnel closure is not necessarily closed under complementation. Wilkie's theorem of the complement states that an additional assumption, which says that every set in the collection can be written as a projection of a zero set of some function, is enough to verify the closure under complementation. This is the reason for the name "theorem of the complement ". In this diploma thesis we will examine and improve Wilkie's theorem of the complement by reducing the requirements on the functions.

This theorem can also be applied to the Pfaffian closure and using this we prove that the Charbonnel closure of the Pfaffian closure of any o-minimal structure is again an o-minimal structure. Therefore we follow some ideas of Karpinski and Macintyre in [KM99], but in their paper there are some mistakes and some proofs are not as obvious as they look at first sight. These proofs are carried out and corrected here.

Content

In the first chapter the subject is introduced and we give the general definitions of o-minimal structures and Wilkie's modification to o-minimal weak structures. These modificated structures, so-called weak structures, have some properties of structures. However, the o-minimality conditions for weak structures are quite stronger than those for structures. Hence for example the number of connected components of intersections with an affine hyperplane must be bounded. The requirement to an o-minimal structure $\mathcal{S}$ only expects that the number of connected components of sets in $\mathcal{S}$ and in $\mathbb{R}$ must be finite. We also recall the instrument of cell decomposition, which means dividing sets into cells, that are constructed with intervals and C^N functions. Furthermore, we prove that all (o-minimal) structures are (o-minimal) weak structures.

The second chapter deals with the Charbonnel closure, which is the closure of a weak structure under projection, union and algebraic closure. We compare different equivalent definitions of Wilkie in [Wil99] and Berarducci and Servi in [BS04] and recall the fact that if a weak structure is o-minimal, then the Charbonnel closure is an o-minimal weak structure as well.

In the third chapter we formulate an improved version of Wilkie's theorem of the complement and prove it in several steps, similar as Wilkie did in [Wil99]. We examine Wilkie's *determined by smooth functions* condition and the DC^N condition with C^N functions of Karpinski and Macintyre in [KM99] and obtain a so-called DPC^N condition by replacing the smooth functions by partially defined C^N functions with closed graph. So the DPC^N condition states that each set is a projection of a union of zero sets of partial defined C^N functions in the Pfaffian closure with closed graph. To prove the modified theorem, we first approximate the sets in the weak structure and the Charbonnel closure by C^N functions in the Charbonnel closure. Therefore we use the DPC^N condition and do an induction along the construction of the sets in the Charbonnel closure of our given weak structure. The projection case is the most interesting and complicated part, there we need to differentiate the functions to get the right approximant. The second step is to prove with the help of Sard's lemma that for every set X in the Charbonnel closure $\widetilde{\mathcal{S}}$ of our weak structure, there is a set in $\widetilde{\mathcal{S}}$ with empty interior containing the boundary of X. After that we can apply the same cell decomposition argument as Wilkie in [Wil99].

The fourth chapter deals with the converse of the theorem of the complement, i.e. that we can write every structure as Charbonnel closure of some o-minimal weak structure satisfying the DC^N condition. We simply prove that every o-minimal structure satisfies this condition. The Charbonnel closure now is the structure itself.

In the fifth and last chapter the theorem of the complement is applied to the Pfaffian closure. At the beginning we consider the basic definitions of Rolle leaves and of the Pfaffian closure of an o-minimal structure. In order to show that the Pfaffian closure is again an o-minimal structure, we apply the theorem of the complement, which we proved in Chapter 3. First we have to verify that the Pfaffian closure of an o-minimal structure is an o-minimal weak structure, what is done with the help of some results Speisegger proved in [Spe99]. The difficult part is to verify the DPC^N condition. Therefore we examine the cell decomposition of the basic sets and distingiush between open and closed cells. At last we put all functions together and obtain the DPC^N condition for an arbitrary set in the Pfaffian closure.

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Contents

1 Basics of O-Minimal (Weak) Structures

In this chapter we will recall the definitions of structures and o-minimality. Furthermore, we will introduce the concept of (o-minimal) weak structures, as Wilkie does in [Wil99]. In general, a weak structure has less properties than a structure. For instance it is not necessarily closed under complementation. O-minimality of a weak structure requires indeed some more assumptions than o-minimality of a structure. These assumptions are needed for the expansion of an o-minimal weak structure to an o-minimal structure in Wilkie's theorem of the complement. Additionally, we recall the instrument of cell decomposition for proofs on o-minimal structures in section 1.2. At last, we prove that every o-minimal structure is also an o-minimal weak structure.

1.1 Structures and Weak Structures, O-Minimality

The easiest o-minimal structure in Wilkie's sense contains all semi-algebraic sets, so semi-algebraic sets are a quite important concept for the following definitions.

Definition 1.1.1 (Semi-Algebraic Set)**.** A *semi-algebraic set* is a finite union of sets in $\mathbb{R}^n$ of the form

$$\{\overline{x} \in \mathbb{R}^n \mid f_1(\overline{x}) = \cdots = f_k(\overline{x}) = 0 \ \wedge \ g_1(\overline{x}) > 0 \ \wedge \cdots \wedge \ g_l(\overline{x}) > 0\},$$

where $f_i, g_j \in \mathbb{R}[X_1, \ldots, X_n]$ are polynomials in the variables $x_1, \ldots, x_n$ and $\overline{x} = (x_1, \ldots, x_n)$.

Convention 1.1.2. We will use the following conventions in the rest of the thesis.

- The *boundary* of a set A is the set $\partial A = \overline{A} - int(A)$, while $\overline{A}$ is the topological closure and $int(A)$ is the interior of A.
- Let π denote the natural projection to the first n coordinates, so $\pi : \mathbb{R}^m \to \mathbb{R}^n$. We write π_n^m if the indices are not clear from the context.

For structures and o-minimality, we will use the following definition by Wilkie, given in [Wil99].

Definition 1.1.3 (Wilkie)**.** A *structure* on $\mathbb{R}$ is a sequence $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$, where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$ such that for all $n, m \geq 1$:

(S1) $\mathcal{S}_n$ is a boolean algebra, i.e. it is closed under union, intersection and complement and contains the whole and the empty set;

(S2) $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

(S3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;

(S4) if $m \geq n, A \in \mathcal{S}_m$, then $\pi[A] \in \mathcal{S}_n$ where $\pi : \mathbb{R}^m \to \mathbb{R}^n$ is projection onto the first n coordinates.

The structure $\mathcal{S}$ is called *o-minimal* if

(S5) the boundary of every set in $\mathcal{S}_1$ is finite.

Remark 1.1.4. This notation differs from the usual model theory notation of an o-minimal structure. It is stronger, since it demands that all semi-algebraic sets are in the structure.

In Definition (2.1) in [vdD98], Chapter 1, van den Dries demands that all sets of the form $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_n\}$ are in $\mathcal{S}_n$, instead of (S2). According to van den Dries' definition it is sufficient for o-minimality that $\{(x, y) \in \mathbb{R}^2 \mid x < y\} \in \mathcal{S}_2$ and that the sets in $\mathcal{S}_1$ are exactly the finite unions of intervals and points.

Thus it is possible to prove that the semi-algebraic sets form an o-minimal structure in van den Dries' sense.

On the other hand the o-minimality structure conditions of Wilkie implies the classical o-minimality conditions, since $\{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_1 = x_n\}$ and $\{(x, y) \in \mathbb{R}^2 \mid x < y\}$ are semi-algebraic sets. Let A be a set in $\mathcal{S}_1$, in an o-minimal structure in Wilkie's sense. Then the boundary has to be finite by (S5). Now we check the classic o-minimality condition. Every connected component has at most two points in the boundary, so there are only finitely many connected components of A. A connected component in $\mathbb{R}$ must be a point or an interval, so A is a finite union of points and intervals. On the other hand a singleton $\{a\}$ is the zero set of the polynomial $x - a$ and an interval (a, b) is equal to $\{x \in \mathbb{R} \mid a - x < 0 \wedge x - b < 0\}$, so just another semi-algebraic set. So every finite union of points and intervals is in $\mathcal{S}_1$.

Thus Wilkie's definition is a specialized version on $\mathbb{R}$ of the o-minimal structure in [vdD98].

Example 1.1.5. The easiest example for an o-minimal structure $\mathcal{S}$ on $\mathbb{R}$ is given by the collection of all semi-algebraic sets.

Most properties of an o-minimal structure are easy to verify:

(S1) Finite unions are by definition in $\mathcal{S}$, intersections are easy, too. Furthermore, $\emptyset = \{\overline{\mathrm{x}} \mid \overline{\mathrm{x}} > 0 \wedge -\overline{\mathrm{x}} > 0\} \in \mathcal{S}$, the same with the whole set. The complement of any set is in $\mathcal{S}$: By $f(\overline{\mathrm{x}}) \neq 0 \Leftrightarrow f(\overline{\mathrm{x}}) < 0 \vee f(\overline{\mathrm{x}}) > 0$ and $g(\overline{\mathrm{x}}) \leq 0 \Leftrightarrow -g(\overline{\mathrm{x}}) > 0 \vee g(\overline{\mathrm{x}}) = 0$ and $(A \cap B)^C = A^C \cup B^C$ the claim can be verified with an easy calculation.

(S2) Trivial.

(S3) Trivial.

(S5) Let A be a semi-algebraic set in $\mathcal{S}$. If all occurring polynomials f_i are zero, then A has no boundary (since it is empty or whole $\mathbb{R}^n$), otherwise the boundary is a finite union of zero sets of polynomials. Furthermore zero sets of non-zero polynomials are finite.

It is more difficult to verify (S4), i.e. that a projection of a semi-algebraic set is again a semi-algebraic set. This is the statement of the Tarski-Seidenberg theorem; a proof can be found in [vdD98], Chapter 2.

The o-minimal structure theory helps us to characterize the definable sets of an ordered field as we see in the following corollary.

Corollary 1.1.6 (Corollary (2.11) in [vdD98], Chapter 2)**.** *The definable sets in the model-theoretic structure* $\overline{\mathbb{R}} = (\mathbb{R}, <, 0, 1, +, -, \cdot)$ *(definable with parameters) are exactly the semi-algebraic sets.*

Proof. The definable sets (with parameters) are the sets in the smallest structure containing the constants $0, 1$, the relation $<$ and the graphs of $+, -, \cdot$. The set $\{(x, y) \mid x < y\} = \{(x, y) \mid y - x > 0\}$ is semi-algebraic and so are $\{0\}, \{1\}$ and the graphs of $+, -, \cdot$. Since the semi-algebraic sets form a structure, the structure of the definable sets must be contained in the structure of the semi-algebraic sets.

Every semi-algebraic set is obviously definable and so containend in the structure of the definable sets. $\square$

Have a look at another example, presented in [vdD96].

Example 1.1.7. [Example (c) in [vdD96], Section 0] Look at the class of sets definable in the structure $\mathbb{R}_{an} := (\overline{\mathbb{R}}, (f))$, where f ranges over all restricted analytic functions, i.e. over all functions $f : \mathbb{R}^n \to \mathbb{R}$ such that $f \restriction [-1,1]^n$ is analytic and f is identically 0 outside $[-1,1]^n$. By Gabrielov's Theorem of the Complement $\mathbb{R}_{an}$ is model complete and by results of Łojasiewicz it is an o-minimal structure.

The o-minimality of the structure leads to a characterization of definable sets. The set $X \subseteq \mathbb{R}^m$ is definable in $\mathbb{R}_{an}$ if and only if for some n, some real polynomial p in $m+n$ variables and some analytic function $f : [-1,1] \to \mathbb{R}$ we have $Y = \{(\overline{x}, \overline{y}) \in \mathbb{R}^m \times [-1,1]^n \mid p(\overline{x}, \overline{y}) = 0 \ \wedge \ f(\overline{y}) = 0\}$ and $X = \pi^{m+n}_m[Y]$.

For the references to these facts see Example (c) in [vdD96], Section 0.

Example 1.1.8. Lessly, we look at the collection of sets definable in $\mathbb{R}_{an,\exp}$, where we expand the structure $\mathbb{R}_{an}$ by the exponential function $x \mapsto e^x$.

We will give a proof of the fact that these sets form an o-minimal structure in Chapter 5. It is an example for the application of Wilkie's theorem of the complement and the Pfaffian closure.

In Wilkie's theorem of the complement, the basic idea is that some properties of a (weak) structure are easier to verify than the fact that the structure contains a complement of any set. Thus we introduce a new notation for an o-minimal weak structure, that has basic properties of a structure. For the o-minimality of a weak structure we need some more properties than in an o-minimal structure.

Definition 1.1.9. Suppose $n \geq 1$ and let $A \subseteq \mathbb{R}^n$. Then $cc(A)$ denotes the number of connected components of A.

Let $\gamma(A)$ be the smallest natural number N with the following property: For any affine subspace X of $\mathbb{R}^n$, the number of connected components $cc(A \cap X)$ is restricted by N. If no such N exists we define $\gamma(A) = +\infty$.

Definition 1.1.10 (Definition 1.1 in [Wil99]). A *weak structure* is a sequence $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$, where each $\mathcal{S}_n$ is a collection of subsets of $\mathbb{R}^n$, such that for all $n, m \geq 1$:

(WS1) if $A, B \in \mathcal{S}_n$, then $A \cap B \in \mathcal{S}_n$;

(WS2) $\mathcal{S}_n$ contains every semi-algebraic subset of $\mathbb{R}^n$;

(WS3) if $A \in \mathcal{S}_n$ and $B \in \mathcal{S}_m$, then $A \times B \in \mathcal{S}_{n+m}$;

(WS4) if $A \in \mathcal{S}_m$, then $\sigma[A] \in \mathcal{S}_n$ where $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection.

A weak structure $\mathcal{S}$ is called *o-minimal* if, in addition to (WS1)-(WS4), it satisfies:

(WS5) for all $n \geq 1$ and $A \in \mathcal{S}_n$ it is $\gamma(A) < \infty$;

(WS6) for all $n \geq 1$ and $A \in \mathcal{S}_n$, there exist an $m \geq n$ and a closed set $B \in \mathcal{S}_m$ such that $A = \pi_n^m[B]$.

Remark 1.1.11. In some literature as in [Max98] an o-minimal weak structure is called *tame* $\mathbb{R}$*-system*.

The following lemma and Theorem 1.3.1 show that this definition of an (o-minimal) weak structure is in line with the definition of an (o-minimal) structure.

Lemma 1.1.12. *Every structure is a weak structure.*

Proof. Let $\mathcal{S}$ be a structure. (WS2) and (WS3) are exactly the same conditions as (S2) and (S3). (WS1) follows from the fact, that $\mathcal{S}$ is a boolean algebra.

To show (WS4) let $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection and let A be in $\mathcal{S}_n$. The set $\{(\overline{\mathrm{y}}, \overline{\mathrm{x}}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \overline{\mathrm{y}} - \sigma(\overline{\mathrm{x}}) = 0\}$ is a semi-algebraic subset of $\mathbb{R}^{n+n}$ and so in $\mathcal{S}_{n+n}$. Look at

$$\pi_n^{n+n}[\{(\overline{y}, \overline{x}) \in \mathbb{R}^n \times \mathbb{R}^n \mid \overline{\mathrm{y}} - \sigma(\overline{\mathrm{x}}) = 0\} \cap (\mathbb{R}^n \times A)] = \sigma[A].$$

This set is constructed by Cartesian product (WS3), intersection (WS1) and projection (S4) of sets in $\mathcal{S}$ and so itself a set in $\mathcal{S}_n$. □

Remark 1.1.13. In [BS04] (WS2) is replaced by the weaker condition that $\mathcal{S}_n$ contains every set of the form $\{\overline{\mathrm{x}} \in \mathbb{R}^n \mid p_1(\overline{\mathrm{x}}) = 0, \ldots, p_m(\overline{\mathrm{x}}) = 0\}$, where $p_1, \ldots, p_m \in \mathbb{Z}[\overline{\mathrm{x}}]$. However, this does not affect the following proofs.

1.2 Cell Decomposition

The cell decomposition is a basic instrument to prove results about structures; we will recall it in this section, since we will use it several times. First, we give the definitions of a cell and a (cell) decomposition, then we state the cell decomposition theorem. Here, we use the version with C^N functions presented in [vdDM96].

Definition 1.2.1. Let $S \subseteq \mathbb{R}^n \times \mathbb{R}^k$, $\overline{y} \in \mathbb{R}^k$. Define the *section*

$$S_{\overline{y}} := \{\overline{\mathrm{x}} \in \mathbb{R}^n \mid (\overline{\mathrm{x}}, \overline{y}) \in S\}.$$

We define cells, which are some kind of basic building blocks in an o-minimal structure, in the following.

Definition 1.2.2 (Definition (2.3) in [vdD98], Chapter 3 with C^N cells)**.** Let $FC^N = \{f : C \to \mathbb{R} \mid f \text{ is } C^N \wedge f \in \mathcal{S}\} \cup \{\infty, -\infty\}$. We say that $f \in \mathcal{S}$, if the graph of f, $\Gamma(f) \in \mathcal{S}$. Sometimes we write $f = \pm\infty$ for the function which satisfies $f(\overline{\mathrm{x}}) = \pm\infty$ for all $\overline{\mathrm{x}}$.

Define a C^N *cell* C in $\mathbb{R}^n$ by the following inductive definition.

- For every N, a C^N cell in $\mathbb{R}$ is either a singleton $\{a\} \in \mathcal{S}_1$ or an open interval $(a, b) \in \mathcal{S}_1$.
- Let C be an arbitrary C^N cell in $\mathbb{R}^{n-1}$. Then a C^N *cell* in $\mathbb{R}^n$ is either a graph of a function $\Gamma(f)$, where $f : C \to \mathbb{R}$ is C^N and in $\mathcal{S}$ or a set of the form $(f, g)_C$, where $f, g \in FC^N$ such that $f < g$. We define

$$(f, g)_C := \{(\overline{\mathrm{x}}, y) \mid \overline{\mathrm{x}} \in C, f(\overline{\mathrm{x}}) < y < g(\overline{x})\}.$$

Definition 1.2.3 (Definition (2.10) in [vdD98], Chapter 3)**.** A C^N *(cell) decomposition* of $\mathbb{R}^n$ is a special kind of partition of $\mathbb{R}^n$ into finitely many cells. The definition works by induction on n:

- A *decomposition* of $\mathbb{R}$ is a collection

$$\big\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, \infty), \{a_1\}, \ldots, \{a_k\}\big\},$$

  where $a_1, \ldots, a_k \in \mathbb{R}$.
- A decomposition of $\mathbb{R}^{n+1}$ is a finite partition of $\mathbb{R}^{n+1}$ into C^N cells C such that the projections $\pi_n^{n+1}[C]$ form a decomposition of $\mathbb{R}^n$.

Definition 1.2.4. Let $\mathcal{S}$ be an o-minimal structure and $A \in \mathcal{S}_n$. Let $\mathcal{D}$ be a C^N decomposition of $\mathbb{R}^n$. The decomposition $\mathcal{D}$ is called *compatible with* A (or it *partitions* the set A as in [vdD98]) if for every cell $C \in \mathcal{D}$ holds $C \subseteq A$ or $C \cap A = \emptyset$.

Remark 1.2.5. If the decomposition $\mathcal{D}$ is compatible with $A \in \mathcal{S}$, we can write $A = \bigcup\{C \mid C \text{ cell in } \mathcal{D} \ \wedge \ C \subseteq A\}$.

Theorem 1.2.6. *[Cell Decomposition (4.2) in [vdDM96]] Let $\mathcal{S}$ be an o-minimal structure.*

a) Given $A_1, \ldots, A_k \in S_n$, there is a C^N decomposition of $\mathbb{R}^n$ compatible with $A_1, \ldots, A_k$.

b) For every function $f : A \to \mathbb{R}$ belonging to $\mathcal{S}$ where $A \subseteq \mathbb{R}^n$, there is a C^N decomposition $\mathcal{D}$ of $\mathbb{R}^n$ compatible with A such that $f|D : D \to \mathbb{R}$ is of class C^N for each $D \in \mathcal{D}$ with $D \subseteq A$.

1.3 O-Minimal Structures are O-Minimal Weak Structures

In this section we will prove the continuity of the property of o-minimality in a weak structure. Hence we prove the o-minimality of a structure in the weak structure sense. Furthermore we give some technical lemmas to check the membership of a set in a (weak) structure.

Theorem 1.3.1. *Every o-minimal structure $\mathcal{S}$ is an o-minimal weak structure.*

To prove this theorem, we have to verify the two conditions for o-minimality in a weak structure.

Lemma 1.3.2. *Every o-minimal structure satisfies (WS5).*

Proof. We can define any set A in an o-minimal structure by a formula of the corresponding language. (If the structure consists of definable sets in a certain language, the language is clear, otherwise we can construct a language by taking additional relation symbols for sets in the structure.) An affine subspace X is in general a zero set of a linear polynomial $p(\overline{\mathrm{x}}, \overline{\mathrm{a}}) = a_0 + a_1 \cdot x_1 + \cdots + a_n \cdot x_n$, where $\overline{\mathrm{a}}$ are parameters in the structure. This zero set of a polynomial can be described by a formula $\phi(\overline{\mathrm{x}}, \overline{\mathrm{a}})$. So the number of connected components of $A \cap X$ is bounded by the following result of Knight, Pillow and Steinhorn. □

Theorem 1.3.3. *[Theorem 0.3 in [KPS86]]*

a) Let M be o-minimal, i.e. the definable sets in M form an o-minimal structure and let $\phi(\overline{\mathrm{x}}, \overline{\mathrm{y}})$ be any formula of the language of M. Then there is a $K \in \mathbb{N}$ such that for any $\overline{b} \in M^m$, the set $X(\overline{\mathrm{x}}, \overline{b})^M = \{\overline{a} \in M^n \mid M \vDash \phi(\overline{a}, \overline{b})\}$ has at most K definably connected components.

b) If M is a o-minimal expansion of $(\mathbb{R}, <)$, then we can replace definably connected by connected.

Lemma 1.3.4. *Every o-minimal structure satisfies (WS6).*

To prove the above lemma, we need some rules to show that some sets are in a structure or the Charbonnel Closure of a weak structure. In the next lemmas we state some rules that will also help us in the next chapters.

Lemma 1.3.5. *Let $\mathcal{S}$ be an o-minimal weak structure.*

a) *Let* $f : B \to \mathbb{R}$, $B \subseteq \mathbb{R}^n$ *be in* $\mathcal{S}$ *and let* $p \in \mathbb{R}[X_1, \dots, X_{n+1}]$ *be a polynomial. Then* $\{(\overline{x}, f(\overline{x})) \in B \times \mathbb{R} \mid p(\overline{x}, f(\overline{x})) = 0\}$ *is in* $\mathcal{S}$*. In general in semi-algebraic sets we can replace arbitrary many coordinates by values of* $\mathcal{S}$*-functions while maintaining the property of being in* $\mathcal{S}$*.*

b) *Every polynomial function is in* $\mathcal{S}$*, i.e. its graph is in* $\mathcal{S}$*.*

c) *If a function* $f > 0$ *is in* $\mathcal{S}$*, then so is the function* $\frac{1}{f}$*.*

Proof. a) We construct the forementioned set as follows:

$$\{(\overline{x}, f(\overline{x})) \in B \times \mathbb{R} \mid p(\overline{x}, f(\overline{x})) = 0\} = \Gamma(f) \cap \{(\overline{x}, y) \mid p(\overline{x}, y) = 0\}.$$

So we intersect a semi-algebraic set which is in $\mathcal{S}$ by (WS2) with $\Gamma(f)$, which is in $\mathcal{S}$ by assumption and we can apply (WS1).

b) The graph of a polynomial is a semi-algebraic set.

c) The graph of the function $\Gamma(\frac{1}{f}) = \{(\overline{x}, y) \mid \frac{1}{f(\overline{x})} = y\} = \{(\overline{x}, y) \mid f(\overline{x}) \cdot y = 1\}$ is by a) in $\mathcal{S}$. □

Lemma 1.3.6. *Let* $\mathcal{S}$ *be an o-minimal structure.*

a) *By interpreting the logical operators as terms of operations (union, intersection) we can apply logical operators to define a set in* $\mathcal{S}$*.*

b) *If* $A \in \mathcal{S}$*, then the topological closure* $\overline{A} \in \mathcal{S}$*.*

c) *If* $g : B \times U \to \mathbb{R}$ *is a continuous function in* $\mathcal{S}$*, where* $B \subseteq \mathbb{R}^n$ *and* $U \subseteq \mathbb{R}^m$ *such that* $g \geq 0$*, then* $f : B \times \mathbb{R} \to \mathbb{R}$*, defined by* $f(\overline{x}) = inf\{g(\overline{x}, \overline{y}) \mid \overline{y} \in U\}$ *is again in* $\mathcal{S}$*.*

d) *For* $S \subseteq B \times A$*, we define* $S_a := \{b \in B \mid (b, a) \in S\}$*. If* $S, B \in \mathcal{S}$*, then so is* S_a*.*

Proof. a) Clear.

b) Look at Lemma (3.4) in [vdD98], Chapter 1.

c) Now define $A := \pi^{2n+1}_{n+1}[\{(\overline{x}, r, \overline{y}) \mid \overline{x} \in B \wedge \overline{y} \in U \wedge r = g(\overline{x}, \overline{y})\}]$. By a) this is a set in $\mathcal{S}$. Next, we want to minimize r, the value of the function. Therefore look at

$$\begin{aligned} \Gamma(f) &:= \{(\overline{x}, r) \in \overline{A} \mid r = \inf\{g(\overline{x}, \overline{y}) \mid \overline{y} \in U\}\} \\ &= \{(\overline{x}, r) \in \overline{A} \mid \forall r' \ (\overline{x}, r') \in A \Rightarrow r < r'\} \end{aligned}$$

As this is realizable as projection of an intersection of members in $\mathcal{S}$ it is again in $\mathcal{S}$. The infimum exists everywhere, since $g \geq 0$ and $\mathbb{R}$ is complete.

d) Note that $S_a = \pi[B \times \{a\} \cap S)]$ by definition. $\square$

We can now prove (WS6) for open sets.

Lemma 1.3.7. *Let $\mathcal{S}$ be an o-minimal structure. For every open set $U \in \mathcal{S}_n$ there exists a closed set $B \in \mathcal{S}_{n+1}$ such that $\pi[B] = U$ and $B = \Gamma(f)$ for a C^N function $f : U \to \mathbb{R}$, which is in $\mathcal{S}$.*

Proof. Notate U^C for the complement of U in $\mathbb{R}^n$. Define $f : U \to \mathbb{R}$ through

$$f(\overline{x}) = d(\overline{x}, U^C)^2 := \inf\{|\,\overline{x} - \overline{y}\,|^2 \mid \overline{y} \in \mathbb{R}^n - U\}.$$

Define $B := \Gamma(\frac{1}{f})$. So $\pi[B] = U$ and obviously $\frac{1}{f}$ is a C^N function on U.

Furthermore, $B \in \mathcal{S}_{n+1}$: By Lemma 1.3.6, b) and c) we have $f \in \mathcal{S}$. An appliance of Lemma 1.3.5 c) supplies $B \in \mathcal{S}_{n+1}$.

Additionally B is closed: Let $\overline{x}_n \to \overline{x} \in \partial U$. Then $\frac{1}{f(\overline{x}_n)} \to \infty$. By Remark 1.3.8 $\Gamma(\frac{1}{f}) = B$ is closed. $\square$

The following remark is often needed to check, whether a function has a closed graph. That is why we state and prove it separately here.

Remark 1.3.8. Let U be an open set in $\mathbb{R}^n$ and $f : U \to \mathbb{R}^k$ be a C^1 function, then the following holds: $\Gamma(f)$ is (topological) closed if and only if for all sequences $(\overline{x}_n)_n$ with $\overline{x}_n \in U$ and $\overline{x}_n \to \overline{x} \in \partial U$ it is $|f(\overline{x}_n)| \to \infty$.

Proof. $\Leftarrow$: Let $(\overline{x}_n, \overline{y}_n) \in \Gamma(f)$ such that $(\overline{x}_n, \overline{y}_n) \to (\overline{x}, \overline{y}) \in \mathbb{R}^{n+k}$. We show that $(\overline{x}, \overline{y}) \in \Gamma(f)$. Assume $\overline{x} \notin U$, so $\overline{x} \in \partial U$. Then $|\,\overline{y}_n\,| = |f(\overline{x}_n)| \to \infty$, but $|\,\overline{y}_n\,| \to |\,\overline{y}\,| < \infty$. Contradiction. So $\overline{x} \in U$ and since f is continuous $(\overline{x}_n, y_n) = (\overline{x}_n, f(\overline{x}_n)) \to (\overline{x}, f(\overline{x})) \overset{\text{lim unique}}{=} (\overline{x}, \overline{y}) \in \Gamma(f)$.

$\Rightarrow$: Let $\Gamma(f)$ be closed. Assume $\overline{x}_n \to \overline{x}$ with $\overline{x}_n \in U$, $\overline{x} \in \partial U$ and $|f(\overline{x}_n)| \to z < \infty$. So the sequence $(\overline{x}_n, f(\overline{x}_n))$ is bounded and by the Theorem of Bolzano-Weierstrass there exists a convergent subsequence $(\overline{x}_{n_k}, f(\overline{x}_{n_k})) \to (\overline{x}, \overline{y})$. Since $\Gamma(f)$ is closed, we have $(\overline{x}, \overline{y}) \in \Gamma(f)$, particularly $\overline{x} \in U$. However $\overline{x} \notin U$ (since U is open), so $(\overline{x}, \overline{y}) \notin \Gamma(f)$. Contradiction. $\square$

Lemma 1.3.9. *Let C be a C^N cell of $\mathcal{S}$ in $\mathbb{R}^n$. Then there exists a closed set $B \in \mathcal{S}_{n+1}$ such that $\pi[B] = C$ and B is a graph of a C^N function in $\mathcal{S}$ defined on C.*

Proof. Every C^N cell is by a coordinate projection C^N homeomorphic to an open cell in $\mathbb{R}^k$ for some $k < n$. (Look at Lemma (2.7) in [vdD98], Chapter 3.) So let $\phi : C \to D \subseteq \mathbb{R}^k$ be this projection. The inverse function ϕ^{-1} is in $\mathcal{S}$, since all functions defining cells are in $\mathcal{S}$.

Since D is an open cell, there exists by Lemma 1.3.7 a closed set B' in $\mathcal{S}_{k+1}$ such that $\pi[B'] = D$ and B' is a graph of a C^N function. Define $B := \{(\overline{\mathrm{x}}, y) \in C \times \mathbb{R} \mid (\phi(\overline{\mathrm{x}}), y) \in B'\}$. Obviously, $B \in \mathcal{S}$ is still a graph of a C^N function, since ϕ^{-1} is C^N, and $\pi[B] = C$. □

Now we prove Lemma 1.3.4.

Proof. Let $A \in \mathcal{S}_n$. Do a finite cell decomposition of $\mathbb{R}^n$ compatible with A. For each cell C of this decomposition there exists by Lemma 1.3.9 a closed set $B_C \subseteq \mathbb{R}^{n+1}$ such that $\pi[B_C] = C$. The cell decomposition is compatible with A, so each cell C is contained in A or disjoint from A. Hence we can write $A = \bigcup\{C \mid C \text{ cell } \wedge C \subseteq A\}$. Take $B := \bigcup\{B_C \mid C \subseteq A\}$. Of course, B is closed, since the cell decomposition is finite and the B_C are closed. Furthermore $B \in \mathcal{S}$, since the B_C are in $\mathcal{S}$. Obviously, $\pi[B] = A$. □

Now Theorem 1.3.1 follows from Lemma 1.1.12, which proves that each structure is a weak structure and Lemmas 1.3.2 and 1.3.4.

2 The Charbonnel Closure

The Charbonnel closure is an expansion of an o-minimal weak structure that equips it with additional operations. So in the Charbonnel closure we can take the union, the topological closure and the projection of sets. It preserves the properties of an o-minimal weak structure and hence we can use it to build up an o-minimal structure in later chapters.

2.1 Definitions of the Charbonnel Closure

First, we cite two different definitions of the Charbonnel Closure. Then we show that the definitions are equivalent in the case, where we need them.

Wilkie uses in [Wil99] the following definition to prove his theorem of the complement.

Definition 2.1.1 (Definition (1.3) in [Wil99]). Let $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ be a weak structure. Define:

a) $\mathcal{S}^u = \langle \mathcal{S}^u_n : n \geq 1 \rangle$ where $\mathcal{S}^u_n := \{\bigcup_{i=1}^p A_i \mid p \geq 1, A_1, \ldots, A_p \in \mathcal{S}_n\}$;

b) $\mathcal{S}^{pr} = \langle \mathcal{S}^{pr}_n : n \geq 1 \rangle$ where $\mathcal{S}^{pr}_n := \{\pi^m_n[A] \mid m \geq n, A \in \mathcal{S}_m\}$;

c) $\mathcal{S}^{cl} \langle \mathcal{S}^{cl}_n : n \geq 1 \rangle$ where $S^{cl}_n := \{A_0 \cap \bigcap_{i=1}^p \overline{A_i} \mid p \geq 0, A_0, \ldots, A_p \in \mathcal{S}_n\}$.

Let $\mathcal{S}^{(0)} := \mathcal{S}$ and $\mathcal{S}^{(i+1)} := ((\mathcal{S}^{(i)u})^{pr})^{cl}$ for $i \geq 0$. The *Charbonnel closure* is defined as $\widetilde{\mathcal{S}} := \bigcup_{i \geq 0} \mathcal{S}^{(i)}$.

One of the central facts we use in our proof of the theorem of the complement is that a Charbonnel closure is again an (o-minimal) weak structure if the basic structure has these properties.

Lemma 2.1.2 (Charbonnel, Lemma (1.4) in [Wil99]). *If $\mathcal{S}$ is a weak structure, then so are $\mathcal{S}^u, \mathcal{S}^{pr}, \mathcal{S}^{cl}$. If further, $\mathcal{S}$ is o-minimal, then $\mathcal{S}^u, \mathcal{S}^{pr}, \mathcal{S}^{cl}$ are, too.*

Proof. Let $\mathcal{S}$ be a weak structure.

For $\mathcal{S}^u$ (WS1) to (WS4) are trivial. If $\mathcal{S}$ is o-minimal, the boundary of connected components in (WS5) is simply the sum of both boundaries. Let now $A, B \in \mathcal{S}$; then there exist by (WS6) $C, D \in \mathcal{S}$ such that $A = \pi_n^{n+m}[C]$ and $B = \pi_n^{n+m'}[D]$. Without loss of generality, we can assume $m' \leq m$, thus $B \cap (D \times \mathbb{R}^{m-m'})$ is a closed set (with (WS1) to (WS4) in $\mathcal{S}$) satisfying (WS6) for $A \cup B$.

The expansion $\mathcal{S}^{pr}$ satisfies obviously (WS2) to (WS4). So check (WS1): Let $A, B \in \mathcal{S}^{pr}$, i.e. $A = \pi_n^{n+m}[C]$ and $B = \pi_n^{n+m'}[D]$. Then we have $A \cap B = \{(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}) \mid (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in C \wedge (\overline{\mathrm{x}}, \overline{\mathrm{z}}) \in D\}$, i.e. $A \cap B = \pi_n^{n+m+m'}[C \times \mathbb{R}^{m'} \cap \sigma[D \times \mathbb{R}^m]]$, where the permutation σ swaps $\overline{\mathrm{y}}$ and $\overline{\mathrm{z}}$. This is obviously a set in $\mathcal{S}^{pr}$.

If further $\mathcal{S}$ is o-minimal, (WS6) is obvious and (WS5) holds by the proof of Lemma (1.6) in [Max98].

Again, for $\mathcal{S}^{cl}$ (WS1) and (WS2) hold by definition, (WS3) is trivial and (WS4) follows from the fact that linear bijections preserves convergence of sequences. If $\mathcal{S}$ is o-minimal, (WS5) and (WS6) for $\mathcal{S}^{cl}$ are proved in Lemma (1.8) in [Max98]. □

Corollary 2.1.3. *If $\mathcal{S}$ is an (o-minimal) weak structure, then so is $\widetilde{\mathcal{S}}$.*

Proof. This fact follows by induction out of the above lemma. □

In the following we will not only use the definition above, but also the following definition of the Charbonnel closure, which is given in [BS04]. Their definition allows us to follow the proof of the theorem of the complement given in [BS04], which is more clear than the one in [Wil99].

Definition 2.1.4. [Definition (4.4) in [BS04]] Let $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$. The *Charbonnel closure* is the smallest set $\widetilde{\mathcal{S}}$ satisfying the following conditions.

a) Ch(base): $\widetilde{\mathcal{S}}_n$ is a collection of subsets of $\mathbb{R}^n$ and $\mathcal{S}_n \subseteq \widetilde{\mathcal{S}}_n$ for each n;

b) Ch($\cup$): for $A, B \in \widetilde{\mathcal{S}}$ holds $A \cup B \in \widetilde{\mathcal{S}}$;

c) Ch($\cap_l$): if $A \in \widetilde{\mathcal{S}}_n$ and $L \subseteq \mathbb{R}^n$ is a zero set of linear polynomials with coefficients in $\mathbb{Z}$ (L is called $\mathbb{Z}$-*affine set*), then $A \cap L \in \widetilde{\mathcal{S}}_n$;

d) Ch(π): if $A \in \widetilde{\mathcal{S}}_{n+k}$, then the projection $\pi_n^{n+k}[A] \in \widetilde{\mathcal{S}}_n$;

e) Ch($\overline{\mathrm{A}}$): if $A \in \widetilde{\mathcal{S}}_n$, then the topological closure $\overline{A} \in \widetilde{\mathcal{S}}_n$.

Lemma 2.1.5. *If $\mathcal{S}$ is an o-minimal weak structure, then the two definitions of $\widetilde{\mathcal{S}}$ are equivalent.*

Proof. Denote the collection given by the first definition by $\widetilde{\mathcal{S}}^1$, the one given by the second definition by $\widetilde{\mathcal{S}}^2$. We show that both collections are closed under the operations of the other one.

It holds $\widetilde{\mathcal{S}}^1 \subseteq \widetilde{\mathcal{S}}^2$: By Ch($\cup$) and Ch($\pi$) finite unions and projections are also in $\widetilde{\mathcal{S}}^2$. So let $A = A_0 \cap \bigcap_{i=1}^p \overline{A_i} \in \widetilde{\mathcal{S}}^1_n$ such that $A_0, \ldots, A_p \in \widetilde{\mathcal{S}}^2_n$. By Lemma (4.8) in [BS04], the intersection of two sets in $\widetilde{\mathcal{S}}^2$ is again in $\widetilde{\mathcal{S}}^2$. This fact and Ch($\overline{\text{A}}$) supply $A \in \widetilde{\mathcal{S}}^2$.

We have also $\widetilde{\mathcal{S}}^2 \subseteq \widetilde{\mathcal{S}}^1$: It is clear, that sets constructed by Ch(base),Ch(π) and Ch($\cup$) are in $\widetilde{\mathcal{S}}^1$. Let $A \in \widetilde{\mathcal{S}}^1_n$. The topological closure $\overline{\text{A}} = \mathbb{R}^n \cap \overline{\text{A}}$ is by definition in $\widetilde{\mathcal{S}}^1$. Lastly, assume L is an $\mathbb{Z}$-affine set. Hence L is the zero set of a polynomial, so it is a semi-algebraic set and in $\widetilde{\mathcal{S}}^1$. Additionally, $L = \overline{L}$, so $A \cap L = A \cap \overline{L}$ is in $\widetilde{\mathcal{S}}^1$. □

2.2 Induction on the Charbonnel Closure

The following definition of Berarducci and Servi helps us to do an induction along the construction of a set in the Charbonnel closure. We use it to prove a modified version of Wilkie's theorem of the complement in the next chapter.

Definition 2.2.1 (Definition (4.5) in [BS04]). A *Ch-description* of $A \in \widetilde{\mathcal{S}}$ is an expression, that illustrates one of the possible ways to obtain A from sets in $\mathcal{S}$ using the Ch-operations in Definition 2.1.4.

First fix a set of symbols (called labels) Σ with the same cardinality as $\bigcup_n \mathcal{S}_n$ and a surjection from Σ to $\bigcup_{n\in\mathbb{N}} \mathcal{S}_n$, so that every set $A \in \mathcal{S}$ has a label $\mathfrak{A} \in \Sigma$. For every set $A \in \mathcal{S}$ its Ch-description is the corresponding label $\mathfrak{A}$.

Let $B, C \in \widetilde{\mathcal{S}}$ with Ch-descriptions $\mathfrak{B}$, $\mathfrak{C}$. Then

- $\mathfrak{B} \cup \mathfrak{C}$ is a Ch-description for $B \cup C$;
- $\mathfrak{B} \cap L$ is a Ch-description for $B \cap L$ (L $\mathbb{Z}$-affine set);
- $\pi_n^{n+k}\, \mathfrak{B}$ is a Ch-description for $\pi_n^{n+k}[B]$;
- $\overline{\mathfrak{B}}$ is a Ch-description for $\overline{B}$.

The label set Σ must not contain any of these constructed strings. Hence we obtain a surjection from the set containing all Ch-descriptions to $\widetilde{\mathcal{S}}$, by mapping a Ch-description $\mathfrak{A}$ to the corresponding set A.

Note that the Ch-description of a set is not unique.

Definition 2.2.2. [Definition (4.5) in [BS04]] The *rank* ρ of a Ch-description is defined as follows.

- $\rho(\mathfrak{A}) = 0$ if the Ch-description of A is a label in Σ;
- $\rho(\mathfrak{B} \cup \mathfrak{C}) = 1 + \max\{\rho(\mathfrak{B}), \rho(\mathfrak{C})\}$;
- $\rho(\mathfrak{B} \cap L) = 1 + \rho(\mathfrak{B})$;
- $\rho(\pi_n^{n+k}\mathfrak{B}) = 1 + \rho(\mathfrak{B})$;
- $\rho(\overline{\mathfrak{B}}) = 4 + \rho(\mathfrak{B})$.

The reason for the bigger factor for the algebraic closure will get clear in the main proof of Chapter 3.

Remark 2.2.3. In [BS04], the rank of any set $A \in \widetilde{\mathcal{S}}$ is defined as the least possible rank of a Ch-description of A. Here it is enough (and more descriptive) to work directly with the rank of Ch-descriptions.

We need two more facts about the rank of more complicate sets and their descriptions, given in [BS04], Chapter 4.

Lemma 2.2.4. *[Lemma (4.7) in [BS04]] Let $\mathfrak{A}, \mathfrak{B}$ be Ch-descriptions for some sets $A, B \in \widetilde{\mathcal{S}}$. Then $A \times B \in \widetilde{\mathcal{S}}$ and there exists a Ch-description $\mathfrak{C}$ of $C = A \times B$ such that $\rho(\mathfrak{C}) \leq \rho(\mathfrak{A}) + \rho(\mathfrak{B})$.*

Proof. Look at the proof of Lemma (4.7) in [BS04]. □

Lemma 2.2.5. *[Lemma (4.8) in [BS04]] Let $\mathfrak{A}, \mathfrak{B}$ be Ch-descriptions for $A, B \in \widetilde{\mathcal{S}}$. Then $A \cap B \in \widetilde{\mathcal{S}}$ and there exists a Ch-description $\mathfrak{C}$ of $C = A \cap B$ such that $\rho(\mathfrak{C}) \leq 2 + \rho(\mathfrak{A}) + \rho(\mathfrak{B})$.*

Proof. Look at the proof of Lemma (4.8) in [BS04]. □

3 Generalization of Wilkie's Theorem of the Complement

In this chapter we prove a generalized version of Wilkie's theorem of the complement.

To do this we have a closer look at the proof of the theorem of the complement given in [KM99] and we explain in detail how we change the proof of Wilkie's theorem of the complement given in [BS04]. We generalize the definition of DC^N for all N given in [KM99]. This definition provides a general description of sets in a weak structure $\mathcal{S}$ as projections of zero sets of functions. We replace the functions in $\mathcal{S}$ by partial defined functions with closed graph in $\widetilde{\mathcal{S}}$, which is a weaker assumption on the sets in $\mathcal{S}$.

The further generalization of the theorem of the complement is necessary, since in the proof of [KM99] that the Pfaffian closure satisfies the DC^N condition for all N not everything is correct. In particular, it seems to be not easy or impossible to prove that there is a total C^N function satisfying the DC^N condition for all N. To fix this issue, we introduce the weaker condition working with partial functions. It is possible to verify this property for the Pfaffian closure as we will see in the last part of Chapter 5.

3.1 Definitions and Formulation of the Theorem of the Complement

The basic property used in Wilkie's theorem of the complement is given in the next definition.

Definition 3.1.1 (Definition (1.7) in [Wil99]). A prestructure $\mathcal{S} = \langle \mathcal{S}_n : n \geq 1 \rangle$ is *determined by its smooth functions* (DSF) if, for each $n \geq 1$ and $A \in \mathcal{S}_n$, there exist an $m \geq n$ and a C^∞ function $f : \mathbb{R}^m \to \mathbb{R}$ with $f \in \mathcal{S}$ such that $A = \pi_n^m[Z(f)]$ where $Z(f) = \{\overline{x} \in \mathbb{R}^m \mid f(\overline{x}) = 0\}$ is the zero set of f.

Karpinkski and Macintyre prove the theorem of the complement under the following assumption, which weakens the DSF property.

Definition 3.1.2 (Definition (1) in [KM99])**.** A prestructure $\langle \mathcal{S}_n : n \geq 1 \rangle$ satisfies the *DC^N condition (determined by C^N functions) for all N* if for each $A \in \mathcal{S}_n$, there exists an $m \geq n$ such that for each N, there exists a C^N function $f_N : \mathbb{R}^m \to \mathbb{R}$ in $\mathcal{S}$ such that $A = \pi_n^m[Z(f_N)]$.

The next definition is the generalization to some partial functions, which we will need in the following version of the theorem of the complement.

Definition 3.1.3 (Partial DC^N Condition)**.** A prestructure $\langle \mathcal{S}_n : n \geq 1 \rangle$ satisfies *DPC^N (determined by partial C functions) for all N* if for each $A \in \mathcal{S}_n$, there exists an $m \geq n$ such that for each N, there exist finitely many functions $f_1, \ldots, f_r$ such that $A = \pi_n^m[Z(f_1) \cup \cdots \cup Z(f_r)]$ where for each $i = 1, \ldots, r$ the functions $f_i : U_i \to \mathbb{R}$ are C^N and in $\widetilde{\mathcal{S}}$ with $\Gamma(f_i)$ closed and domains U_i, which are open subsets of $\mathbb{R}^m$ for $i = 1, \ldots, r$.

Now we state the modified theorem of the complement. We will prove it in the rest of this chapter.

Theorem 3.1.4. *Suppose $\mathcal{S}$ is an o-minimal weak structure satisfying DPC^N for all N. Then $\widetilde{\mathcal{S}}$ is an o-minimal structure and the smallest structure containing $\mathcal{S}$.*

Recall Corollary 2.1.3, which states that if we take the Charbonnel closure of an o-minimal weak structure, the properties of a weak structure and o-minimality are preserved. This fact reduces the proof of our theorem to the problem that $\widetilde{\mathcal{S}}$ is closed under complementation.

3.2 First Step: Find Approximants

In the first step of the proof we will approximate the sets in $\widetilde{\mathcal{S}}$ by $M^N(\mathcal{S})$-sets, which are based on graphs of $\widetilde{\mathcal{S}}$-functions. This is done by induction on the Ch-description of the sets. Therefore we walk along Chapter 10 of [BS04] and change the proofs, where it is necessary.

We start with the definition of $M^N(\mathcal{S})$ functions, inspired by a similar construction in Definition (6.5) in [BS04], where we replace the C^∞ functions by finitely often differentiable functions and simplify the definition.

Definition 3.2.1. Let $\mathcal{S}$ be an o-minimal weak structure. Let $M^N(\mathcal{S})$ contain all functions $f : U \to \mathbb{R}$, such that there is an $n \geq 1$ and

- $U \subseteq \mathbb{R}^n$ open,
- f is C^N,

- $f \in \widetilde{\mathcal{S}}$, i.e. $\Gamma(f) \in \widetilde{\mathcal{S}}$,
- the graph $\Gamma(f)$ is closed in $\mathbb{R}^{n+1}$.

We write $\mathrm{M}^{\mathrm{N}}(\mathcal{S})_n$ for functions with fixed n, i.e. the domain $U \subseteq \mathbb{R}^n$.

Remark 3.2.2. For all $f : U \to \mathbb{R}$ in $M^N(\mathcal{S})$ the set $U = \pi_n^{n+1}[\Gamma(f)]$ is in $\widetilde{\mathcal{S}}$.

Remark 3.2.3. We can formulate the DPC^N condition as: For every set $A \in \mathcal{S}_n$ there exist $m \geq n$ and for each N finite many functions $f_1, \ldots, f_r \in M^N(\mathcal{S})_m$ such that $A = \bigcup_{i=1}^r \pi_n^m[Z(f_i)]$.

For some proofs it is necessary to work with non-negative functions, so we often use f^2 instead of f. The following remark states that $M^N(\mathcal{S})$ is closed under this operation.

Remark 3.2.4. Let $f : U \to \mathbb{R}$ be in $M^N(\mathcal{S})$. Then $f^2 \in \mathrm{M}^{\mathrm{N}}(\mathcal{S})$, too.

Proof. It is obvious that f^2 is C^N and the domain of f^2 is also U, hence still open. Furthermore,

$$\begin{aligned}
\Gamma(f^2) &= \{(\overline{\mathrm{x}}, f^2(\overline{\mathrm{x}})) \mid \overline{\mathrm{x}} \in U\} \\
&= \{(\overline{\mathrm{x}}, y) \mid \exists z\; y = z^2 \wedge f(\overline{\mathrm{x}}) = z\} \\
&= \pi\left(\{(\overline{\mathrm{x}}, y, z) \mid y - z^2 = 0\} \cap \{(\overline{\mathrm{x}}, y, z) \mid f(\overline{\mathrm{x}}) = z\}\right)
\end{aligned}$$

The second set in the intersection is a permutation of coordinates of $\Gamma(f) \times \mathbb{R}$, so it is in $\widetilde{\mathcal{S}}$. The first set is semi-algebraic and so in $\widetilde{\mathcal{S}}$. Since $\widetilde{\mathcal{S}}$ is a weak structure and fulfills (WS1) and (WS4) and is closed under projection we obtain $\Gamma(f^2) \in \widetilde{\mathcal{S}}$.

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in U with $x_n \to x \in \partial U$. Since $\Gamma(f)$ closed it follows by Remark 1.3.8 that $|f(\overline{\mathrm{x}}_n)| \to \infty$ and also $|(f(\overline{\mathrm{x}}_n))^2| \to \infty$. Another application of Remark 1.3.8 yields that $\Gamma(f^2)$ is closed. ☐

Convention 3.2.5. The next definitions express some notations for the next proofs.

a) Given $A \subseteq \mathbb{R}^n$ and $\varepsilon \in \mathbb{R}_+$, define the *$\varepsilon$-neighborhood* A^ε of A as the set $\{x \in \mathbb{R}^n \mid \exists y \in A\; |x - y| < \varepsilon\}$.

b) Recall also that $S_{\overline{y}} := \{\overline{\mathrm{x}} \in \mathbb{R}^n \mid (\overline{\mathrm{x}}, \overline{y}) \in S\}$.

c) We write $\forall^s \varepsilon\, \phi$ (spoken as *for all sufficient small* ε) as a shorthand for $\exists \mu \forall \varepsilon < \mu\; \phi$, where $\mu, \varepsilon \in \mathbb{R}_+$.

The aim is to approximate all sets in $\widetilde{\mathcal{S}}$ by a set of a special form, obtained by functions in $M^N(\mathcal{S})$. For this we use the following definition.

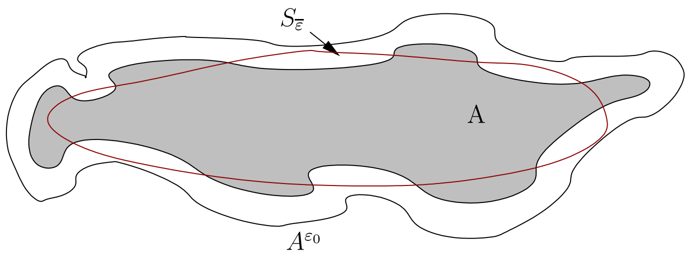

Figure 3.1: The set S approximates A from below.

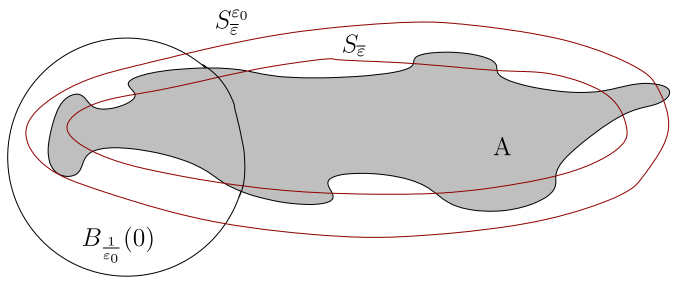

Figure 3.2: The set S approximates A from above on bounded sets.

Definition 3.2.6 (Definition (6.4) in [BS04])**.** Let $A \subseteq \mathbb{R}^n$ and $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$.

a) The set S *approximates* A *from below*, if

$$\forall^s \varepsilon_0 \, \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k \;\; S_{\varepsilon_1,\ldots,\varepsilon_k} \subseteq A^{\varepsilon_0}.$$

b) The set S *approximates* A *from above on bounded sets* if

$$\forall^s \varepsilon_0 \, \forall^s \varepsilon_1 \ldots \forall^s \varepsilon_k \;\; A \cap B_{\frac{1}{\varepsilon_0}}(0) \subseteq S^{\varepsilon_0}_{\varepsilon_1,\ldots,\varepsilon_k}$$

where $B_{\frac{1}{\varepsilon_0}}(0)$ is the compact ball of radius $\frac{1}{\varepsilon_0}$ centered in the origin.

Definition 3.2.7 (Definition (6.6) in [BS04])**.** Let $N \in \mathbb{N}$ and $\mathcal{S}$ be an o-minimal weak structure.

a) An $M^N(\mathcal{S})$-*constituent* is a set of the form

$$\left\{ (\overline{\mathrm{x}}, \overline{\varepsilon}) \in \mathbb{R}^n \times \mathbb{R}^k_+ \mid \exists \overline{y} \in \mathbb{R}^{k-1} \; (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in B \;\wedge\; F(\overline{\mathrm{x}}, \overline{y}) = \overline{\varepsilon} \right\},$$

where there is some open $U \subseteq \mathbb{R}^{n+k-1}$ such that $F : U \to \mathbb{R}^k$ belongs to $M^N(\mathcal{S})_k$.

b) An $M^N(\mathcal{S})$-*set* is a finite union of $M^N(\mathcal{S})$-constituents in $\mathbb{R}^n \times \mathbb{R}^k_+$.

c) For a given set $A \in \widetilde{\mathcal{S}_n}$, an $M^N(\mathcal{S})$-set $S \subseteq \mathbb{R}^{n+k}$ is called an $M^N(\mathcal{S})$- *approximant* for A if S approximates $\partial\overline{A}$ from above on bounded sets and approximates $\overline{A}$ from below.

Now it is possible to formulate the first central statement for the proof of the theorem of the complement. We express that we can approximate each set in a weak structure. Here we replace the DSF condition in Theorem (6.11) in [BS04] by the DPC^N condition for all N.

Theorem 3.2.8 (Approximation, Theorem (6.11) in [BS04], modified)**.** *Suppose $\mathcal{S}$ is an o-minimal weak structure satisfying DPC^N for all N. Let $n \geq 1$ and $A \in \mathcal{S}_n$. Then, for every $N \in \mathbb{N}$, there exists an* $\mathrm{M^N}(\mathcal{S})$*-approximant for A.*

We prove this theorem by induction on the structure of A. The proof will take the rest of this section and the next section, where we deal with the projection case, which requires some more work.

Lemma 3.2.9. *[Lemma of the Union] Let A_1, A_2 be subsets of* $\mathbb{R}^n$ *which have* $\mathrm{M^N}(\mathcal{S})$*-approximants S_1, S_2. Then $S_1 \cup S_2$ is an* $\mathrm{M^N}(\mathcal{S})$*-approximant for $A_1 \cup A_2$.*

Proof. For

$$T = \left\{ (\overline{\mathrm{x}}, \overline{\varepsilon}) \in \mathbb{R}^n \times \mathbb{R}^k_+ \mid \ \exists \overline{y} \in \mathbb{R}^{k-1} (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in U \ \wedge \ F(\overline{\mathrm{x}}, \overline{y}) = \overline{\varepsilon} \right\}$$

define

$$T' = \left\{ (\overline{\mathrm{x}}, \overline{\varepsilon}, \varepsilon_{k+1}) \in \mathbb{R}^n \times \mathbb{R}^{k+1}_+ \mid \qquad \exists \overline{y} \in \mathbb{R}^{k-1}\ (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in U \ \wedge \ F(\overline{\mathrm{x}}, \overline{y}) = \overline{\varepsilon} \right\}$$

Then, for all $\varepsilon_{k+1} \in \mathbb{R}_+$ we have $T_{(\varepsilon_1,\ldots,\varepsilon_k)} = T'_{(\varepsilon_1,\ldots,\varepsilon_k,\varepsilon_{k+1})}$. So if T approximates any set A from above on bounded sets or from below, it is equivalent, that T' approximates A in the same way. So we can assume that all $\mathrm{M^N}(\mathcal{S})$-constituents which form S_1 and S_2 have the same k.

1. The set $S_1 \cup S_2$ is an $\mathrm{M^N}(\mathcal{S})$-set. This is clear by definition.

2. The $\mathrm{M^N}(\mathcal{S})$-set $S_1 \cup S_2$ approximates $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ from below.

Since S_i approximates A_i from below ($i = 1, 2$), we have:

$$\exists \mu_{1,0} \forall\, \varepsilon_{1,0} < \mu_{1,0} \ldots \exists \mu_{1,k} \forall\, \varepsilon_{1,k} < \mu_{1,k}\ S_{1_{\varepsilon_{1,1},\ldots,\varepsilon_{1,k}}} \subseteq \overline{A_1}^{\varepsilon_{1,0}}$$

$$\exists \mu_{2,0} \forall\, \varepsilon_{2,0} < \mu_{2,0} \ldots \exists \mu_{2,k} \forall\, \varepsilon_{2,k} < \mu_{2,k}\ S_{2_{\varepsilon_{2,1},\ldots,\varepsilon_{2,k}}} \subseteq \overline{A_2}^{\varepsilon_{2,0}}$$

We notice that $\mu_{1,j}$ and $\mu_{2,j}$ depend on the choice of $\varepsilon_0, \ldots, \varepsilon_{j-1}$. So define for the union $\mu_j(\varepsilon_0, \ldots, \varepsilon_{j-1}) := \min\{\mu_{1,j}(\varepsilon_0, \ldots, \varepsilon_{j-1})), \mu_{2,j}(\varepsilon_0, \ldots, \varepsilon_{j-1})\}$ for $0 \leq j \leq k$. Then clearly,

$$\forall^s \varepsilon_0 \ldots \forall^s \varepsilon_k \; \forall^s \varepsilon_0 \ldots \forall^s \varepsilon_k \, S_1 \cup S_{2\varepsilon_1,\ldots,\varepsilon_k} \subseteq \overline{A_1}^{\varepsilon_0} \cup \overline{A_2}^{\varepsilon_0}.$$

Since $\overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2}$ the union $S_1 \cup S_2$ approximates $\overline{A_1 \cup A_2}$ from below.

3. The $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-set $S_1 \cup S_2$ approximates $\partial\overline{A_1 \cup A_2}$ from above.

The following calculation holds for all sufficient small $\overline{\varepsilon}$.

$$\begin{array}{cl} & \partial\overline{A_1 \cup A_2} \cap B_{\frac{1}{\varepsilon_0}}(0) \\ \subseteq & \left(\partial\overline{A_1} \cap B_{\frac{1}{\varepsilon_0}}(0)\right) \cup \left(\partial\overline{A_2} \cap B_{\frac{1}{\varepsilon_0}}(0)\right) \\ \overset{S_i \text{ approximates } \partial A_i \text{ from above}}{\subseteq} & {S_1}^{\varepsilon_0}_{\varepsilon_1,\ldots,\varepsilon_k} \cup {S_2}^{\varepsilon_0}_{\varepsilon_1,\ldots,\varepsilon_k} \\ = & (S_1 \cup S_2)^{\varepsilon_0}_{\varepsilon_1,\ldots,\varepsilon_k}. \end{array}$$

□

The following lemma helps to check that the condition of the closed graph is preserved in several of the following proofs.

Lemma 3.2.10. *Let $f : U \to \mathbb{R}^k$ be a continuous function in $\widetilde{\mathcal{S}}$ with $U \subseteq \mathbb{R}^n$ and $\Gamma(f)$ closed. Let $g : U \times \mathbb{R} \to \mathbb{R}$ be a second continuous function. Then the function $(f, g) : U \times \mathbb{R} \to \mathbb{R}^{k+1}$ defined by $(f, g)(\overline{\mathrm{x}}, y) = (f(\overline{\mathrm{x}}), g(\overline{\mathrm{x}}, y))$ has a closed graph.*

Proof. By Remark 1.3.8 we have to show that for each $(\overline{\mathrm{x}}_n, y_n) \in U \times \mathbb{R}$ with $(\overline{\mathrm{x}}_n, y_n) \to (\overline{\mathrm{x}}, y) \in \partial(U \times \mathbb{R})$ the limit $|(f, g)(\overline{\mathrm{x}}_n, y_n)| \to \infty$. But since $\Gamma(f)$ is closed and $\overline{\mathrm{x}}_n \to \overline{\mathrm{x}} \in \partial U$, it is clear that $|f(\overline{\mathrm{x}}_n)| \to \infty$. Hence $|(f, g)(\overline{\mathrm{x}}_n, y_n)|^2 \leq |f(\overline{\mathrm{x}}_n)|^2 + |g(\overline{\mathrm{x}}_n, y_n)|^2 \to \infty$. □

The next lemma is about $M^N(\mathcal{S})$-approximations to zero sets of $M^N(\mathcal{S})$ functions.

Lemma 3.2.11. *[Lemma (10.3) in [BS04] with a partial function] If $f : U \to \mathbb{R}$ is in $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$ and $U \subseteq \mathbb{R}^n$, then its zero-set $Z(f)$ has an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant $S \in \widetilde{S}_{n+2}$.*

Proof. The proof follows mainly the proof of Berarducci and Servi in [BS04], but it is necessary to make some modifications to keep the lemma true for partial functions.

Without loss of generality, we can assume that U is connected: Due to the fact that by (WS5) it has only finitely many connected components, we can take the union of the corresponding $\mathrm{M^N}(\mathcal{S})$-approximants to obtain an $\mathrm{M^N}(\mathcal{S})$-approximant for U, what is possible by Lemma 3.2.9.

Additionally we can assume that $f \geq 0$. Otherwise replace f by f^2, which is still in $M^N(\mathcal{S})$ by Remark 3.2.4.

Exactly as in [BS04], let

$$S := \left\{ (\overline{\mathrm{x}}, \varepsilon_1, \varepsilon_2) \in U \times \mathbb{R}_+^2 \mid |(1, \overline{\mathrm{x}})|^2 \leq \frac{1}{\varepsilon_1} \wedge f(\overline{\mathrm{x}}) = \varepsilon_2 \right\}.$$

1. The set S is an $\mathrm{M^N}(\mathcal{S})$-set.

Note that $|(1, x_1, \ldots, x_n)|^2 \leq \frac{1}{\varepsilon_1}$ if and only if $\exists y (1 + x_1^2 + \cdots + x_n^2 + y^2)^{-1} = \varepsilon_1$. The function $g : U \times \mathbb{R} \to \mathbb{R}, \quad (x_1, \ldots, x_n, y) \mapsto (1 + x_1^2 + \cdots + x_n^2 + y^2)^{-1}$ belongs to $\mathrm{M^N}(\mathcal{S})$ for all N, since the graph is a semi-algebraic set as we can see easily. The function f is in $\mathrm{M^N}(\mathcal{S})$ and C^N. Define $F = (f, g) : U \times \mathbb{R} \to \mathbb{R}_+^2$.

The graph of F is closed by Lemma 3.2.10. $\Gamma(F)$ is in $\widetilde{\mathcal{S}}$, because $f \in \widetilde{\mathcal{S}}$ by assumption and $\Gamma(g)$ is in $\widetilde{\mathcal{S}}$ and so $\Gamma(F) = \{(\overline{\mathrm{x}}, y, z, z') \mid (\overline{\mathrm{x}}, z) \in \Gamma(f) \wedge (\overline{\mathrm{x}}, y, z') \in \Gamma(g)\}$ is at last an intersection of sets in $\widetilde{\mathcal{S}}$ which is in $\widetilde{\mathcal{S}}$ by (WS1). Hence

$$S = \left\{ (\overline{\mathrm{x}}, \varepsilon_1, \varepsilon_2) \in \mathbb{R}^n \times \mathbb{R}_+^2 \mid \exists y \in \mathbb{R} \ \left((\overline{\mathrm{x}}, y) \in U \times \mathbb{R} \ \wedge \ F(\overline{\mathrm{x}}, y) = (\varepsilon_1, \varepsilon_2) \right) \right\}$$

is an $\mathrm{M^N}(\mathcal{S})$-set.

2. The set S approximates $Z(f)$ from below.

We have to show $\forall^{\mathrm{s}} \varepsilon_0 \, \forall^{\mathrm{s}} \varepsilon_1 \, \forall^{\mathrm{s}} \varepsilon_2 \, S_{\varepsilon_1, \varepsilon_2} \subseteq Z(f)^{\varepsilon_0}$.

Fix ε_1. Then $K := \left\{ (\overline{\mathrm{x}} \in \mathbb{R}^n \mid |(1, \overline{\mathrm{x}})|^2 \leq \frac{1}{\varepsilon_1} \right\}$ is compact and contains (with a look at the definition of S) the set $S_{\varepsilon_1, \varepsilon_2}$ for all ε_2.

This set K is defined in the same way in [BS04]. They continue with the argument that $K - Z(f)^{\varepsilon_0}$ is compact and the minimum of $f(\overline{\mathrm{x}})$ is taken in this set. But this works only if f is defined on the whole set, what we cannot assume in our case, where f is only defined on some set U. Since U is open, it is still not enough to intersect with U. To obtain a compact set and finish the proof as in [BS04], we have to add one dimension and use the fact that the graph of f is closed.

Define $K' := \left\{ (\overline{\mathrm{x}}, y) \mid \overline{\mathrm{x}} \in K \wedge y \leq \frac{1}{\varepsilon_0} \right\}$. Since K is compact, so is K'. Now let

$$C := (\Gamma(f) - (Z(f)^{\varepsilon_0} \times \mathbb{R})) \cap K'.$$

Then C is compact: $\Gamma(f)$ is closed and $Z(f)^{\varepsilon_0} \times \mathbb{R}$ is open, K' is closed, so C is closed. It is bounded, since K' is bounded.

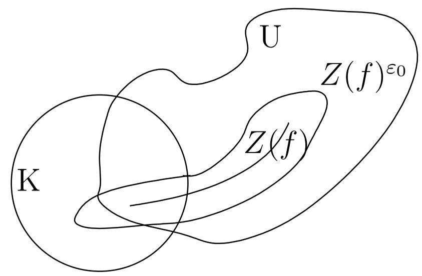

Figure 3.3: Sets needed for the approximation from below

Suppose, $\varepsilon_2 < \min\{f(\overline{x}) \mid (\overline{x}, y) \in C\}$ (the minimum always exists since C is compact) and $\varepsilon_2 < \frac{1}{\varepsilon_0}$. So ε_2 depends on ε_0 and ε_1.

Now let $\overline{x} \in S_{\varepsilon_1,\varepsilon_2}$, thus $\overline{x} \in U$, $\overline{x} \in K$ and $f(\overline{x}) = \varepsilon_2$. Since $\overline{x} \in K$ we have $f(\overline{x}) = \varepsilon_2 < \frac{1}{\varepsilon_0}$, hence $(\overline{x}, f(\overline{x})) \in K'$. So $(\overline{x}, f(\overline{x})) \in \Gamma(f) \cap K'$. Because of the choice of ε_2 we have $(\overline{x}, f(\overline{x})) \notin C = (\Gamma(f) - (Z(f)^{\varepsilon_0} \times \mathbb{R})) \cap K'$, hence $(\overline{x}, f(\overline{x}))$ must be in $\Gamma(f) \cap (Z(f)^{\varepsilon_0} \times \mathbb{R}) \cap K'$. Particularly $\overline{x} \in Z(f)^{\varepsilon_0}$, what we wanted to show.

3. S approximates $\partial\overline{Z(f)}$ from above on bounded sets.

We have to show that $\forall^s\varepsilon_0\, \forall^s\varepsilon_1\, \forall^s\varepsilon_2\, \partial Z(f) \cap B_{\frac{1}{\varepsilon_0}}(0) \subseteq S^{\varepsilon_0}_{\varepsilon_1,\varepsilon_2}$. Fix ε_0 and let $\varepsilon_1 = \varepsilon_1(\varepsilon_0)$ be small enough, such that K (as above) contains $B_{\frac{1}{\varepsilon_0}}(0)$.

The following claim is Lemma (10.2) in [BS04]; at some point in the proof we have to care about the partial defined functions and we need that U is open and $\Gamma(f)$ is closed, so we cannot take over the proof of Berarducci and Servi directly.

Claim : The graph $\Gamma(f) = \{(\overline{x}, \varepsilon_2) \mid f(\overline{x}) = \varepsilon_2\}$ approximates $\partial\overline{Z(f)}$ from above on bounded sets, that is

$$\forall^s\varepsilon_0\, \forall^s\varepsilon_2\;\; \partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0) \subseteq \{\overline{x} \mid f(\overline{x}) = \varepsilon_2\}^{\varepsilon_0}.$$

Proof: Let $\varepsilon_0 > 0$. We assume that there exists a sequence $t_n \to 0$ such that $\partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0) \nsubseteq \{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0}$. So for every $n \in \mathbb{N}$ there is a point $\overline{x}_n \in \partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0)$ such that $\overline{x}_n \notin \{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0} = f^{-1}(t_n)^{\varepsilon_0}$. Since $B_{\frac{1}{\varepsilon_0}}(0)$ is compact and $\partial\overline{Z(f)}$ is closed, we can choose a subsequence $\overline{x}_n$ which converges to some $\overline{x} \in \partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0)$.

Let $O_{\frac{\varepsilon_0}{2}}(\overline{x})$ be the open ball with radius $\frac{\varepsilon_0}{2}$ around $\overline{x}$. We can assume $O_{\frac{\varepsilon_0}{2}}(\overline{x}) \subseteq U$: Since $\overline{x} \in U$ and U is open, there exists an $\varepsilon_0' < \varepsilon_0$ such that $O_{\frac{\varepsilon_0'}{2}}(\overline{x}) \subseteq U$. The

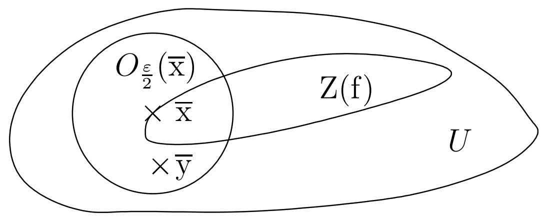

Figure 3.4: How to find $\overline{y}$ such that $f(\overline{y}) > 0$.

properties of $(\overline{x}_n)_n$ are conserved: Since $\{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0'} \subseteq \{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0}$ we obtain $\overline{x}_n \notin f^{-1}(t_n)^{\varepsilon_0'}$ and since $B_{\frac{1}{\varepsilon_0}}(0) \subseteq B_{\frac{1}{\varepsilon_0'}}(0)$ we have $\overline{x}_n, \overline{x} \in \partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0'}}(0)$.

Berarducci and Servi take on this point of the proof directly an $y \in O_{\frac{\varepsilon_0}{2}}(\overline{x})$ where f takes a positive value, but for this purpose we have to check that f is not the zero function on its domain intersected with $O_{\frac{\varepsilon_0}{2}}(\overline{x})$. We will see that we need that the domain is open.

Notice that $\partial\overline{Z(f)} \subseteq U$: Let $\overline{z} \in \partial\overline{Z(f)}$ such that $\overline{z}_n \to \overline{z}$ with $f(\overline{z}_n) = 0$, then if $\overline{z} \in U$, $f(\overline{z}) = 0$ since f is continuous. But $\overline{z}$ must be in U, since $(\overline{z}_n, 0) \in \Gamma(f)$ and $\Gamma(f)$ is closed, so the limit $(\overline{z}, 0)$ must be in $\Gamma(f)$, too.

As $O_{\frac{\varepsilon_0}{2}}(\overline{x})$ is open and $\overline{x}$ is in the boundary of the zero set we have $O_{\frac{\varepsilon_0}{2}}(\overline{x}) - Z(f) \neq \emptyset$ and since $f \geq 0$ a positive value is taken, hence there exists a $\overline{y} \in O_{\frac{\varepsilon_0}{2}}(\overline{x}) \subseteq U$ with $f(\overline{y}) = \eta > 0$. Now we can proceed as in the proof of Lemma (10.3) in [BS04].

We have $f(\overline{y}) = \eta$ and $f(\overline{x}) = 0$. As f is continuous, f takes all values in the interval $[0, \eta]$ in $O_{\frac{\varepsilon_0}{2}}$. Take n big enough that $t_n < \eta$ and $\overline{x}_n \in O_{\frac{\varepsilon_0}{2}}(\overline{x})$; then the value t_n is taken at $\overline{y}_n \in O_{\frac{\varepsilon_0}{2}}(\overline{x})$, so $f(\overline{y}_n) = t_n$. So $\overline{y}_n \in O_{\frac{\varepsilon_0}{2}}(\overline{x}) \cap \{\overline{x} \mid f(\overline{x}) = t_n\}$ and since $\forall \overline{z} \in O_{\frac{\varepsilon_0}{2}}(\overline{x})\ |\overline{z} - \overline{y_n}| < \varepsilon_0$ and $\overline{x}_n \in O_{\frac{\varepsilon_0}{2}}(\overline{x})$, it follows that $\overline{x}_n \in O_{\frac{\varepsilon_0}{2}}(\overline{x}) \subseteq \{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0}$. But we assumed $\overline{x}_n \notin \{\overline{x} \mid f(\overline{x}) = t_n\}^{\varepsilon_0}$. Contradiction! □(Claim)

Hence we have $\partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0) \subseteq \{\overline{x} \mid f(\overline{x}) = \varepsilon_2\}^{\varepsilon_0}$.

Furthermore $\partial\overline{Z(f)} \cap B_{\frac{1}{\varepsilon_0}}(0) \subseteq K = \left\{ \overline{x} \in \mathbb{R}^n \mid |(1, \overline{x})|^2 \leq \frac{1}{\varepsilon_1} \right\}$ and as

$$S^{\varepsilon_0}_{\varepsilon_1,\varepsilon_2} = \{\overline{x} \mid f(\overline{x}) = \varepsilon_2\}^{\varepsilon_0} \cap K$$

the proof is finished. □

Lemma 3.2.12. *[Approximant for the Projection, Lemma (10.6) in [BS04], modified] If $A \subseteq \mathbb{R}^{n+1}$ has an $M^{N+1}(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^k_+$, then there exists an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{k+1}_+$ for $\pi^{n+1}_n[A] \subseteq \mathbb{R}^n$.*

The proof of this lemma is quite lengthy, so we move it into the next section.

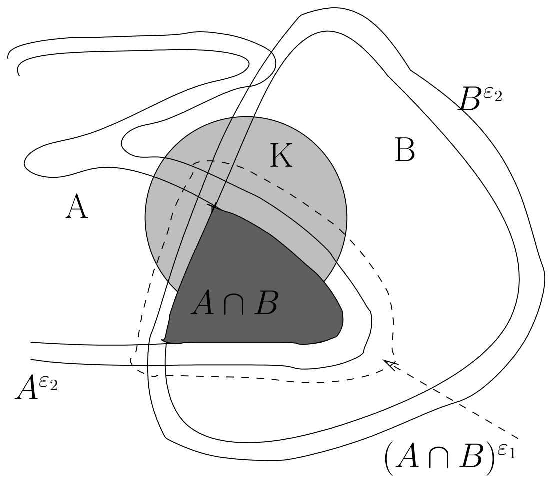

Figure 3.5: Intersection and ε-neighborhoods (Lemma 3.2.13)

Lemma 3.2.13. *[Lemma (10.7) in [BS04]] Let $A, B \subseteq \mathbb{R}^n$ be closed sets, $K \subseteq \mathbb{R}^n$ compact. Then*

$$\forall^{\mathrm{s}} \varepsilon_1 \, \forall^{\mathrm{s}} \varepsilon_2 \quad A^{\varepsilon_2} \cap B^{\varepsilon_2} \cap K \subseteq (A \cap B)^{\varepsilon_1}$$

Proof. By the sketch in Figure 5, the definition of

$$\varepsilon_2 := \frac{1}{3} \min \left\{ d(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \mid \overline{\mathrm{x}}, \overline{\mathrm{y}} \in K \;\wedge\; \overline{\mathrm{x}}, \overline{\mathrm{y}} \notin (A \cap B)^{\varepsilon_1} \;\wedge\; \overline{\mathrm{x}} \in A \;\wedge\; \overline{\mathrm{y}} \in B \right\},$$

and a few calculations the fact is clear. □

Lemma 3.2.14. *[Approximant for Intersection with Affine Set, Lemma (10.8) in [BS04], modified] Let $A \in \widetilde{\mathcal{S}}_n$ have an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^n \times \mathbb{R}^k_+$ and let L be an $(n-1)$ dimensional $\mathbb{Z}$-affine set such that $\overline{A} \cap L = \partial \overline{A} \cap L$. Then there is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{k+2}_+$ for $\overline{A} \cap L$.*

Proof. Look at the proof of Lemma 10.8. in [BS04]. We take the same definitions, only with partial functions and see that the proof works.

$$S'_{\varepsilon_1, \ldots, \varepsilon_{k+2}} := S_{\varepsilon_3, \ldots, \varepsilon_{k+2}} \cap L(\varepsilon_2) \cap K_{\varepsilon_1}$$

where $L(\varepsilon_2) = \left\{ \overline{\mathrm{x}} \mid \exists x_{n+k+1} \; l(x_1, \ldots, x_n)^2 + x_{n+k+1}^2 = \varepsilon_2 \right\}$, where l is a polynomial with $L = Z(l)$ and $K_{\varepsilon_1} = \left\{ \overline{\mathrm{x}} \mid \exists x_{n+k} (1 + \sum_{i=1}^n x_i^2 + x_{n+k}^2)^{-1} = \varepsilon_1 \right\}$. So it is $S' := \left\{ (\overline{\mathrm{x}}, \varepsilon_1, \ldots, \varepsilon_{k+2}) \in \mathbb{R}^n \times \mathbb{R}^{k+2} \mid (\overline{\mathrm{x}}, \varepsilon_3, \ldots, \varepsilon_{k+2}) \in S \;\wedge\; \overline{\mathrm{x}} \in L(\varepsilon_2) \;\wedge\; \overline{\mathrm{x}} \in K_{\varepsilon_1} \right\}$

1. The set S' is an $\mathrm{M^N}(\mathcal{S})$-set. Let $S = \bigcup_{i=1}^{s} T_i$ with T_i the $\mathrm{M^N}(\mathcal{S})$-constituents described through $\mathrm{M^N}(\mathcal{S})$-functions $f_i : U_i \to \mathbb{R}^k$. Then if T_i' is defined in the same way as S', we have

$$\begin{aligned} T_i' = \ & \Big\{(x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_{k+2}) \mid \exists x_{n+1}, \dots, x_{n+k-1} \ \big(\overline{\mathrm{x}} \in U_i \\ & \quad \wedge\ f_i(\overline{\mathrm{x}}) = (\varepsilon_3, \dots, \varepsilon_{k+1})\big) \\ & \quad \wedge\ \exists x_{n+k+1} l(x_1, \dots, x_n)^2 + x_{n+k+1}^2 = \varepsilon_2 \\ & \quad \wedge\ \exists x_{n+k}\big(1 + \sum_{i=1}^{n} x_i^2 + x_{n+k}^2\big)^{-1} = \varepsilon_1 \Big\} \\ \Leftrightarrow\ T_i' = \ & \Big\{(x_1, \dots, x_n, \varepsilon_1, \dots, \varepsilon_{k+2}) \mid \\ & \quad \exists x_{n+1}, \dots, x_{n+k+1} \ \big(\overline{\mathrm{x}} \in U_i \times \mathbb{R} \times \mathbb{R} \\ & \quad \wedge\ \big(1 + \sum_{i=1}^{n} x_i^2 + x_{n+k}^2\big)^{-1} = \varepsilon_1 \\ & \quad \wedge\ l(x_1, \dots, x_n)^2 + x_{n+k+1}^2 = \varepsilon_2 \\ & \quad \wedge\ f_i(x_1, \dots, x_{n+k-1}) = (\varepsilon_3, \dots, \varepsilon_{k+1})\big)\Big\} \end{aligned}$$

Obviously $S' = \bigcup_{i=1}^{s} T_i'$, all additional occurring functions are C^∞ and in $\widetilde{\mathcal{S}}$ and since $U_i \times \mathbb{R} \times \mathbb{R}$ is open the T_i' are $\mathrm{M^N}(\mathcal{S})$-constituents if we can prove that the graph of (h, g, f_i) is closed where $g(\overline{\mathrm{x}}) := l(x_1, \dots, x_n)^2 + x_{n+k+1}^2$ and $h(\overline{\mathrm{x}}) := (1 + \sum_{i=1}^{n} x_i^2 + x_{n+k}^2)$ and $(h, g, f_i) : U_i \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^{k+2}$ is defined by $(f, g, h)(\overline{\mathrm{x}}) := (h(\overline{\mathrm{x}}), g(\overline{\mathrm{x}}), f(x_1, \dots, x_n))$. To see this, apply the Lemma of closed graphs 3.2.10 two times. Then S' is an $\mathrm{M^N}(\mathcal{S})$-set.

2. The set S' approximates $\overline{A} \cap L$ from below. By Lemma 3.2.13 $\forall^{\mathrm{s}} \varepsilon_0 \, \forall^{\mathrm{s}} \varepsilon_2 \ \overline{A}^{\varepsilon_2} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0)^{\varepsilon_2} \cap L(\varepsilon_2) \subseteq (\overline{A} \cap L)^{\varepsilon_0}$. And since S approximates $\overline{A}$ from above on bounded sets, we have $\forall^{\mathrm{s}} \varepsilon_0 \dots \forall^{\mathrm{s}} \varepsilon_{k+2} \ \ S'_{\varepsilon_1, \dots, \varepsilon_{k+1}} \subseteq S_{\varepsilon_3, \dots, \varepsilon_{k+2}} \subseteq \overline{A}^{\varepsilon_2}$. Furthermore $S'_{\varepsilon_1, \dots, \varepsilon_{k+1}} \subseteq K_{\varepsilon_1} \cap L(\varepsilon_2)$, hence $S'_{\varepsilon_1, \dots, \varepsilon_{k+1}} \subseteq (\overline{A} \cap L)^{\varepsilon_0}$.

3. The set S' approximates $\partial(\overline{A} \cap L)$ from above on bounded sets. By precondition we have $\forall^{\mathrm{s}} \varepsilon_2 \dots \forall^{\mathrm{s}} \varepsilon_{k+2} \ \partial \overline{A} \cap B_{\frac{1}{\varepsilon_2}}(0) \subseteq S^{\varepsilon_2}_{\varepsilon_3, \dots, \varepsilon_{k+2}}$. Of course,

$$\forall^{\mathrm{s}} \varepsilon_1 \, \forall^{\mathrm{s}} \varepsilon_2 \ \ B_{\frac{1}{\varepsilon_2}}(0) \subseteq K_{\varepsilon_1}.$$

So we have $\forall^{\mathrm{s}} \varepsilon_0 \dots \forall^{\mathrm{s}} \varepsilon_{k+2}$

$$\begin{aligned} & \partial(\overline{\mathrm{A}} \cap L) \subseteq \overline{A} \cap L \\ \overset{\textit{precondition}}{=} \ & \partial \overline{A} \cap L \subseteq (S^{\varepsilon_2}_{\varepsilon_3, \dots, \varepsilon_{k+2}} \cap L(\varepsilon_2) \cap K_{\varepsilon_1})^{\varepsilon_0} \\ = \ & (S'_{\varepsilon_1, \dots, \varepsilon_{k+2}})^{\varepsilon_0'}, \end{aligned}$$

where $\varepsilon_2 + \varepsilon_0 \leq \varepsilon_0'$, but this is possible to achieve by choosing the ε_0 and ε_2 depend on ε_0'. □

Now we prove Theorem 3.2.8, which was stated at the beginning of this section, along the proof of Theorem (6.11) in [BS04]. Some arguments are given here in more details than in [BS04]. However, since all lemmas proved in this section are nearly the same as in [BS04], Section 11, the principles are very similar. We just have to replace DSF by DPC^N for all N.

Proof. The proof works by induction on the rank of the Ch-description of $A \in \mathcal{S}_n$, recall Definition 2.2.2. Let $\mathfrak{A}$ be a Ch-description of A such that $\rho(\mathfrak{A}) = k$.

Induction hypothesis: Assume that if $\mathfrak{B}$ is a Ch-description of a set B with $\rho(\mathfrak{B}) < k$, then there exists for all $N \in \mathbb{N}$ an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for B.

First fix $N \in \mathbb{N}$. Then examine the different possibilities of $\mathfrak{A}$.

a) $\mathfrak{A} \in \Sigma$, so $A \in \mathcal{S}_n$. By assumption $\mathcal{S}$ has DPC^N for all N, so there exists an $m \geq n$ such that for all N' there exist $C^{N'}$ functions $f_i : U_i \to \mathbb{R}$ in $M^{N'}(\mathcal{S})_m$ for $i = 1, \ldots, r$ (where r is an arbitrary finite index) such that $A = \pi_n^m[Z(f_1) \cup \cdots \cup Z(f_r)]$. Particularly, there exist C^{N+m-n} functions $f_i : U \to \mathbb{R}$ in $\widetilde{\mathcal{S}}$ with $U_i \subseteq \mathbb{R}^m$ open for $i = 1, \ldots, r$ such that $A = \pi_n^m[Z(f_1) \cup \cdots \cup Z(f_r)]$ and $\Gamma(f_i)$ closed. By the Lemma of the zero-set 3.2.11 there is an $M^{N+m-n}(\mathcal{S})$-approximant for $Z(f_i)$ for every $i = 1, \ldots, r$. Then we apply the Lemma of the union 3.2.9 and the Lemma of the projection 3.2.12 $m-n$-times and obtain an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for A. This is the only place where the DPC^N condition for all N is used.

b) $\mathfrak{A} = \pi_n^{n+h}\mathfrak{B}$ such that $\mathfrak{B}$ is a Ch-description of a set $B \in \widetilde{\mathcal{S}}$ with $\rho(\mathfrak{A}) = 1 + \rho(\mathfrak{B})$. So $\rho(\mathfrak{B}) < k$. By induction hypothesis there exists an $M^{N+h}(\mathcal{S})$-approximant for B. Apply now the Lemma of the projection 3.2.12 h times.

c) $\mathfrak{A} = \mathfrak{A}_1 \cup \mathfrak{A}_2$. Then $\rho(\mathfrak{A}) = 1 + \max\{\rho(\mathfrak{A}_1), \rho(\mathfrak{A}_2)\}$, so $\rho(\mathfrak{A}_1), \rho(\mathfrak{A}_2) < k$. By induction hypothesis there exist $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximants for A_1 and A_2. Apply the Lemma for the union 3.2.9 on A_1 and A_2.

d) $\mathfrak{A} = \overline{\mathfrak{B}}$. Hence $\rho(\mathfrak{B}) = \rho(\mathfrak{A}) - 4 < k$. By the definition of the approximants, which works with the closure of any set, the $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for B (which we have by induction hypothesis) is already one for A.

e) $\mathfrak{A} = \mathfrak{B} \cap L$ where L is a $\mathbb{Z}$-affine set. Note that $k = \rho(\mathfrak{A}) = \rho(\mathfrak{B}) + 1$. Here we have to examine several cases.

- $\mathfrak{B} \in \Sigma$, so $B \in \mathcal{S}$. Then $B \cap L \in \mathcal{S}$, since L is a semi-algebraic set. Apply the case a).

- $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$ with $\rho(\mathfrak{B}_1), \rho(\mathfrak{B}_2) < \rho(\mathfrak{B})$. Then $B \cap L = (B_1 \cap L) \cup (B_2 \cap L)$ and $\rho(\mathfrak{B}_i \cap L) = \rho(\mathfrak{B}_i) + 1 < \rho(\mathfrak{B}) + 1 = \rho(\mathfrak{B} \cap L)$. Hence we can apply the induction hypothesis on $B_i \cap L$ with Ch-description $\mathfrak{B}_i \cap L$. The union case supplies an approximant for $B \cap L$.

- $\mathfrak{B} = \overline{\mathfrak{B}_1}$. Hence $B = \overline{B}_1$. Since L is a zero set of a polynomial and therefore finite and hence closed, we obtain $B \cap L = \overline{B_1 \cap L}$. As $\rho(\mathfrak{B}_1 \cap L) < \rho(\mathfrak{B} \cap L)$ we can apply the induction hypothesis. The closure case provides the desired result.

- $\mathfrak{B} = \pi_n^{n+h}\,\mathfrak{D}$. Recognize, that

$$\pi_n^{n+h}[D] \cap L = \pi_n^{n+h+n}[(D \times L) \cap \Delta]$$

where $\Delta = \{(\overline{x}, \overline{y}, \overline{x}) \mid \overline{x} \in \mathbb{R}^n \wedge \overline{y} \in \mathbb{R}^h\}$ is obviously an $\mathbb{Z}$-affine set and particularly in $\mathcal{S}$. Also, $L \in \mathcal{S}$, so there exists a label $\mathfrak{L} \in \Sigma$ (i.e. $\rho(\mathfrak{L}) = 0$), which is a Ch-description of L. By Lemma 2.2.4 there exists a Ch-description $\mathfrak{C}$ of $C := D \times L$ such that $\rho(\mathfrak{C}) \leq \rho(\mathfrak{D}) + \rho(\mathfrak{L}) = \rho(\mathfrak{D}) = \rho(\mathfrak{B}) - 1 < \rho(\mathfrak{B})$. So $\rho(\mathfrak{C} \cap \Delta) = \rho(\mathfrak{C}) + 1 \leq \rho(\mathfrak{B}) < \rho(\mathfrak{A})$. Hence we can use the induction hypothesis, which supplies for all N' an $M^{N'}(\mathcal{S})$-approximant for $C \cap \Delta$. Take an $M^{N+h+n}(\mathcal{S})$-approximant and apply the lemma of the projection $h+n$-times and obtain an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-constituent for A.

- $\mathfrak{B} = \mathfrak{D} \cap L'$ where L' is a $\mathbb{Z}$-affine set and $D \in \widetilde{\mathcal{S}}$. Then $L' \cap L$ is a $\mathbb{Z}$-affine set and $A = D \cap (L \cap L')$ with $\rho(\mathfrak{D}) < \rho(\mathfrak{B}) < \rho(\mathfrak{A})$. So apply the induction hypothesis on $\mathfrak{D} \cap (L \cap L')$.

- $\mathfrak{B} = \overline{\mathfrak{D}}$. Write $L = Y_1 \cap \cdots \cap Y_k$ with $\mathbb{Z}$-affine sets Y_i with co-dimension 1, so $Y_1 = Z(l)$, the zero set of a polynomial l. With $Y_1^+ = \{\overline{x} \in \mathbb{R}^n \mid l(\overline{x}) > 0\}$ and $Y_1^- = \{\overline{x} \in \mathbb{R}^n \mid l(\overline{x}) < 0\}$ we obtain

$$\overline{\mathrm{D}} \cap Y_1 = \overline{D \cap Y_1} \cup (\overline{D \cap Y_1^+} \cap Y_1) \cup (\overline{D \cap Y_1^-} \cap Y_1).$$

(This is a simple calculation: Take $\overline{x} \in \overline{D \cap Y_1} \Rightarrow \exists (\overline{x}_n)_{n \in \mathbb{N}} \subseteq D \cap Y_1 \;\; \overline{x}_n \rightarrow \overline{x} \overset{Y_1 \text{ closed}}{\Rightarrow} \overline{x} \in \overline{\mathrm{D}} \;\wedge\; \overline{x} \in Y_1$. Also clear is that $\overline{x} \in \overline{D \cap Y_1^+} \cap Y_1 \Rightarrow \overline{x} \in$

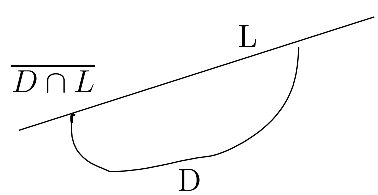

Figure 3.6: For a $\mathbb{Z}$-affine set L regard $\overline{\mathrm{D}} \cap L \neq \overline{D \cap L}$.

$\overline{\mathrm{D}} \cap Y_1$. Now let $\overline{\mathrm{x}} \in \overline{\mathrm{D}} \cap Y_1 \Rightarrow \exists \overline{\mathrm{x}}_n \in D \;\; \overline{\mathrm{x}}_n \to \overline{\mathrm{x}} \;\wedge\; \overline{\mathrm{x}} \in Y_1$. By taking a subsequence, we can assume

$$(\forall n\, \overline{\mathrm{x}}_n \in D \cap Y_1) \vee (\forall n\, \overline{\mathrm{x}}_n \in D \cap Y_1^+) \vee (\forall n\, \overline{\mathrm{x}}_n \in D \cap Y_1^-)$$

and then $\overline{\mathrm{x}}$ is in the right side of the equation.)

The set Y_1^+ is semi-algebraic, hence in $\mathcal{S}$ and so there exists a Ch-description $\mathfrak{Y}_1^+ \in \Sigma$ for Y_1^+ such that $\rho(\mathfrak{Y}_1^+) = 0$. By Lemma 2.2.5 there exists a Ch-description $\mathfrak{C}$ of $C := D \cap Y_1^+$ such that $\rho(\mathfrak{C}) \leq 2 + \rho(\mathfrak{D}) + \rho(\mathfrak{Y}_1^+) = 2 + \rho(\mathfrak{D}) < 4 + \rho(\mathfrak{D}) = \rho(\overline{\mathfrak{D}}) < k$. Hence, we can apply the induction hypothesis on $\mathfrak{D} \cap \mathfrak{Y}_1^+$. Now, $\overline{D \cap Y_1^+} \cap Y_1 = \partial \left(\overline{D \cap Y_1^+} \right) \cap Y_1$, so we can apply the Lemma of the approximation of the intersection with an affine set 3.2.14 and obtain an approximant for $\overline{D \cap Y_1^+} \cap Y_1$.

Analogously, there exists an approximant for $\overline{D \cap Y_1^-} \cap Y_1$. Furthermore, by induction hypothesis there is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for $D \cap Y_1$, since $\rho(\mathfrak{D} \cap Y_1) = 1 + \rho(\mathfrak{D}) = \rho(\mathfrak{B}) - 3 < k$. By definition, it is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for $\overline{D \cap Y_1}$.

The union case c) supplies an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant for $\overline{\mathrm{D}} \cap Y_1$.

The set $\overline{\mathrm{D}} \cap Y_1$ has empty interior, so we can apply the Lemma of the approximation of the intersection with an affine set 3.2.14 on $(\overline{\mathrm{D}} \cap Y_1) \cap Y_2$. This again has empty interior, so we can proceed and at last obtain an approximant for $\overline{\mathrm{D}} \cap Y_1 \cap \cdots \cap Y_k = A$.

□

3.3 Approximant for the Projection

To approximate the projection of a set we need some more work. We access several lemmas in [Wil99] and [BS04].

The following lemma helps us to maintain the property of an empty interior while taking the closure of a set in the o-minimal weak structure $\mathcal{S}$.

Lemma 3.3.1. *[Charbonnel, Lemma (2.1) in [Wil99]] Let $\mathcal{S}$ an o-minimal weak structure. Suppose $n \geq 1$ and $A \in \widetilde{\mathcal{S}}_n$. Then the following are equivalent:*

a) A has no interior points;

b) A has measure zero (in the Lebesgue measure on $\mathbb{R}^n$);

c) $\overline{A}$ has no interior points;

d) $\overline{A}$ *has measure zero.*

Proof. This fact is proved in Corollary (2.8) in [Max98]. □

For the approximant of the projection we use the partial derivatives of the function, hence we need some propositions about the derivatives and their singular and regular values.

Lemma 3.3.2. *[Theorem (2.6) Closure under Differentiation in [Wil99]] Let* $\mathcal{S}$ *be an o-minimal weak structure. Suppose* $n \geq 1$, $U \in \widetilde{\mathcal{S}}_n$ *is open and let* $f : U \to \mathbb{R}$ *be a* C^1*-function in* $\widetilde{\mathcal{S}}$*. Then the parital derivatives of* F *are also in* $\widetilde{\mathcal{S}}$*.*

The Theorem of weak selection (Theorem (2.3) in [Wil99]) and an improvement with the Theorem of the almost everywhere smoothness of functions (Theorem (2.4) in [Wil99]) imply the following lemma.

Lemma 3.3.3. *[Lemma of Weak Selection, Remark (2.5) in [Wil99]] Let* $\mathcal{S}$ *an o-minimal weak structure. Suppose* $A \in \widetilde{\mathcal{S}}_n, B \in \widetilde{\mathcal{S}}_{n+m}$ *and that* A *contains an interior point. Suppose further that* $\forall\, \overline{\mathrm{x}} \in A\ \exists\, \overline{\mathrm{y}} \in \mathbb{R}^m\ (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in B$*. Then there exists an open set* $U \in \widetilde{\mathcal{S}}_n$ *with* $U \subseteq A$*, and a function* $\phi : U \to \mathbb{R}^m$ *which is* C^N *(for a given* N*) such that* $\forall\, \overline{\mathrm{x}} \in U(\overline{\mathrm{x}}, \phi(\overline{\mathrm{x}})) \in B$*.*

The next theorem of Wilkie has to be changed because we want to use it for partial defined functions.

Lemma 3.3.4. *[Theorem (2.8) in[Wil99] with partial functions] Let* $\mathcal{S}$ *be an o-minimal weak structure. Suppose* $n > m \geq 1$ *and consider* C^1*-functions* $F : U \to \mathbb{R}^k$ *and* $f : U \to \mathbb{R}$ *in* $\widetilde{\mathcal{S}}$*, where* $U \subseteq \mathbb{R}^{n+k}$ *open and* $\Gamma(F)$ *closed,* $F, f \in \widetilde{\mathcal{S}}$*. Let* $\overline{\mathrm{a}} \in \mathbb{R}^k$ *be a regular value of* F*. Then there are at most finitely many* $b \in \mathbb{R}$ *such that* $(\overline{\mathrm{a}}, b)$ *is a singular value of the function* $(F, f) : U \to \mathbb{R}^{k+1}$*. Moreover the function* (F, f) *has a closed graph.*

Proof. The proof of this lemma is given in [Wil99], the restriction to partial functions does not change the proof, as we will see here. Let $F = (F_1, \ldots, F_k)$. The point $(\overline{\mathrm{a}}, b)$ is a singular value of (F, f) if and only if

$$\exists\, \overline{\mathrm{x}} \in U\ \ f(\overline{\mathrm{x}}) = \overline{\mathrm{a}}\ \wedge\ f(\overline{\mathrm{x}}) = b\ \wedge\ d_{\overline{\mathrm{x}}}F_1, \ldots, d_{\overline{\mathrm{x}}}F_k, d_{\overline{\mathrm{x}}}f \text{ are linearly dependent.}$$

By the closure under differentiation (Lemma 3.3.2) we know $d_{\overline{\mathrm{x}}}F_1, \ldots, d_{\overline{\mathrm{x}}}F_k, d_{\overline{\mathrm{x}}}f \in \widetilde{\mathcal{S}}$. Since we can express the linearly dependence through a formula and replace variables by values of functions by Lemma 1.3.5 it is $A := \{b \in \mathbb{R} \mid (\overline{\mathrm{a}}, b) \text{ singular value of } (F, f)\} \in \widetilde{\mathcal{S}}$.

We show now by contradiction that A must be finite. Assume, that A is infinite. By (WS5), A has only finitely many connected components, so it must contain an open interval, particularly an interior point. Define

$$S := \{(t, \overline{x}) \mid t \in A \wedge (F, f)(\overline{x}) = (\overline{a}, t) \wedge (\overline{a}, t) \text{ singular value of } (F, f)\}.$$

This is a set in $\widetilde{\mathcal{S}}$. Then for all $t \in A$ there exists $\overline{x} \in \mathbb{R}^{n+k}$ such that $(t, \overline{x}) \in S$, so we can apply the weak selection (Lemma 3.3.3) and obtain an interval $(\alpha, \beta) \in \widetilde{\mathcal{S}}$ and a function $\phi : (\alpha, \beta) \to \mathbb{R}^{n+k}$ such that $\forall t \in (\alpha, \beta)\ (t, \phi(t)) \in S$, i.e.

$$(*) \qquad f(\phi(t)) = t \text{ and}$$
$$(**) \qquad F(\phi(t)) = \overline{a}$$

and $d_{\phi(t)}F_1, \ldots, d_{\phi(t)}F_k, d_{\phi(t)}$ are linearly dependant.

However, by differentiation we obtain from (*) that $d_{\phi(t)}f \circ d_t\phi = 1$ and out of (**) that $d_{\phi(t)}F_i \circ d_t\phi = 0$ for $i \in \{1, \ldots, m\}$. This leads already to the fact that $d_{\phi(t)}F_1, \ldots d_{\phi(t)}F_k$ are linearly dependant for all $t \in (\alpha, \beta)$, which is a contradiction to the fact that $F(\phi(t)) = \overline{a}$ is a regular value.

The graph of the function (F, f) is closed, since the graph of F is closed and and because of a similar calculation as in Lemma 3.2.10. □

Lemma 3.3.5. *[Corollary (2.9) in [Wil99], modified] Let $\mathcal{S}$ be an o-minimal weak structure. Let $n, k \geq 1$ and $F : U \to \mathbb{R}^k$ a C^1 function in $\widetilde{\mathcal{S}}$ with $U \subseteq \mathbb{R}^{n+k}$ open and $\Gamma(F)$ closed. Let $\overline{a}$ be a regular value of F and U be an open ball in $\mathbb{R}^n$ such that that the set $X := F^{-1}(\overline{a}) \cap (U \times \mathbb{R}^k)$ is non-empty and bounded. Then either*

a) $\pi_n^{n+k}[X] = U$ or

b) there exists $\eta > 0$ and distinct $i_1, \ldots i_k \in \{1, \ldots, m+k\}$ such that

$$det\left(\frac{\partial(F_1, \ldots, F_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}\right)^2 \upharpoonright X$$

takes all values in the interval $[0, \eta]$, where $F = (F_1, \ldots, F_k)$.

Without loss of generality, we can assume in b) that $1 \leq i_1 < \cdots < i_k \leq m+k$.

Proof. In this proof we need to check some details, since our function is only partial defined. Particularly we need that F is defined on an open set.

Case 1: $\pi[X]$ is finite.

Let $\overline{b} \in \pi[X]$. Then there exists $\overline{c} \in \mathbb{R}^k$ such that $(\overline{b}, \overline{c}) \in X$. It is contained in a connected component of $F^{-1}(\overline{a})$, call it Y.

We show $\pi(Y) = \{\overline{b}\}$: If another point $\overline{b}' \neq \overline{b}$ is contained in $\pi(Y)$, there must exist $\overline{c}'$ with $(\overline{b}', \overline{c}') \in Y$ and since Y is connected, there is a path $\gamma : [0,1] \to F^{-1}(\overline{a})$ between the points $(\overline{b}, \overline{c})$ and $(\overline{b}', \overline{c}')$. Since U is open and $\overline{b} \in U$ there exists $t > 0$ such that $\gamma(t) \neq \overline{b}$ and $\pi[\gamma([0,t])] \subseteq U$ and $\gamma([0,t]) \subseteq F^{-1}(\overline{a})$, therefore $\pi[\gamma([0,t])] \subseteq \pi[X]$. Since $\pi[\gamma([0,t])]$ is connected, it is infinite, which is a contradiction to our assumption that $\pi[X]$ is finite.

Thus we have $Y \subseteq X$ and since X is bounded, Y is bounded, too.

In the proof of the fact that Y is compact, we have to take care of the only partial defined function, since $F^{-1}(\overline{a})$ is in general not closed for partial functions. (For example take $f : (0,1) \to \mathbb{R}$, defined by $f(x) = 1$. Then $f^{-1}(1) = (0,1)$ is open.)

The set Y is a connected component of $D := \pi^{-1}(\overline{b}) \cap F^{-1}(\overline{a})$, which is a closed set in U (since F and π are continuous and $\{\overline{b}\}, \{\overline{a}\}$ are closed). Now we prove that D is closed: Take a sequence $\overline{x}_n \to \overline{x}$ such that $\overline{x}_n \in F^{-1}(\overline{a})$. So $F(\overline{x}_n) = \overline{a}$ for all $n \in \mathbb{N}$. Therefore $(\overline{x}_n, \overline{a}) \to (\overline{x}, \overline{a})$. Since $(\overline{x}_n, \overline{a}) \in \Gamma(F)$, which is closed, $(\overline{x}, \overline{a}) \in \Gamma(F)$. In particular, $\overline{x} \in F^{-1}(\overline{a})$. This shows that $F^{-1}(\overline{a})$ is closed. The set $\pi^{-1}(\{\overline{b}\})$ is closed, since $\{\overline{b}\}$ is closed and π continuous on $\mathbb{R}^{n+k}$. So D is closed as an intersection of two closed sets.

Then Y as a connected component of a closed set is closed itself. Hence Y is compact.

Since $\overline{a}$ is a regular value of F and $F(\overline{b}, \overline{c}) = 0$ the derivation has full rank, so there exists $1 \leq i_1 < \cdots < i_k \leq n+k$ such that $\det\left(\frac{\partial F}{\partial(x_{i_1}, \ldots, x_{i_k})}\right)(\overline{b}, \overline{c}) = \eta \neq 0$. The next part will be a bit more detailed than in [Wil99], since it was difficult to understand his proof.

Permute coordinates and write $(\overline{u}, \overline{v}) \in \mathbb{R}^n \times \mathbb{R}^k$ with $\overline{v} = (x_{i_1}, \ldots, x_{i_k})$ and $\overline{u}$ collects the rest coordinates, call it $(x_{j_1}, \ldots, x_{j_n})$. Since Y is compact, the maximum of $\overline{x} \to x_{j_1}$ is taken, let us call the corresponding point $(\overline{y}, \overline{z})$ (coordinates as said above) with $\overline{y}_{j_1}$ is maximal.

Claim : The partial $\frac{\partial F}{\partial v}$ is not invertable in $(\overline{y}, \overline{z})$.

Proof: Assume, $\det\left(\frac{\partial F}{\partial v}\right)(\overline{y}, \overline{z}) \neq 0$. Since F is defined on an open set, there exists some open $U_0 \subseteq \mathbb{R}^n, V_0 \subseteq \mathbb{R}^k$ such that $(\overline{y}, \overline{z}) \in U_0 \times V_0$ and F is defined on $U_0 \times V_0$.

By the implicit function theorem now there exist some open neighborhoods of U' of $\overline{y}$ and V of $\overline{z}$ and a C^N function $g : U' \to V$ such that $g(\overline{y}) = \overline{z}$ and $F(\overline{y}', g(\overline{y}')) = \overline{a}$ for all $\overline{y}' \in U'$. So $F^{-1}(\overline{a}) \supseteq \Gamma(g)$. Since U' is open, there exists an $\varepsilon > 0$ such that $\overline{y} + \varepsilon\, e_{j_1} \in U'$, so $\overline{c} = (\overline{y} + \varepsilon\, e_{j_1}, g(\overline{y} + \varepsilon\, e_{j_1})) \in F^{-1}(\overline{a})$. If ε is small enough, $\overline{c}$ is in the same connected components as $(\overline{y}, \overline{z})$ and $\pi(\overline{c}) \in U'$ and therefore in Y. But $\pi_{j_1}(\overline{c}) = \overline{y}_{j_1} + \varepsilon > \overline{y}_{j_1}$, which was chosen as maximum. Contradiction. □(Claim)

Thus $\det\left(\frac{\partial F}{\partial v}\right)(\overline{\mathrm{y}},\overline{\mathrm{z}}) = 0$, so $\det\left(\frac{\partial F}{\partial v}\right)$ must take all values between 0 and η (since it is continuous), what was to prove.

Case 2: $\pi[X]$ is infinite.

Then there exists an $i \in \{1,\ldots,n\}$ such that $\pi_i \circ \pi(X)$ is infinite (with π_i projection on the i-th coordinate). Apply now Lemma 3.3.4 on $f = \pi_i \circ \pi$. So we can choose $b \in \pi_i(U)$ such that $(\overline{\mathrm{a}}, b)$ is a regular value of $(F, \pi_i \circ \pi) : U \to \mathbb{R}^{k+1}$. Define

$$\hat{U} = \left\{(x_1,\ldots,x_{n-1}) \in \mathbb{R}^{n-1} \mid (x_1,\ldots,x_{i-1},b,x_i,\ldots,x_{n-1+k}) \in U\right\}$$

and $\hat{F} : \hat{U} \to \mathbb{R}$ by $\hat{F}(x_1,\ldots,x_{n-1+k}) = F(x_1,\ldots,x_{i-1},b,x_i,\cdots x_{n-1+k})$.

It is clear, that $\hat{U}$ is an open ball and in $\widetilde{\mathcal{S}}$, since U is open and in $\widetilde{\mathcal{S}}$. The graph of $\hat{F}$ is closed: Assume that $(x_1^{(i)},\ldots,x_{n-1+k}^{(i)}, \hat{F}(\overline{\mathrm{x}}^{(i)}) \to (x_1,\ldots,x_{n-1+k},\overline{\mathrm{z}})$. Then

$$\left(x_1^{(i)},\ldots,b,\ldots,x_{n-1+k}^{(i)},\hat{F}(\overline{\mathrm{x}}^{(i)})\right) \to (x_1,\ldots,b,\ldots,x_{n-1+k},\overline{\mathrm{z}}),$$

since $(x_1^{(i)},\ldots,b,\ldots,x_{n-1+k}^{(i)},\hat{F}(\overline{\mathrm{x}}^{(i)})) \in \Gamma(F)$, which is closed and so we obtain that the limit $(x_1,\ldots,x_{n-1+k},\overline{\mathrm{z}}) \in \Gamma(\hat{F})$.

Now $\overline{\mathrm{a}}$ is a regular value of $\hat{F}$: By assumption $(\overline{\mathrm{a}}, b)$ is a regular value of $(F, \pi_i \circ \pi)$, so we get that the vectors $\frac{\partial(F,\pi_i\circ\pi)}{\partial x_{j_1}},\ldots,\frac{\partial(F,\pi_i\circ\pi)}{\partial x_{j_{k+1}}}$ are linearly independent. But only for $j_l = i$ we have $\frac{\pi_i\circ\pi}{\partial x_i} \neq 0$. (There exists an l with $j_l = i$, otherwise $\frac{\partial F}{\partial x_{j_1}},\ldots,\frac{\partial F}{\partial x_{j_{k+1}}}$ would be independent, but these are $k+1$ vectors of dimension k.) Hence we can take the rest of the vectors, which are derivations of $\hat{F}$, too and they are linear independent.

At last define $\hat{X} = \hat{F}^{-1}(\overline{\mathrm{a}}) \cap (\hat{U} \times \mathbb{R}^k)$ and $\hat{\pi} : \mathbb{R}^{n-1+k} \to \mathbb{R}^{n-1}$.

Now there are again two cases:

a. $\hat{\pi}[\hat{X}]$ is finite and $n > 1$.

Apply the argument of Case 1 to $\hat{F}, \hat{U}$ and $\hat{X}$ and obtain the second conclusion for $\hat{F}$. It is obvious that the same $i_1,\ldots,i_k$ satisfy the condition for F.

b. $\hat{\pi}[\hat{X}]$ is infinite and $n > 1$ or $n = 1$.

Apply Case 2 to $\hat{F}, \hat{U}$ and $\hat{X}$ (if $n > 1$).

Continue in this way until Case 1 is reached and therefore the second conclusion or until we find $\overline{\mathrm{b}} \in U$ such that $(\overline{\mathrm{a}},\overline{\mathrm{b}})$ is a regular value of the function $(F,\pi) = (F, \pi_1 \circ \pi,\ldots,\pi_n \circ \pi)$. (This happen at least if we reach $n = 1$.) So there exists a $\overline{\mathrm{c}} \in \mathbb{R}^k$ such that $(F,\pi)(\overline{\mathrm{b}},\overline{\mathrm{c}}) = (\overline{\mathrm{a}},\overline{\mathrm{b}})$ and so $(\overline{\mathrm{b}},\overline{\mathrm{c}}) \in X$ and $\det\left(\frac{\partial F}{\partial x_{n+1},\ldots,x_{n+k}}\right)(\overline{\mathrm{b}},\overline{\mathrm{c}}) \neq 0$.

Regard the connected component Y of $F^{-1}(\overline{\mathrm{a}})$ containing $(\overline{\mathrm{b}}, \overline{\mathrm{c}})$. Y is bounded, since X is bounded.

Now it is possible that $\pi(X) = U$, then we reached the first condition.

So assume $\pi(X) \neq U$, hence there exists $\overline{\mathrm{b}}' \in U$ such that $\overline{\mathrm{b}}' \notin F^{-1}(\overline{\mathrm{a}})$. By assumption X is bounded and $F^{-1}(\overline{\mathrm{a}})$ is closed, so $\pi(X)$ must be closed in U. (If $\overline{\mathrm{x}}_n \to \overline{\mathrm{x}}$ in U, $\overline{\mathrm{x}}_n \in \pi(X) \cap U$, so there exists $\overline{\mathrm{z}}_n$ such that $(\overline{\mathrm{x}}_n, \overline{\mathrm{z}}_n) \in X$. As X is bounded, there is a subsequence which converges, so we can assume $(\overline{\mathrm{x}}_n, \overline{\mathrm{z}}_n) \to (\overline{\mathrm{x}}, \overline{\mathrm{z}})$ with $(\overline{\mathrm{x}}_n, \overline{\mathrm{z}}_n) \in F^{-1}(\overline{\mathrm{a}})$ and since this is closed we obtain $(\overline{\mathrm{x}}, \overline{\mathrm{z}}) \in F^{-1}(\overline{\mathrm{a}})$, i.e. $\overline{\mathrm{x}} \in \pi(X)$.)

Let $\gamma : [0,1] \to U$ be a path from $\overline{\mathrm{b}}$ to $\overline{\mathrm{b}}'$. Since $\pi(F^{-1}(\overline{\mathrm{a}}))$ is closed in U, there exist $t \in [0,1]$ such that $\gamma(t) \in \pi(F^{-1}(\overline{\mathrm{a}}))$ and $\gamma(s) \notin \pi(F^{-1}(\overline{\mathrm{a}})$ for all $s > t$.

Now there exists $(\overline{\mathrm{y}}, \overline{\mathrm{z}}) \in U \times \mathbb{R}^k$ and $\overline{\mathrm{y}} = \gamma(t)$ and if we assume

$$\det\left(\frac{\partial F}{\partial x_{n+1}, \ldots x_{n+k}}\right)(\overline{\mathrm{y}}, \overline{\mathrm{z}}) \neq 0$$

it follows by the implicit function theorem that $(\gamma(t+\varepsilon), g(\gamma(t+\varepsilon))) \in F^{-1}(\overline{\mathrm{a}})$ (similar to the first case). This is a contradiction to our choice of γ and t. □

To use this lemma in our context, Berarducci and Servi modified it. It is no difference with partial functions on open sets. They do not give a proof, so we do here. In this form the lemma helps us to find conditions for the new approximants of projections.

Lemma 3.3.6. *[Lemma (10.4) in [BS04], slightly modified] Let $\mathcal{S}$ be an o-minimal weak structure. Let $F : U \to \mathbb{R}^k$ be a C^1 function in $\widetilde{\mathcal{S}}$, $U \in \widetilde{\mathcal{S}}_{m+k}$ open and $\Gamma(F)$ closed. Let $a \in \mathbb{R}^k$ be a regular value of F. Define $V := F^{-1}(a)$. Let O be an open ball in $\mathbb{R}^m$ such that $O \cap \partial\pi_m^{m+k} V \neq \emptyset$. Then for every sufficient small $\varepsilon > 0$ is $O \cap \pi_m^{m+k} V[\varepsilon] \neq \emptyset$ where $V[\varepsilon] \subseteq V$ is defined as the set of points $(x_1, \ldots, x_{m+k}) \in V$ satisfying one of the following conditions:*

- $|(1, x_{m+1}, \ldots, x_{m+k})| = \frac{1}{\varepsilon}$
- $\det(\frac{\partial F}{\partial x_{i_1}, \ldots, x_{i_k}})^2 = \varepsilon$ *for some* $1 \leq i_1 < \cdots < i_k \leq m+k$.

The set $V[\varepsilon]$ is called the critical part of V.

Proof. By assumption $O \cap \partial\pi_m^{m+k}[F^{-1}(\overline{a})] \neq \emptyset$. Then $X := F^{-1}(\overline{a}) \cap (O \times \mathbb{R}^k)$ is not empty.

1. Case: X bounded.

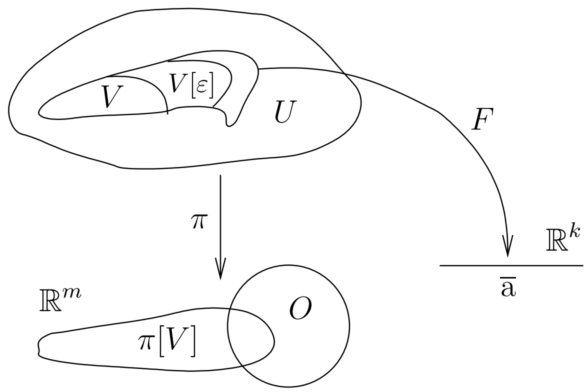

Figure 3.7: Sketch to Lemma 3.3.6.

Apply Lemma 3.3.5 on F and O. Assume that the first condition $\pi_m^{m+k}(X) = O$ holds. We have $V \supseteq X$, so $\pi[F^{-1}(\overline{\mathrm{a}})] \supseteq \pi[X] = O$. Since O is open, we obtain $(\partial\pi[F^{-1}(\overline{\mathrm{a}})]) \cap O = \emptyset$, which is contradiction to our assumption.

So the second condition must hold and there exist $\eta < 0$ and $1 \leq i_1 < \cdots < i_k \leq m+k$ such that $\det\left(\frac{\partial F}{\partial(x_{i_1},\ldots,x_{i_k})}\right)^2 \restriction X$ takes all values in the interval $[0,\eta]$. Hence for all sufficient small $\varepsilon > 0$ the value ε is taken in X and so $\pi_m^{m+k}(V[\varepsilon]) \cap O = \pi_m^{m+k}(F^{-1}(\overline{\mathrm{a}})[\varepsilon]) \cap O \neq \emptyset$.

2. Case: X **unbounded**. Due to the fact that O is bounded, the other components must be unbounded, so for a sufficient small ε there have to be $x_1, \ldots, x_{m+k}$ such that $(x_1, \ldots, x_{m+k}) \in X$ and $|(1, x_{m+1}, \ldots, x_{m+k})|^2 = \frac{1}{\varepsilon}$. So $O \cap \pi(V[\varepsilon]) \neq \emptyset$. □

The next two lemmas helps us to provide regularity, that we need to apply the above lemma.

Lemma 3.3.7. *[Lemma (3.4) in [Wil99]] Let $\mathcal{S}$ be an o-minimal weak structure, $k \geq 1$, $A \in \widetilde{\mathcal{S}}_k$ and suppose A contains no interior points. Then $\forall^{\mathrm{s}}\varepsilon_1 \ldots \forall^{\mathrm{s}}\varepsilon_k\ (\varepsilon_1, \ldots, \varepsilon_k) \notin A$.*

Proof. Look at Lemma (3.4) in [Wil99]. □

Before we can start to prove that there exist approximants for projections, we need one last helpful lemma about the singular values of the functions we need in the approximants, again to guarantee regularity.

Lemma 3.3.8. *[Singular Values, Lemma (2.7) in [Wil99]] Let $\mathcal{S}$ be an o-minimal weak structure. Suppose that $n \geq m \geq 1$ and that $F : U \to \mathbb{R}^m$ is a C^1 function in $\widetilde{\mathcal{S}}$, where U is an open subset of $\mathbb{R}^n$. Then the subset $Sing(F)$ of $\mathbb{R}^m$ consisting of the singular values of F is a set in $\widetilde{\mathcal{S}}$ containing no interior points.*

Proof. Look at Lemma (2.7) in [Wil99]. □

Now we have collected all necessary statements and can prove that there is an approximant for a projection of a set that has already an approximant. Recall the corresponding lemma.

Lemma 3.3.9 (Approximant for Projections, Lemma 3.2.12)**.** *Let $\mathcal{S}$ be an o-minimal weak structure. If $A \subseteq \mathbb{R}^{n+1}$ has an $M^{N+1}(\mathcal{S})$-approximant $S \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^k_+$, then there is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant $S' \subseteq \mathbb{R}^n \times \mathbb{R}^{k+1}_+$ for $\pi^{n+1}_n[A] \subseteq \mathbb{R}^n$.*

Proof. Look at $S = \bigcup_{i=1}^s S_i$, where S_i is a $M^{N+1}(\mathcal{S})$-constituent, let

$$S_i = \left\{(\overline{\mathrm{x}}, \overline{\varepsilon}) \in \mathbb{R}^{n+1} \times \mathbb{R}^k_+ \mid \exists \overline{y} \in \mathbb{R}^{k-1}\ (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in U_i \wedge F_i(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = \overline{\varepsilon}\right\},$$

where $F_i : U_i \to \mathbb{R}^k$ is a $M^{N+1}(\mathcal{S})$-function, particularly C^{N+1}. Note that

$$\begin{aligned} S_{i,\varepsilon_1,\dots,\varepsilon_k} &= \left\{\overline{\mathrm{x}} \in \mathbb{R}^{n+1} \mid \exists \overline{y} \in \mathbb{R}^{k-1}\ (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in U_i \ \wedge\ F_i(\overline{\mathrm{x}}, \overline{y}) = \overline{\varepsilon}\right\} \\ &= \pi^{n+1+k-1}_{n+1}[F_i^{-1}(\overline{\varepsilon})] \end{aligned}$$

Then we have obviously $S_{\varepsilon_1,\dots,\varepsilon_k} = \bigcup_{i=1}^s S_{i,\varepsilon_1,\dots,\varepsilon_k}$.

We can assume that $\overline{\varepsilon}$ is regular for all F_i: By Lemma 3.3.8 about singular values $Sing(F_i)$ contains no interior points and by Lemma 3.3.7 for all sufficient small $\varepsilon_0, \dots, \varepsilon_k$ we have $(\varepsilon_1, \dots, \varepsilon_k) \notin Sing(F_i)$. Taking the minimum of the boundaries of the $\overline{\varepsilon}_i$ over the finite count of the F_i, we can assume that $\overline{\varepsilon}$ is regular for all F_i for all sufficient small $\overline{\varepsilon}$.

Let $V_i := F_i^{-1}(\varepsilon_1, \dots, \varepsilon_k)$ and $V_i[\varepsilon_{k+1}]$ be as defined in Lemma 3.3.6.

Define S' by the sections $S'_{\varepsilon_1,\dots,\varepsilon_{k+1}} \subseteq \mathbb{R}^n$:

$$S'_{\varepsilon_1,\dots,\varepsilon_{k+1}} = \pi^{n+1}_n \left[\bigcup_{i=1}^s \pi^{n+1+k-1}_{n+1}\left[F_i^{-1}(\varepsilon_1, \dots, \varepsilon_k)[\varepsilon_{k+1}]\right]\right]$$

1. S' is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-set.

We use the following abbreviations to simplify the calculation: Set $\overline{\mathrm{x}} = (x_1, \ldots, x_n)$, $\overline{\mathrm{y}} = (x_{n+2}, \ldots, x_{n+k})$, and $\overline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_k)$. Assume that there is only one constituent for S and take the union afterwards.

$$
\begin{aligned}
& (\overline{\mathrm{x}}, \overline{\varepsilon}, \varepsilon_{k+1}) \in S' \\
\Leftrightarrow\ & \exists x_{n+1} (\overline{\mathrm{x}}, x_{n+1}, \overline{\varepsilon}, \varepsilon_{k+1}) \in V[\varepsilon_{k+1}] \\
\Leftrightarrow\ & \exists x_{n+1} (\overline{\mathrm{x}}, x_{n+1}, \overline{\varepsilon}) \in V \ \wedge \\
& \left(|(1, \overline{\mathrm{y}})| < \frac{1}{\varepsilon_{k+1}} \vee \det \left(\frac{\partial F}{\partial x_{i_1}, \ldots x_{i_{k-1}}} \right) = \varepsilon_{k+1} \right) \\
\Leftrightarrow\ & \exists \overline{\mathrm{y}} \Big((\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) \in U \ \wedge \ F(\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) = \overline{\varepsilon} \\
& \quad \wedge \big(\exists x_{n+1+k} |(1, \overline{\mathrm{y}}, x_{n+1+k})| \big)^{-1} = \varepsilon_{k+1} \\
& \quad\quad \vee \det \left(\frac{\partial F}{\partial x_{i_1}, \ldots x_{i_{k-1}}} \right) = \varepsilon_{k+1} \Big) \Big) \\
\Leftrightarrow\ & \Big(\exists x_{n+1} \exists \overline{\mathrm{y}}\, \exists x_{n+1+k} \big((\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}, x_{n+1+k}) \in U \times \mathbb{R} \\
& \quad \wedge F(\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) = \overline{\varepsilon} \wedge |(1, \overline{\mathrm{y}}, x_{n+1+k})|^{-1} = \varepsilon_{k+1} \big) \Big) \\
& \vee \Big(\exists x_{n+1} \exists \overline{\mathrm{y}} \big((\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) \in U \wedge F(x_1, \ldots, x_{n+1+k-1}) = \overline{\varepsilon} \\
& \quad \wedge \det \left(\frac{\partial F_i}{\partial x_{i_1}, \ldots x_{i_{k-1}}} \right) = \varepsilon_{k+1} \big) \Big) \\
\Leftrightarrow\ & (\overline{\mathrm{x}}, \overline{\varepsilon}, \varepsilon_{k+1}) \in \Big\{ (\overline{\mathrm{x}}, \overline{\varepsilon}, \varepsilon_{k+1}) \mid \exists x_{n+1} \exists \overline{\mathrm{y}}\, \exists x_{n+1+k} \\
& \quad \big((\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}, x_{n+1+k}) \in U \times \mathbb{R} \wedge F(\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) = \overline{\varepsilon} \\
& \quad \wedge |(1, \overline{\mathrm{y}}, x_{n+1+k})|^{-1} = \varepsilon_{k+1} \big) \Big\} \\
& \cup \Big\{ (\overline{\mathrm{x}}, \overline{\varepsilon}, \varepsilon_{k+1}) \mid \exists x_{n+1} \exists \overline{\mathrm{y}} \big((\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}) \in U \\
& \quad \wedge F(x_1, \ldots, x_{n+1+k-1}) = \overline{\varepsilon} \wedge \det \left(\frac{\partial F}{\partial x_{i_1}, \ldots x_{i_{k-1}}} \right) = \varepsilon_{k+1} \big) \Big\}
\end{aligned}
$$

The function $g(\overline{\mathrm{x}}, x_{n+1}, \overline{\mathrm{y}}, x_{n+1+k}) := |(1, \overline{\mathrm{y}}, x_{n+1+k})|^{-1}$ is in $M^N(\mathcal{S})$ for all N and it is defined on $U \times \mathbb{R}$, which is an open set in $\widetilde{\mathcal{S}}$. The graph of (F, g) is closed by the Lemma of the closed graph 3.2.10. So the first part of the union is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-constituent.

The function $\det\left(\frac{\partial F_i}{\partial x_{i_1}, \ldots x_{i_{k-1}}}\right)$ is C^N, since F_i was C^{N+1}, and so it is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-function. Again by the Lemma of the closed graph 3.2.10 also the composed function $(F_i, \det\left(\frac{\partial F_i}{\partial x_{i_1}, \ldots x_{i_{k-1}}}\right)) : U_i \to \mathbb{R}^{k+1}$ has a closed graph. Hence S' is an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-set.

2. S' **approximates** $\overline{\pi_n^{n+1}(A)}$ **from below.**

We have to show $\forall^s\varepsilon_0 \dots \forall^s\varepsilon_{k+1}\ S'_{\varepsilon_1,\dots,\varepsilon_{k+1}} \subseteq \overline{\pi_n^{n+1}[A]}^{\varepsilon_0}$. We know already that S approximates $\overline{A}$ from below, thus we have $\forall^s\varepsilon_0 \dots \forall^s\varepsilon_k\, S_{\varepsilon_1,\dots,\varepsilon_k} \subseteq \overline{A}^{\varepsilon_0}$. By the definition $V_i[\varepsilon_{k+1}] \subseteq V_i$, so $S'_{\varepsilon_1,\dots,\varepsilon_{k+1}} \subseteq \pi_n^{n+1}(S'_{\varepsilon_1,\dots,\varepsilon_k}) \subseteq \pi_n^{n+1}[\overline{A}]^{\varepsilon_0} \subseteq \overline{\pi_n^{n+1}[A]}^{\varepsilon_0}$.

3. S' **approximates** $\partial\overline{\pi_n^{n+1}A}$ **from above on bounded sets.**

We have to show $\forall^s\varepsilon_0 \dots \forall^s\varepsilon_{k+1}\ \partial\overline{\pi_n^{n+1}[A]} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0) \subseteq S'^{\varepsilon_0}_{\varepsilon_1,\dots,\varepsilon_k}$. Let $\varepsilon_0 > 0$. The bounded set $\partial\overline{\pi_n^{n+1}[A]} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0)$ is compact, so it is possible to find open balls $O_1, \dots, O_m$ of radius $\frac{\varepsilon_0}{2}$ such that $\partial\overline{\pi_n^{n+1}[A]} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0) \subseteq O_1 \cup \dots \cup O_m$ and $O_i \cap \partial\overline{\pi_n^{n+1}[A]} \neq \emptyset$. Therefore $O_i \nsubseteq \overline{\pi_n^{n+1}[A]}$ for $i = 1, \dots, m$.

Claim : The ball O_i is not included in $\pi(S_{\varepsilon_1,\dots,\varepsilon_k})$ for all sufficient small $\overline{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_k)$.

Proof: Assume that $\forall\mu_1\exists\,\varepsilon_1 < \mu_1 \dots \forall\mu_k\exists\,\varepsilon_k < \mu_k$ such that $O_i \subseteq \pi(S_{\overline{\varepsilon}})$. We know that S approximates $\overline{A}$ from below, so for all sufficient small δ we have $\forall^s\varepsilon_1 \dots \forall^s\varepsilon_k\ S_{\overline{\varepsilon}} \subseteq \overline{A}^{\delta}$. So there exist for every i some $\varepsilon_1, \dots, \varepsilon_k$ with $O_i \subseteq \pi_n^{n+1}[S_\varepsilon] \subseteq \pi_n^{n+1}[\overline{A}^{\delta}] \subseteq \pi_n^{n+1}[\overline{A}]^{2\delta}$. Since $\delta > 0$ was arbitrary we have $O \subseteq \overline{\pi(A)}$, which contradicts to $O_i \cap \partial\overline{\pi_n^{n+1}[A]} \neq \emptyset$, since $\overline{\pi_n^{n+1}[A]}$ is closed and O_i open. □(Claim)

Furthermore $O_i \cap \pi_n^{n+1}[S_{\varepsilon_1,\dots,\varepsilon_k}] \neq \emptyset$, hence $O_i \cap \partial\pi_n^{n+1}[S_{\varepsilon_1,\dots,\varepsilon_k}] \neq \emptyset$. Now by Lemma 3.3.6 $\forall^s\varepsilon_1, \dots, \forall^s\varepsilon_{k+1}\, O_i \cap \pi_n^{n+1}[S_{\varepsilon_1,\dots,\varepsilon_k}[\varepsilon_{k+1}]] = O_i \cap S'_{\varepsilon_1,\dots,\varepsilon_{k+1}} \neq \emptyset$, so $O_i \subseteq (S'_{\varepsilon_1,\dots,\varepsilon_{k+1}})^{\varepsilon_0}$.

Since $\partial\overline{\pi(A)} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0)$ is covered by the balls O_i, it is contained in $(S'_{\varepsilon_1,\dots,\varepsilon_{k+1}})^{\varepsilon_0}$.

□

This finishes the projection case and so the proof of Theorem 3.2.8.

3.4 Second Step: A Set containing the Boundary

Now we prove the second central statement, that we need to proceed by the cell decomposition argument Wilkie gives in [Wil99] and prove the theorem of the complement. For every set A in the Charbonnel closure of an o-minimal weak structure $\widetilde{\mathcal{S}}$ we construct a closed set in $\widetilde{\mathcal{S}}$ with empty interior which contains the boundary of A.

Here the partial defined functions do not lead to additional problems, so we can follow the proof given in Chapter 6 of [BS04].

Theorem 3.4.1. *Let $\mathcal{S}$ be a o-minimal weak structure which satisfies DPC^N for all N. Then for every closed set $A \in \widetilde{\mathcal{S}}_n$ there exists a closed set $B \in \widetilde{\mathcal{S}}_n$ such that B has empty interior and $\partial A \subseteq B$.*

For the empty interior property we need the Morse-Sard-Theorem and apply it on functions in the Charbonnel closure of our o-minimal weak structure.

Theorem 3.4.2 (Morse-Sard Theorem (1.3) in [Hir94], Chapter 3)**.** *Let M, N be manifolds, let $r > max\{0, dim(M) - dim(N)\}$ and let $f : M \to N$ be a $C^r(M, N)$ map. Then the set of the critical values of f, $Sing(f)$ has measure zero.*

Proof. Look at [Hir94], Chapter 3. □

As a direct consequence of applying the above lemma to an open subset $B \subseteq \mathbb{R}^k$ as M and $N = \mathbb{R}^n$, we obtain the following corollary.

Corollary 3.4.3. *Let $\mathcal{S}$ be an o-minimal weak structure. If $f : B \to \mathbb{R}^n$ is a C^1 function, $B \in \widetilde{\mathcal{S}}_k$ open with $n > k$ and $f \in \widetilde{\mathcal{S}}$, then $im(f)$ has empty interior.*

The next step is a statement about $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-sets. We need it to proceed with the approximants, which are $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-sets.

Lemma 3.4.4. *[Lemma (6.7) in [BS04]] Let $\mathcal{S}$ be an o-minimal weak structure and let $N \geq 2$. Every $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-set $S \subseteq \mathbb{R}^{n+k}$ is in $\widetilde{\mathcal{S}}_{n+k}$ and has empty interior.*

Proof. It is clear that $S \in \widetilde{\mathcal{S}}_{n+k}$.

Let $T = \{(\overline{\mathrm{x}}, \overline{\varepsilon}) \mid \exists \overline{\mathrm{y}} \in \mathbb{R}^{k-1} F(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = \overline{\varepsilon}\}$ be an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-constituent of S. Then $T = im(h)$, where $h : \mathbb{R}^{n+k-1} \to \mathbb{R}^{n+k}$ is defined by $h(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = (\overline{\mathrm{x}}, F(\overline{\mathrm{x}}, \overline{\mathrm{y}}))$. By Corollary 3.4.3 we obtain that $\mathrm{im}(h)$ has empty interior and so T and S have empty interior. □

Lemma 3.4.5. *[Charbonnel, Theorem (2.2) in [Wil99]] Let $\mathcal{S}$ be an o-minimal weak structure. Suppose that $A \in \widetilde{\mathcal{S}}_{n+1}$ and $A \subseteq \mathbb{R}^n \times \mathbb{R}_+$. Define a set $B := \{\overline{\mathrm{x}} \mid (\overline{\mathrm{x}}, 0) \in \overline{A}\}$. Then $B \in \widetilde{\mathcal{S}}_n$ and if A contains no interior points then nor does B.*

Proof. Look at Lemma (4.3) in [Max98]. □

Lemma 3.4.6. *[Lemma (3.3) in [Wil99]] Let $\mathcal{S}$ be an o-minimal weak structure. Let $A \in \widetilde{\mathcal{S}}_n$ and suppose $S \in \mathcal{S}_{n+k}$ has empty interior and approximates $\partial\overline{A}$ from above on bounded sets. Then there exists a closed set $B \in \widetilde{\mathcal{S}}_n$ with empty interior such that $\partial\overline{A} \subseteq B$.*

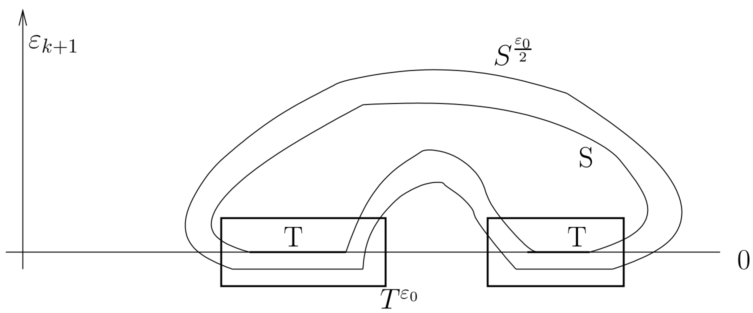

Figure 3.8: How $S^{\frac{\varepsilon_0}{2}}_{\overline{\varepsilon},\varepsilon_{k+1}}$ is contained in $T^{\varepsilon_0}_{\overline{\varepsilon}}$

Proof. The proof works by induction on k as in Lemma 3.3. in [Wil99] using Lemma 3.4.5.

Consider $k = 0$. Take $B = \overline{S}$. By Lemma 3.3.1 B has empty interior like S. Since S approximates $\partial\,\overline{\mathrm{A}}$ from above, $\forall^{\mathrm{s}}\varepsilon_0\ \partial\,\overline{\mathrm{A}} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0) \subseteq S^{\varepsilon_0}$. Taking the limit $\varepsilon_0 \to 0$ we obtain $\partial\,\overline{\mathrm{A}} \subseteq \overline{S} = B$.

Induction hypothesis: Suppose the lemma is true for all $A \in \widetilde{\mathcal{S}}_n$ with approximants in $\mathcal{S}_{n+k}$.

Now let $k \to k+1$. Let $A \in \widetilde{\mathcal{S}}_n$ and let $S \in \mathcal{S}_{n+k+1}$ with empty interior approximate $\partial\,\overline{\mathrm{A}}$ from above on bounded sets. Define

$$T := \overline{\mathrm{S}}_0 = \{(\overline{\mathrm{x}}, \varepsilon_1, \dots, \varepsilon_k) \mid (\overline{\mathrm{x}}, \varepsilon_1, \dots, \varepsilon_k, 0) \in \overline{\mathrm{S}}\}$$

By Lemma 3.4.5 $T \in \widetilde{\mathcal{S}}_{n+k}$ and T contains no interior points.

The set T approximates $\partial\,\overline{\mathrm{A}}$ from above on bounded sets: We have to show that

$$\forall^{\mathrm{s}}\varepsilon_0 \dots \forall^{\mathrm{s}}\varepsilon_k\ \ \partial\,\overline{\mathrm{A}} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0) \subseteq T^{\varepsilon_0}_{\overline{\varepsilon}} = \overline{\mathrm{S}}^{\varepsilon_0}_0 .$$

We know that S approximates $\partial\,\overline{\mathrm{A}}$ from above on bounded sets, replace there ε_0 by $\frac{\varepsilon_0}{2}$, then $\forall^{s}\frac{\varepsilon_0}{2}\forall^{\mathrm{s}}\varepsilon_1 \dots \forall^{\mathrm{s}}\varepsilon_k$ and a given $\varepsilon > 0$ exists $\varepsilon_{k+1} < \varepsilon$ such that $\partial\,\overline{\mathrm{A}} \cap \mathrm{B}_{\frac{1}{\varepsilon_0}}(0) \subseteq S^{\frac{\varepsilon_0}{2}}_{\overline{\varepsilon},\varepsilon_{k+1}}$. If ε is small enough (that for all $(\overline{\mathrm{y}}, \delta) \in S_{\overline{\varepsilon}}$ there exists an $\overline{\mathrm{x}} \in T$ such that $|(\overline{\mathrm{y}}, \overline{\varepsilon}, \varepsilon_{k+1}) - (\overline{\mathrm{x}}, \overline{\varepsilon}, 0)| < \varepsilon_0$) we can assume $S^{\frac{\varepsilon_0}{2}}_{\overline{\varepsilon},\varepsilon_{k+1}} \subseteq T^{\varepsilon_0}_{\overline{\varepsilon}}$. Hence T approximates $\partial\,\overline{\mathrm{A}}$ from above on bounded sets.

Now we can apply the induction hypothesis on T and obtain the wished set B. ☐

Now we proof Theorem 3.4.1.

Proof. Take for A an $\mathrm{M}^{\mathrm{N}}(\mathcal{S})$-approximant S, where $N \geq 2$. Such an S exists by Theorem 3.2.8. By Lemma 3.4.4 the set S has empty interior. Of course it approximates $\partial\overline{A}$ from above on bounded sets. Thus we can apply Lemma 3.4.6 and find a set $B \in \widetilde{\mathcal{S}}_n$ with empty interior satisfying $\partial A \subseteq \partial\overline{A} \subseteq B$. □

3.5 Third Step: Closed under Complementation

In this last step we finish the proof of the theorem of the complement. We cite therefor some cell decomposition arguments of Wilkie, given in [Wil99]. However, we will not look at the details of his proof, since we do not need to do any modifications.

Theorem 3.5.1. *Let $\mathcal{S}$ be an o-minimal weak structure. Then the Charbonnel closure $\widetilde{\mathcal{S}}$ is closed under complementation.*

Proof. For the proof look at the cell decomposition arguments in Chapter 4 of [Wil99], or in Chapter 7 of [BS04]. We will give a short sketch of the main arguments here.

At first Wilkie defines $\widetilde{\mathcal{S}}$-cells and a $\widetilde{\mathcal{S}}$ cell decomposition by replacing the functions occurring in the definition of cells (look at Definition 1.2.2) by functions in $\widetilde{\mathcal{S}}$. Next he proves the following theorem.

Theorem 3.5.2. *[Theorem (4.5) in [Wil99]] Let $n \geq 1$ and suppose that D is an $\widetilde{\mathcal{S}}$-cell in $\mathbb{R}^n$ and $A \in \widetilde{\mathcal{S}}_n$ such that $A \subseteq D$ closed in D. Then there exists an $\widetilde{\mathcal{S}}$-cell decomposition $\mathcal{D}$ of D which is compatible with A.*

Proof. Therefor two statements are proved simultaneously by induction on n: Firstly that for every $\widetilde{\mathcal{S}}$-cell and a closed $\widetilde{\mathcal{S}}$-subset of this cell, there exists a compatible $\widetilde{\mathcal{S}}$-cell decomposition and secondly that finitely many $\widetilde{\mathcal{S}}$-cell decompositions can be combined. For this proof the result of Theorem 3.1. in [Wil99] is used. Our Theorem 3.4.1 has exactly the same conclusion, but with different, weaker assumptions on $\mathcal{S}$. So we can adapt Wilkie's proof of this theorem in Chapter 4 of [Wil99] without changes. □

To prove Theorem 3.1.4 from Theorem 3.5.2 for a set B, we use a semi-algebraic diffeomorphism θ_p from $\mathbb{R}^p$ to $(-1,1)^p$ to reduce the problem to a bounded set $(-1,1)^p$. Next, we apply the above theorem to the cell $(-1,1)^m$ so that we obtain an $\widetilde{\mathcal{S}}$-cell decomposition compatible with $\theta_m(B)$, which can be transformed back into a $\widetilde{\mathcal{S}}$-cell decomposition compatible with B which is of course also compatible with the complement of B. We obtain that the complement of B is again in $\widetilde{\mathcal{S}}$. For more details look at the proof of Theorem (4.5) applies Theorem (1.8) in [Wil99].

It is also clear, that $\widetilde{\mathcal{S}}$ is the smallest structure containing $\mathcal{S}$: A structure $\mathcal{J}$ which contains $\mathcal{S}$ must contain the projection and union of sets by the definition of a structure. Since the intersections with $\mathbb{Z}$-affine sets are intersections with semi-algebraic sets, they are in $\mathcal{J}$ and the closure of any set by is by Lemma (3.4) in [vdD98], Chapter 1 in $\mathcal{J}$. So $\mathcal{J}$ must contain $\widetilde{\mathcal{S}}$. $\square$

This finishes the proof of the modified Theorem of the Complement 3.1.4.

4 The Converse of the Theorem of the Complement

The aim of this chapter is to prove that every o-minimal structure can be characterized by the Charbonnel closure of some o-minimal weak structure, which additionally satisfies the DC^N condition for all N. This is in some sense the converse to the theorem of the complement. Actually, every o-minimal structure satisfies DC^N for all N itself.

Theorem 4.3. *[Theorem 2 in [KM99]] Any o-minimal structure $\mathcal{J}$ is of the form $\widetilde{\mathcal{S}}$, where $\mathcal{S}$ is an o-minimal weak structure satisfying DC^N for all N.*

The proof goes along the ideas given by Karpinski and Macintyre in [KM99]. First we need the following lemma of van den Dries and Miller about closed sets, that are zero sets.

Lemma 4.4. *[Theorem (4.22) in [vdDM96]] Let $\mathcal{S}$ be an o-minimal structure. Let $A \in \mathcal{S}_n$ be closed. Then for every N there exists a C^N function $f : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{S}$ with $A = Z(f)$.*

Proof. Look at [vdDM96]. □

In the next step, we write all sets as a boolean combination of closed sets, so we can apply Lemma 4.4.

Lemma 4.5. *Let $\mathcal{S}$ be an o-minimal structure. Every set $A \in \mathcal{S}$ is a boolean combination of closed sets in $\mathcal{S}$.*

Proof. Let $A \in \mathcal{S}_n$. Then there exists by Theorem 1.2.6 a finite cell decomposition compatible with A, so $A = \bigcup\{C \mid C \text{ cell} \wedge C \subseteq A\}$. So it is enough to show that every cell C is a boolean combination of closed sets. We prove the following Claim.

Claim : For every cell C there exists an open set $U \in \mathcal{S}$ such that $C = \overline{\mathrm{C}} \cap U$.

Proof: We prove this claim by induction on the cell, first let C be a cell in $\mathbb{R}$. If C is a point, $C = \overline{\mathrm{C}} \cap \mathbb{R}$. If $C = (a, b)$ is an interval, $C = \overline{\mathrm{C}} \cap (a, b)$.

Now let C' be a cell with dimension $n-1$. Then by induction hypothesis we have $C' = \overline{\mathrm{C}}' \cap U'$. Examine now an n-dimensional cell C over C'.

Case 1: Let $C = \Gamma(f)$ with a C^N function $f : C' \to \mathbb{R}$ in $\mathcal{S}$. Then $\overline{\Gamma(f)} \cap U' \times \mathbb{R} = \Gamma(f) = C$.

Case 2: Let $C = (f, g)$ with $f, g \in FC^N$ as in the definition of cells. Then $C = \overline{C' \times \mathbb{R}} \cap \{(\overline{\mathrm{x}}, y) \mid \overline{\mathrm{x}} \in U' \wedge f(\overline{\mathrm{x}}) < y < g(\overline{\mathrm{x}})\} = \overline{\mathrm{C}} \cap U$, where $U = \{(\overline{\mathrm{x}}, y) \mid \overline{\mathrm{x}} \in U' \wedge f(\overline{\mathrm{x}}) < y < g(\overline{\mathrm{x}})\}$ is obviously an open set in $\mathcal{S}$. □(Claim)

Hence $C = \overline{\mathrm{C}} \cap (\mathbb{R}^n - U^C)$ is a boolean combination of closed sets. □

The following lemma proves directly Theorem 4.3, if we take $\mathcal{S} = \mathcal{J}$. Due to the fact that $\mathcal{J}$ is an o-minimal structure directly follows $\mathcal{J} = \widetilde{\mathcal{J}}$.

Lemma 4.6. *Let $\mathcal{S}$ be an o-minimal structure. Then $\mathcal{S}$ satisfies DC^N for all N.*

Proof. By Lemma 4.5 we can write each set $A \in \mathcal{S}$ as boolean combination of closed sets. By Lemma 4.4 for each closed A set there exists a C^N function $f : \mathbb{R}^n \to \mathbb{R}$ in $\mathcal{S}$ such that $A = Z(f)$.

Hence we have to look at the different boolean combinations. A boolean combination corresponds to a logical formula without quantifiers. We can assume it is in disjunctive normal form. So the complement is taken only on sets which are closed.

Let $A \in \mathcal{S}_n$ be a closed set and let the corresponding function $f_A : \mathbb{R}^n \to \mathbb{R}$ be C^N and let $A = Z(f_A)$ (by Lemma 4.4). Define for the complement $f : \mathbb{R}^{n+1} \to \mathbb{R}$ by $f(\overline{\mathrm{x}}, y) = y \cdot f_A(\overline{\mathrm{x}}) - 1 = 0$. Then

$$\begin{aligned} \overline{\mathrm{x}} \in \pi[Z(f)] &\Leftrightarrow \exists y \; y \cdot f_A(\overline{\mathrm{x}}) - 1 = 0 \\ &\Leftrightarrow \exists y \; y \cdot f_A(\overline{\mathrm{x}}) = 1 \\ &\Leftrightarrow f_A(\overline{\mathrm{x}}) \neq 0 \Leftrightarrow \overline{\mathrm{x}} \notin Z(f_A) \\ &\Leftrightarrow \overline{\mathrm{x}} \notin A. \end{aligned}$$

Now assume there are functions $f_A : \mathbb{R}^{n+m_A} \to \mathbb{R}, f_B : \mathbb{R}^{n+m_B} \to \mathbb{R}$ which are C^N and in $\mathcal{S}$ such that $\pi_n^{n+m_A}[Z(f_A)] = A$ and $\pi_n^{n+m_B}[Z(f_B)]$. Without loss of generality assume $m_A = m_B = m$. (Adding some additional coordinates has no influence.)

- For $A \cup B$: Define $f : \mathbb{R}^{n+m} \to \mathbb{R}$ by $f(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = f_A(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \cdot f_B(\overline{\mathrm{x}}, \overline{\mathrm{y}})$. Then

$$\begin{aligned} \overline{\mathrm{x}} \in \pi[Z(f)] &\Leftrightarrow \exists \overline{\mathrm{y}} \, f(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = 0 \Leftrightarrow \exists \overline{\mathrm{y}} \, f_A(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = 0 \vee f_B(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = 0 \\ &\Leftrightarrow \overline{\mathrm{x}} \in \pi[Z(f_A)] \vee \overline{\mathrm{x}} \in \pi[Z(f_B)] \Leftrightarrow \overline{\mathrm{x}} \in A \cup B. \end{aligned}$$

- For $A \cap B$: Define $f : \mathbb{R}^{n+m+m} \to \mathbb{R}$ by $f(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}) = f_A^2(\overline{\mathrm{x}}, \overline{\mathrm{y}}) + f_B^2(\overline{\mathrm{x}}, \overline{\mathrm{z}})$. Then

$$\begin{aligned} \overline{\mathrm{x}} \in \pi[Z(f)] &\Leftrightarrow \exists \overline{\mathrm{y}}, \overline{\mathrm{z}}\, f(\overline{\mathrm{x}}, \overline{\mathrm{y}}, \overline{\mathrm{z}}) = 0 \\ &\Leftrightarrow \exists \overline{\mathrm{y}}, \overline{\mathrm{z}}\, f_A(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = 0 \wedge f_B(\overline{\mathrm{x}}, \overline{\mathrm{z}}) = 0 \\ &\Leftrightarrow \overline{\mathrm{x}} \in \pi[Z(f_A)] \wedge \overline{\mathrm{x}} \in \pi[Z(f_B)] \Leftrightarrow \overline{\mathrm{x}} \in A \cap B. \end{aligned}$$

$\square$

This finishes the proof of the converse of the theorem of the complement, Theorem 4.3.

5 Application to the Pfaffian Closure

In this chapter we apply the theorem of the complement to extend an o-minimal structure. It is possible to extend a structure by zero sets of Pfaffian functions. We will see that the structure generated by this is contained in the Pfaffian Closure, which is constructed by intersecting with special manifolds, the Rolle leaves. The theorem of the complement shows that the Charbonnel closure of the Pfaffian Closure is again an o-minimal structure.

5.1 Rolle Leaves and the Pfaffian Closure

We begin with the basic definitions of Rolle leaves and the Pfaffian closure. In the following, fix an arbitrary o-minimal structure $\mathcal{S}$.

Definition 5.1.1. Let $U \subseteq \mathbb{R}^n$ be open and let $\omega = a_1 dx_1 + \cdots + a_n dx_n$ be a 1-form on U of class C^1, i.e. each $a_i : U \to \mathbb{R}$ is a C^1 function.

a) $S(\omega) := \{\overline{\mathrm{x}} \in U \mid \omega(\overline{\mathrm{x}}) = 0\} = \bigcap_{i=1}^{n} a_i^{-1}(0)$

b) Let $\overline{\mathrm{x}} \in U - S(\omega)$. The *kernel* of $\omega(\overline{\mathrm{x}})$ is defined as

$$\ker(\omega(\overline{\mathrm{x}})) := \{\overline{\mathrm{y}} \in \mathbb{R}^n \mid a_1(\overline{\mathrm{x}})y_1 + \cdots + a_n(\overline{\mathrm{x}})y_n = 0\}.$$

In the following we assume in general that $S(\omega) = \emptyset$. This is possible since $S(\omega)$ is closed and so $U - S(\omega)$ is still an open set in $\mathbb{R}^n$. Thus we can restrict ω to $U - S(\omega)$.

A Rolle leaf is some nice manifold, which respects a given vector field.

Definition 5.1.2. Let U and ω be as in the above definition.

a) An *integral manifold* M of $\omega = 0$ is an $(n-1)$-dimensional immersed C^1 manifold of $U \subseteq \mathbb{R}^n$ such that $T_{\overline{\mathrm{x}}}M = \ker(\omega(\overline{\mathrm{x}}))$.

b) A *leaf* of $\omega = 0$ is a maximal connected integral manifold of $\omega = 0$.

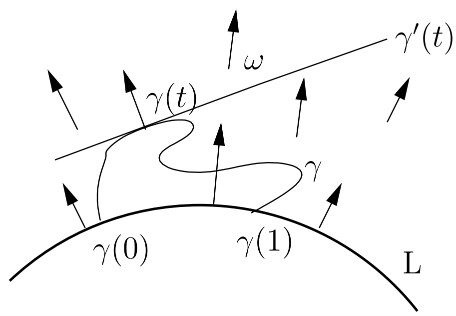

Figure 5.1: Example for a Rolle leaf

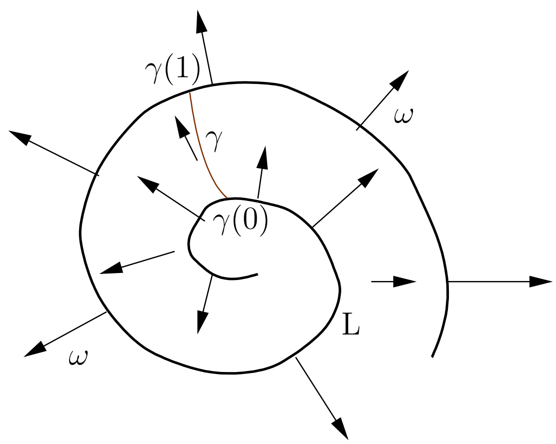

Figure 5.2: Example for a leaf which is not a Rolle leaf

c) A leaf L is a *Rolle leaf* if L is an embedded submanifold of U, which is closed in U, such that each C^1 curve $\gamma : [0,1] \to U$ with $\gamma(0), \gamma(1) \in L$ is tangential to the hyperplane field defined by $\omega = 0$ at some point, i.e. $\exists t \in [0,1]\ \omega(\gamma(t))(\gamma'(t)) = 0$.

At next, we define the Pfaffian closure, which is generated by an o-minimal structure and Rolle leaves on this structure.

Definition 5.1.3. Let $\mathcal{S}$ be an o-minimal structure. Let $n \in \mathbb{N}$

- Define a *basic* $\mathbb{R}$-*Pfaffian set* as a set of the form

  $$A \cap L_1 \cap \cdots \cap L_k$$

  where $k \in \mathbb{N}, A \in \mathcal{S}_n$ and every L_i is a Rolle leaf, whose base set U_i and 1-form ω_i are in $\mathcal{S}$.

- A finite union of basic $\mathbb{R}$-Pfaffian sets in $\mathbb{R}^n$ is called *Pfaffian set*.
- Define $\text{Rolle}(\mathcal{S})_n$ as the Pfaffian sets in $\mathbb{R}^n$. The collection $\langle \text{Rolle}(\mathcal{S})_n \rangle_{n\in\mathbb{N}}$ is called *Pfaffian closure*.

Now we can formulate the central theorem of this chapter.

Theorem 5.1.4. *[Theorem (4.1) in [Spe99]] The Charbonnel closure of the Pfaffian closure $\widetilde{Rolle}(\mathcal{S})$ is an o-minimal structure.*

In order to prove this theorem we use our version of the Theorem of the complement 3.1.4. In the next sections we will show that $\text{Rolle}(\mathcal{S})$ is an o-minimal weak structure and that it satisfies DPC^N for all N. Afterwards we can apply the theorem of the complement.

Before we do this, let us take a look at the connection between Rolle leaves and Pfaffian functions, which gave the name to the Pfaffian closure.

Definition 5.1.5 (Pfaffian Function, [Wil99], footnote on p.398)**.** A C^1 function $f : \mathbb{R}^n \to \mathbb{R}$ is called *Pfaffian* if there exist C^1 functions $f_1, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ with $f = f_k$, such that for each $1 \leq i \leq k$ and $1 \leq j \leq n$, $\frac{\partial f_i}{\partial x_j}$ is expressible as a polynomial in $x_1, \ldots, x_n, f_1, \ldots f_i$.

The following Lemma of Khovanskii connects Pfaffian functions with Rolle leaves.

Lemma 5.1.6. *Let f be a Pfaffian function over an arbitrary expansion of $\overline{\mathbb{R}}$. Then the graph of f is a Rolle leaf.*

Proof. Look at Example 1.3 in [Spe99]. □

We do not want to give the proof of this lemma, but here is the special case for the exponential function. It is similar to Example 1.3 in [Spe99].

Example 5.1.7. The graph of exp is a Rolle leaf.

Define $U = \mathbb{R}^2$ and $\omega(x, y) = ydx - 1dy$. Obviously, $S(\omega) = \emptyset$ and $\Gamma(\exp)$ is closed and an embedded 1-dimensional C^1 manifold of $\mathbb{R}^2$. So for every $(x, y) = (x, \exp(x)) \in \Gamma(f)$ look at

$$\begin{aligned} T_{(x,y)}\Gamma(f) &= \langle (x, \frac{\partial \exp}{\partial x}(x)) \rangle = \langle (x, \exp(x)) \rangle \\ &= \{(v, \exp(x)v \mid v \in \mathbb{R}\} = \{(v, w) \mid \exp(x)v - w = 0\} \\ &= \ker(\omega(x, \exp(x)). \end{aligned}$$

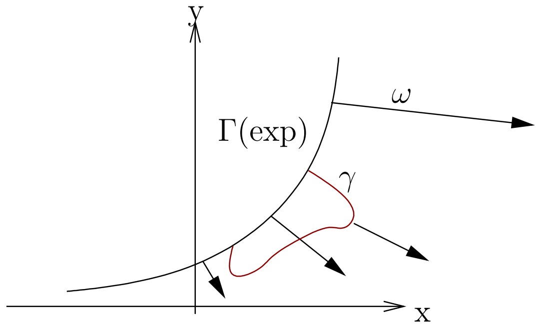

Figure 5.3: The graph Γ(exp) is a Rolle leaf.

Hence L is a leaf.

Now check the Rolle condition: Let $\gamma : [0,1] \to \mathbb{R}^2$ such that the values $\gamma(0), \gamma(1)$ are in L. The set $U - \Gamma(f)$ has two connected components: $C_1 = \{(x,y) \in U \mid y < f(x)\}$ and $C_2 = \{(x,y) \in U \mid y > f(x)\}$. Of course, we can assume that $\omega(\gamma(0))(\gamma'(0)) \neq 0$, $\omega(\gamma(1))(\gamma'(1)) \neq 0$ and that $\gamma[(0,1)]$ is in one of the two connected components.

Claim : The values $\omega(\gamma(0))(\gamma'(0))$ and $\omega(\gamma(1))(\gamma'(1))$ have a different sign.

Proof: Without loss of generality, assume that $\omega(\gamma(0))(\gamma'(0)) > 0$. Then there exists some $\varepsilon > 0$ such that $\gamma[(0,\varepsilon)] \subseteq C_1$ and so by assumption, $\gamma[(0,1)] \subseteq C_1$. If also $\omega(\gamma(1))(\gamma'(1)) > 0$, then there is a $0 < \delta < 1$ such that $\gamma[(\delta,1)] \subseteq C_2$, which is a contradiction. □(Claim)

The claim implies, since ω is continuous, that there exists a $t \in [0,1]$ such that $\omega(\gamma(t))(\gamma'(t)) = 0$.

To motivate Theorem 5.1.4 we apply it to Example 1.1.8, which was mentioned in the beginning of this diploma thesis and involves the exponential function.

Example 5.1.8. By Example 1.1.7 the definable sets in the structure $\mathbb{R}_{an}$, i.e. $\overline{\mathbb{R}}$ expanded with restricted analytic functions form an o-minimal structure. We examine $\mathbb{R}_{an,\exp} = (\overline{\mathbb{R}}, (f), \exp)$ as expansion of $\mathbb{R}_{an}$. Look at the definable sets

$$\mathcal{S}_n := \{A \subseteq \mathbb{R}^n \mid A \text{ definable over } \mathbb{R}_{an,\exp} \text{ with parameters}\},\ n \geq 1$$

At first, notice that $\mathcal{S}$ is a structure: If two sets $A = \{\overline{x} \in \mathbb{R}^n \mid \phi(\overline{x})\}$ and $B = \{\overline{x} \in \mathbb{R}^n \mid \psi(\overline{x})\}$ are in $\mathcal{S}$, trivially $A \cap B, A \cup B, A \times B$ and A^C are in $\mathcal{S}$ and $\pi^n_m[A] = \{\overline{y} \in \mathbb{R}^m \mid \exists \overline{z} \in \mathbb{R}^{n-m}\ \phi(\overline{y},\overline{z})\}$ is in $\mathcal{S}$, too. Thus (S1),(S3) and (S4) are satisfied. The semi-algebraic sets are already definable in $\mathbb{R}_{an}$, so they are obviously in $\mathcal{S}$.

By the last example $\Gamma(\exp)$ is a Rolle leave on $(\mathbb{R}^2, \omega)$ and thus a set in $\widetilde{\text{Rolle}(\mathcal{S})}$. Note that by the above example, $\mathbb{R}^2$ and ω are definable in $\mathbb{R}_{an}$. Hence $\mathbb{R}_{an,\exp} \subseteq \widetilde{\text{Rolle}(\mathbb{R}_{an})}$. By Theorem 5.1.4 $\widetilde{\text{Rolle}(\mathbb{R}_{an})}$ is an o-minimal structure. Trivially, every substructure $\mathcal{A}$ of an o-minimal structure $\mathcal{B}$ is also o-minimal. (If $A \subseteq \mathbb{R}$ is in $\mathcal{A}$ it is also in $\mathcal{B}$, so by the o-minimality of $\mathcal{B}$ it is a finite union of points and intervals.) Hence the structure $\mathbb{R}_{an,\exp} \subseteq \widetilde{\text{Rolle}(\mathbb{R}_{an})}$ is o-minimal.

Corollary 5.1.9. *The sets definable in $\mathbb{R}_{an,\exp}$ form an o-minimal structure.*

5.2 Transforming Rolle Leaves

We begin the proof of Theorem 5.1.4 with some facts on Rolle leaves. In some of the following proofs we need to transform Rolle leaves via a diffeomorphism or a projection into new Rolle leaves on other basic sets. Some properties of the new basic sets help us to check some facts about the Rolle leaves.

First we need some terms and facts of differential topology.

Definition 5.2.1. Let $\phi : V \to U$ be a map between some open sets V, U on manifolds.

- For every $\overline{x} \in V$ the *pushforward* $\phi_*(\overline{x}) : T_{\overline{x}}V \to T_{\phi(\overline{x})}U$ is defined by $\phi_*(\overline{x}) = d\phi_{\overline{x}}$, or in other words, if $v \in T_{\overline{x}}V$, $v = c'(0)$ with $c : [-\varepsilon, \varepsilon] \to V$, then $\phi_*(v) = (\phi \circ c)'(0)$.
- For a differential form, the *pullback* $\phi^* : T^*U \to T^*V$ is $\omega \in T^*U$ defined by $\phi^*(\omega)(\overline{x})(v) = \omega(\phi(\overline{x}))(\phi_*(\overline{x})(v))$ $(\overline{x} \in V,\ v \in T_{\overline{x}}V)$.

The next lemma helps us to transform Rolle leaves into other spaces, look also at Remark 1.7 in [Spe99]. To prove them, we need the following statement about manifolds.

Definition 5.2.2 (p.22 in [Hir94])**.** Let M, N be manifolds and $A \subseteq N$ be a submanifold. A map $f : M \to N$ is called transverse to A if and only if for all $\overline{x} \in f^{-1}(A)$ holds $\text{im}(df_{\overline{x}}) + T_{f(\overline{x})}A = T_{f(\overline{x})}N$.

Theorem 5.2.3. *[Theorem 3.3. in [Hir94], Chapter 1] Let $f : M \to N$ be a C^r map between two manifolds and $A \subseteq N$ be a C^r submanifold. If f is transverse to A, then $f^{-1}(A)$ is a C^r submanifold of M. The co-dimension of $f^{-1}(A)$ is the same as the co-dimension of A in N.*

Proof. Look at Chapter 1 in [Hir94]. □

Lemma 5.2.4. *Let $V \subseteq \mathbb{R}^n$ and $U \subseteq \mathbb{R}^m$ be open sets and let $\phi : V \to U$ be a submersion, i.e. the differential is surjective in every point. Furthermore let L be a (Rolle) leaf on U and some 1-form ω. Then every connected component of $\phi^{-1}(L)$ is a (Rolle) leaf on $(V, \phi^*\omega)$.*

Proof. Without loss of generality we can assume that $\phi^{-1}(L)$ is connected, otherwise we examine only the connected components. (If some set is a manifold, so is every connected component.)

1. The preimage $\phi^{-1}(L)$ is a manifold of dimension $n-1$.

Let $\overline{\mathrm{x}} \in L$. Since ϕ is a submersion $d\phi_{\overline{\mathrm{x}}}$ is surjective and so $\mathrm{im}(d\phi_{\overline{\mathrm{x}}}) = T_{\phi(\overline{\mathrm{x}})}U$ and thus ϕ is transverse to L. Hence by Theorem 5.2.3 $\phi^{-1}(L)$ is a submanifold of V with co-dimension 1, like L.

2. The manifold $\phi^{-1}(L)$ is a leaf on $(V, \phi^*\omega)$.

We have to show

$$T_{\overline{\mathrm{x}}}\phi^{-1}(L) = \ker(\phi^*\omega(\overline{\mathrm{x}}))$$

Let $\overline{\mathrm{v}} \in T_{\overline{\mathrm{x}}}\phi^{-1}(L)$. Hence $\overline{\mathrm{v}} = c'(0)$, where $c : [-\varepsilon, \varepsilon] \to \phi^{-1}(L)$ is a differentiable curve such that $c(0) = \overline{\mathrm{x}}$. Then $\phi \circ c$ is a differentiable curve such that $\phi \circ c(0) = \phi(\overline{\mathrm{x}})$. Thus $(\phi \circ c)'(0) \in T_{\phi(\overline{\mathrm{x}})}L$. Furthermore $(\phi \circ c)'(0) = \phi_*(\overline{\mathrm{x}})(\overline{\mathrm{v}})$ by definition, i.e. $\phi_*(\overline{\mathrm{x}})(\overline{\mathrm{v}}) \in T_{\phi(\overline{\mathrm{x}})}L$. So $\phi^*\omega(\overline{\mathrm{x}})(\overline{\mathrm{v}}) = \omega(\phi(\overline{\mathrm{x}}))(\phi_*(\overline{\mathrm{x}})(\overline{\mathrm{v}})) = 0$, since $T_{\phi(\overline{\mathrm{x}})}L = \ker(\omega(\phi(\overline{\mathrm{x}})))$ by assumption. Hence $\overline{\mathrm{v}} \in \ker(\phi^*\omega(\overline{\mathrm{x}}))$.

Since ϕ is a submersion, ϕ_* has full rank for every $\overline{\mathrm{x}}$. Hence, we obtain the following equality:

$$\begin{aligned}
\dim(\mathrm{im}(\phi^*\omega(\overline{\mathrm{x}}))) &= \dim\{\phi^*\omega(\overline{\mathrm{x}})(\overline{\mathrm{v}}) \mid \overline{\mathrm{v}} \in T_{\overline{\mathrm{x}}}V\} \\
&= \dim\{\omega(\phi(\overline{\mathrm{x}}))\phi_*\,\overline{\mathrm{v}} \mid \overline{\mathrm{v}} \in T_{\overline{\mathrm{x}}}V\} \\
&\overset{\phi_*\text{ surjective}}{=} \dim\left\{\omega(\phi(\overline{\mathrm{x}}))\,\overline{\mathrm{u}} \mid \overline{\mathrm{u}} \in T_{\phi(\overline{\mathrm{x}})}U\right\} \\
&= \dim(\mathrm{im}(\omega(\phi(\overline{\mathrm{x}}))))
\end{aligned}$$

As L is a Rolle leaf and so an integral manifold of dimension $m-1$, for every $\overline{\mathrm{x}} \in \phi^{-1}(L)$ the tangential space $T_{\phi(\overline{\mathrm{x}})}L = \ker(\omega(\overline{\mathrm{x}}))$ has also dimension $m-1$. Therefore $\dim(\mathrm{im}(\omega(\phi(\overline{\mathrm{x}})))) = m-(m-1) = 1$ and by the above calculation $\dim(\mathrm{im}(\phi^*\omega(\overline{\mathrm{x}}))) = 1$, which implies $\dim(\ker(\phi^*\omega(\overline{\mathrm{x}}))) = n-1$.

Since $\phi^{-1}(L)$ is an $(n-1)$-dimensional manifold the tangential space $T_{\overline{\mathrm{x}}}\phi^{-1}L$ has dimension $n-1$ and since we know already $T_{\overline{\mathrm{x}}}\phi^{-1}(L) \subseteq \ker(\phi^*\omega(\overline{\mathrm{x}}))$ we obtain

$$T_{\overline{\mathrm{x}}}\phi^{-1}(L) = \ker(\phi^*\omega(\overline{\mathrm{x}})).$$

3. The leaf $\phi^{-1}(L)$ satisfies the Rolle condition, if L does it.

As a preimage of a closed set under a continuous function, $\phi^{-1}(L)$ is closed.

Let $\gamma : [0,1] \to V$ be a path such that $\gamma(0), \gamma(1) \in \phi^{-1}(L)$. Then $\phi \circ \gamma : [0,1] \to U$ is a curve with $\phi \circ \gamma(0), \phi \circ \gamma(1) \in L$. Since L is Rolle leaf, there exists $t \in [0,1]$ such that $\omega(\phi \circ \gamma(t))(\phi \circ \gamma)'(t) = 0$. We obtain $\phi^*\omega(\gamma(t)) = \omega(\phi \circ \gamma(t))(\phi \circ \gamma)'(t) = 0$. □

Corollary 5.2.5. *If $\phi : V \to U$ is a diffeomorphism and L a Rolle leaf on (U, ω), then $\phi^{-1}(L)$ is a Rolle leaf on $(V, \phi^*\omega)$.*

Corollary 5.2.6. *Let L be a Rolle leaf on (U, ω) with $\omega = \sum_{i=1}^n a_i dx_i$ and let $V \subseteq \mathbb{R}^m$ be an open and connected set. Define $\widetilde{\omega} := \pi^*\omega$. Then $L \times V$ is a Rolle leaf on $(U \times V, \widetilde{\omega})$.*

Proof. Recognize that the projection $\pi : U \times V \to U$ is a submersion. Furthermore V is connected and so $L \times V$ is connected. □

5.3 The Pfaffian sets form an o-minimal weak structure

In this section we prove that Rolle($\mathcal{S}$) is an o-minimal weak structure. This is the first condition we need to apply the theorem of the complement. For the rest of this chapter, fix an o-minimal structure $\mathcal{S}$.

The next remark about the basic sets of some Rolle leaves helps us to simplify the notation in the following proofs.

Remark 5.3.1. [Remark (3.4) in [KM99]] A set of the form $A \cap L_1 \cap \cdots \cap L_k \subseteq \mathbb{R}^n$ with L_i Rolle leaf on (U_i, ω_i) and $A \in \mathcal{S}$ can be represented also by a set of the form $\pi_n^{n \cdot k}[A' \cap L_1' \cap \cdots \cap L_k']$ where the L_i' are Rolle leaves on $(U, \tilde{\omega}_i)$ (with the same set U!). We can also assume that U is open and connected.

Proof. We can assume that each U_i is connected, since each L_i is connected and we can restrict to the connected component of U_i containing L. To show the claim, define

$$U = U_1 \times \cdots \times U_k.$$

This set is still open and connected. Furthermore for $\omega_j = \sum_{i=1}^n a_{ji}(x_1, \ldots, x_n) dx_i$ define

$$\tilde{\omega}_j = \sum_{i=1}^n a_{ji}(x_{j\cdot(n-1)+1}, \ldots, x_{j\cdot n}) dx_{j\cdot(n-1)+i}$$

and

$$L'_j = U_1 \times \cdots \times U_{j-1} \times L_j \times U_{j+1} \times \cdots \times U_k.$$

Of course, each L'_j is a Rolle leaf for $\tilde{\omega}_j$ on U and with $\Delta = \{(\overline{x}, \ldots, \overline{x}) \in (\mathbb{R}^n)^k\}$ the diagonal copy of $\mathbb{R}^n$, which is a semialgebraic set, we obtain $A^k \cap \Delta \in \mathcal{S}$ and

$$A = \pi_n^{n \cdot k}[(A^k \cap \Delta) \cap L'_1 \cap \cdots \cap L'_k]$$

Hence we can assume in the proofs below, that the Rolle leaves are defined on the same open and connected set U. □

Lemma 5.3.2 (Lemma 3 in [KM99]). *The Pfaffian closure* Rolle($\mathcal{S}$) *is a weak structure.*

Proof. (WS1) Trivial.

(WS2) Clear since $\mathcal{S} \subseteq Rolle(\mathcal{S})$.

(WS3) Take $A = A_1 \cap L_1 \cap \cdots \cap L_k \in Rolle(\mathcal{S})_n$ and $B = B_1 \cap L'_1 \cap \cdots \cap L'_l \in Rolle(\mathcal{S})_m$. Assume that the $L_i, i = 1, \ldots k$ are Rolle leaves on (U, ω_i) (by Remark 5.3.1) and the $L'_j, j = 1, \ldots l$ are Rolle leaves on (U', ω'_j) such that U and U' are open and connected. By Corollary 5.2.6 the set $U \times L'_j$ is a Rolle leaf on $(U \times U', (0, \omega'_j))$ and $L_i \times U'$ is a Rolle leaf on $(U \times U', (\omega_i, 0))$.

$$A \times B = A_1 \times B_1 \cap \bigcap_{i=1}^{k} (L_i \times U') \cap \bigcap_{j=1}^{l} (U \times L'_j).$$

So $A \times B$ is a set in $Rolle(\mathcal{S})_{n+m}$.

In general $A = A_1 \cup \cdots \cup A_k$, $B = B_1 \cup \cdots \cup B_l$ such that A_i, B_j are sets as examined above. Then $A \times B = \bigcup_{i=1}^{k} \bigcup_{j=1}^{l} A_i \times B_j$ which is a finite union of sets of the form $A_i \times B_j$ that are in Rolle($\mathcal{S}$) by the above argument.

(WS4) If $\sigma : \mathbb{R}^n \to \mathbb{R}^n$ is a linear bijection (so a diffeomorphism) and $B = A \cap L_1 \cap \cdots \cap L_k \in$ Rolle($\mathcal{S}$). Then from $A \in \mathcal{S}_n$ it follows that $\sigma(A) \in \mathcal{S}$ by (WS4) for $\mathcal{S}$. L_i is a Rolle leaf on (U_i, ω) and so $\sigma(L_i)$ is a Rolle leaf on $(\sigma(U_i), (\sigma^{-1})^* \omega)$ by Lemma 5.2.5. Since $\mathcal{S}$ is a structure $\sigma(U_i)$ and $(\sigma^{-1})^* \omega$ are in $\mathcal{S}$. Hence $\sigma(B) \in$ Rolle($\mathcal{S}$).

As above in the proof of (WS3), the argument can be expanded easily to finite unions of basic Pfaffian sets.

□

The next step is (WS5), i.e. to prove that the number of connected components stays bounded during intersection with Rolle leaves and affine hyperplanes. Speisegger proved a bound of connected components for fibers of Pfaffian sets.

Lemma 5.3.3. *[Corollary 2.7. in [Spe99]] Assume $a \leq m \leq n$ and let $A \subseteq \mathbb{R}^n$ be a set in $\mathcal{S}$. Then there is a $K \in \mathbb{N}$ such that, whenever $\overline{\mathrm{a}} \in \mathbb{R}^m$ and L_i is a Rolle leaf of $\omega_i = 0$ for each i, then the fiber $(A \cap L_1 \cap \cdots \cap L_q)_{\overline{\mathrm{a}}}$ is a union of at most K connected manifolds.*

Proof. Look at [Spe99], Section 1 and 2. □

Lemma 5.3.4 (Lemma 4 in [KM99])**.** *The Pfaffian closure* $\mathrm{Rolle}(\mathcal{S})$ *satisfies (WS5).*

Proof. First assume $B = A \cap L_1 \cap \cdots \cap L_k \in \mathrm{Rolle}(\mathcal{S})_n$. We have to show that there exists a boundary for $cc(B \cap X)$, which is independent from the hyperplane X.

If X is a hyperplane, X is a zero set of a linear polynomial $p(\overline{\mathrm{x}}) = a_0 + a_1 x_1 + \cdots + a_n x_n$. We can consider p as a function depending on $a(X) = \overline{\mathrm{a}} = (a_0, a_1, \ldots, a_n)$ and $\overline{\mathrm{x}} \in \mathbb{R}^n$. Define $m := n+1$. Then $p : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ and $(\overline{\mathrm{x}}, \overline{\mathrm{a}}) \mapsto a_0 + a_1 x_1 + \cdots + a_n x_n$. Let $C := \{(\overline{\mathrm{x}}, \overline{\mathrm{a}}) \in \mathbb{R}^{n+m} \mid p(\overline{\mathrm{x}}, \overline{\mathrm{a}}) = 0\}$. Since p is a polynomial in $(\overline{\mathrm{x}}, \overline{\mathrm{a}})$ this is a semialgebraic set and so in $\mathcal{S}$. Obviously, $X = C_{\overline{\mathrm{a}}(X)}$, which is in $\mathcal{S}$ by Lemma 1.3.6 d).

Define $A' := A \times \mathbb{R}^m$ and $L_i' := L_i \times \mathbb{R}^m$ for $i = 1, \ldots, k$. By Corollary 5.2.6 every L_i' is a Rolle leaf. Next, we can apply Lemma 5.3.3 on $A' \cap C$ and obtain a $K \in \mathbb{N}$ such that whenever $\overline{\mathrm{a}} \in \mathbb{R}^m$ then the fiber $(A' \cap C \cap L_1' \cap \cdots \cap L_k')_{\overline{\mathrm{a}}}$ is a union of at most K connected manifolds. Particularly, whenever there is a hyperplane X and a corresponding $\overline{\mathrm{a}}(X) \in \mathbb{R}^m$, we have $(A' \cap C \cap L_1' \cap \cdots \cap L_k')_{\overline{\mathrm{a}}(X)}$ has at most K connected components. But

$$\begin{aligned}
(A' \cap C \cap L_1' \cap \cdots \cap L_k')_{\overline{\mathrm{a}}(X)} &= ((A \cap L_1 \cap \cdots \cap L_k) \times \mathbb{R}^m) \cap C)_{\overline{\mathrm{a}}(X)} \\
&\overset{\overline{\mathrm{a}}(X) \in \mathbb{R}^m}{=} (A \cap L_1 \cap \cdots \cap L_k) \cap C_{\overline{\mathrm{a}}(X)} \\
&= B \cap X
\end{aligned}$$

So the number of connected components of $B \cap X$ is bounded by K.

Now assume we have a finite union of sets of the form $A \cap L_1 \cap \cdots \cap L_k$, say $B = \bigcup_{i=1}^m B_i$. For each B_i the set $B_i \cap X$ is bounded by some K_i, independent from the hyperplane. So $B \cap X = \bigcup_{i=1}^m (B_i \cap X)$ is bounded by $K := \sum_{i=1}^m K_i \in \mathbb{N}$, still independent from the choice of the hyperplane X. □

The following lemma about the intersection of projections of closed sets will help us not only for the proof of the last condition (WS6).

Lemma 5.3.5. *Let $C_0 \in \mathbb{R}^{n+m_1}, \dots, C_k \in \mathbb{R}^{n+m_k}$ be closed sets in* Rolle($\mathcal{S}$) *with $\pi_n^{n+m_i}(C_i) = B_i$ and $C_i \in$* Rolle($\mathcal{S}$) *for $i = 1, \dots, k$. Then there exists a closed set $D \subseteq \mathbb{R}^{n+m_0+\dots+m_k}$ in* Rolle($\mathcal{S}$) *such that $\pi(D) = B_0 \cap \dots \cap B_k$.*

Proof. Write $\overline{\mathrm{x}} = (x_1, \dots, x_n)$ and $\overline{\mathrm{y}}_i = (y_1, \dots, y_{m_i})$.

$$D \ := \ \{(\overline{\mathrm{x}}, \overline{\mathrm{y}}_0, \dots \overline{\mathrm{y}}_k) \mid (\overline{\mathrm{x}}, \overline{\mathrm{y}}_0) \in C_0 \wedge \dots \wedge (\overline{\mathrm{x}}, \overline{\mathrm{y}}_k) \in C_k\}.$$

The set D is in Rolle($\mathcal{S}$), since it is an intersection of sets of the form $C_i \times \mathbb{R}^h$ (for some h) with permutated coordinates and Rolle($\mathcal{S}$) is a weak structure.

It is clear that

$$\pi(D) = \pi(C_0) \cap \pi(C_1) \cap \dots \cap \pi(C_k) = B_0 \cap B_1 \cap \dots \cap B_k.$$

The set D is closed: Take $\overline{\mathrm{z}}^i \in D$, $\overline{\mathrm{z}}^i \to \overline{\mathrm{z}} = (\overline{\mathrm{x}}, \overline{\mathrm{y}}_0, \dots, \overline{\mathrm{y}}_k) \in \mathbb{R}^{n+m_0+\dots+m_k}$. Then, since C_i is closed, $(\overline{\mathrm{x}}, \overline{\mathrm{y}}_i) \in C_i$ for $i = 0 \dots k$. So $\overline{\mathrm{x}} \in D$. $\square$

Lemma 5.3.6. *[Lemma 5 in [KM99]] Rolle($\mathcal{S}$) satisfies (WS6).*

Proof. First, we reduce the claim to Rolle leaves.

Claim : It is enough to show (WS6) for Rolle leaves.

Proof: Let $A = B \cap L_1 \cap \dots \cap L_k$. If we had a finite union of sets of this form, we can take the finite union of the closed sets that are projected on these sets. So it is enough to examine a basic Pfaffian set A. By Lemma 1.3.4 (WS6) holds for $\mathcal{S}$ and so there exists a closed set $C_0 \in \mathbb{R}^{m_0}$ such that $B = \pi(C)$. If we assume (WS6) for Rolle leaves, there exists $C_i \in \text{Rolle}(\mathcal{S})_{m_i}, i = 1, \dots, k$ closed such that $L_i = \pi(C_i)$. By Lemma 5.3.5 there exists a closed set $D \in \text{Rolle}(\mathcal{S})_{n+m_0+\dots+m_k}$ with $\pi(D) = B \cap L_1 \cap \dots \cap L_k$. $\square$(Claim)

Thus let L be a Rolle leaf on (U, ω). By Lemma 1.3.4 every o-minimal structure satisfies (WS6), hence for $U \in \mathcal{S}$ there exists a closed set B in some $\mathcal{S}_{n+m}$ such that $\pi(B) = U$.

Define $\tilde{\omega}$ on $U \times \mathbb{R}^m$ through $\tilde{\omega} = \pi^*\omega$. By Corollary 5.2.6 is $L' := L \times \mathbb{R}^m$ a Rolle leaf on $(U \times \mathbb{R}^m, \pi^*\omega)$. Hence $B \cap L'$ is in $\text{Rolle}(\mathcal{S})_{n+m}$. Furthermore it is closed: Let $(\overline{\mathrm{x}}_i, \overline{\mathrm{y}}_i) \in B \cap L'$ with $(\overline{\mathrm{x}}_i, \overline{\mathrm{y}}_i) \to (\overline{\mathrm{x}}, \overline{\mathrm{y}})$. Because B is closed, $(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in B$, in particular $\overline{\mathrm{x}} \in U$, so $(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in U'$. Since L' is closed in U', we obtain $(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in L'$, thus $(\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in B \cap L'$. $\square$

5.4 The Pfaffian sets satisfy DPC^N

In this section we prove that Rolle($\mathcal{S}$) satisfies DPC^N for all N. The proof is based on ideas of Karpinski and Macintyre in [KM99], where they use a cell decomposition of the basic sets of a Rolle leaf and distinguish between open cells and cells with lower dimension. It seems to be complex or maybe even impossible to extend a cell on $\mathbb{R}^n$ such that all additional assumptions made on the 1-form ω and the Rolle leaf are preserved. So we work with partial defined functions with closed graph, which is enough to apply the theorem of the complement as we have shown it in Chapter 3.

Fix the dimension n. We have to show that for all $B \in \text{Rolle}(\mathcal{S})_n$ there exists an $m \in \mathbb{N}$ such that for all $N \in \mathbb{N}$ there are some C^N functions $f_1 : B_i \to \mathbb{R}, \ldots, f_r : B_r \to \mathbb{R}$ with closed graph which are in $\widetilde{\text{Rolle}(\mathcal{S})}$ such that the $B_i \subseteq \mathbb{R}^{n+m}$ are open and $\pi_n^m[Z(f_1) \cup \cdots \cup Z(f_r)] = B$. In other words we want to prove that there are $f_i \in M^N(\text{Rolle}(\mathcal{S}))$ for $i = 1, \ldots r$ such that $\pi_n^m[Z(f_1) \cup \cdots \cup Z(f_r)] = B$. We call this statement DPC^N for B with m, recall the definition in Chapter 3.

The first step is to reduce the problem to Rolle leaves; this is done with the following lemma.

Lemma 5.4.1. *If DPC^N holds for all N for Rolle leaves, it holds for all $A \in$ Rolle($\mathcal{S}$). Let $A = \bigcup_{i=1}^r \left(A_i \cap \bigcap_{j=1}^{k_i} L_j^{(i)} \right)$ and assume that for each Rolle leaf $L_j^{(i)} \subseteq \mathbb{R}^n$ the m of the DPC^N condition is bounded by $n+2$. Then DPC^N holds for A for arbitrary N with m bounded by $n + 2 \cdot max\{k_i + 1 \mid i = 1, \ldots, r\}$.*

Proof. **1. Reduce on a set of the form $A' \cap L_1 \cap \cdots \cap L_k \in \widetilde{\text{Rolle}(\mathcal{S})}$.**

If we find functions satisfying the DPC^N condition for $B_i = A_i \cap \bigcap_{j=1}^{k_i} L_j^{(i)}$ with m_i', then we can take the finite union of these functions extended on the dimension $\max\{m_i' \mid i = 1, \ldots, r\}$. As we show in the following, we can bound m_i' by $n+2(k_i+1)$, so m is bounded by $n+2\cdot\max\{k_i+1 \mid i = 1, \ldots, r\}$. Hence we assume that we have a set of the form $A = A' \cap L_1 \cap \cdots \cap L_k \in \text{Rolle}(\mathcal{S})_n$ (for some $k \in \mathbb{N}$).

2. Defining new functions for the DPC^N condition.

By Lemma 4.4 DC^N holds for $A' \in \mathcal{S}$ (and so DPC^N) and so by assumption there exist m_i such that for all N exists C^N functions $f_i^{(1)} : U_i^{(1)} \to \mathbb{R}, \ldots, f_i^{(r_i)} : U_i^{(r_i)} \to \mathbb{R}$ in $\widetilde{\text{Rolle}(\mathcal{S})}$ with closed graph and $U_i^{(j)} \subseteq \mathbb{R}^{n+m_i}$ open for $j = 1, \ldots, r_i$ such that $\pi_n^{n+m_0}(Z(f_0^{(1)})) = A'$ and $L_i = \bigcup_{j=1}^{r_i} \pi_n^{n+m_i}(Z(f_i^{(j)}))$ for $i = 1, \ldots, k$.

Define $m = n + \sum_{i=0}^k m_i$. By our second assumption $m_i \leq 2$, so $m \leq n + 2(k+1)$. Fix $N \in \mathbb{N}$.

Now for each $\overline{a} = (a_0, \ldots, a_k) \in I := \{1, \ldots, r_0\} \times \cdots \times \{1, \ldots r_k\}$ define

$$U_{\overline{a}} := \{(\overline{x}, \overline{y}_1, \ldots, \overline{y}_k) \mid (\overline{x}, \overline{y}_1) \in U_1^{(a_1)} \wedge \cdots \wedge (\overline{x}, \overline{y}_k) \in U_k^{(a_k)}\}$$

$$f_{\overline{a}}(\overline{x}, \overline{y}_0, \ldots, \overline{y}_k) := \sum_{i=0}^{k} \left(f_i^{(a_i)}(\overline{x}, \overline{y}_i) \right)$$

For every $\overline{a} \in I$ the set $U_{\overline{a}} \subseteq \mathbb{R}^m$ is open, since all $U_i^{(j)}$ are open and $f_{\overline{a}} : U_{\overline{a}} \to \mathbb{R}$ is a C^N function, since it is a combination of C^N functions.

3. The functions $f_{\overline{a}}$ are in $M^N(\mathrm{Rolle}(\mathcal{S}))$.

It remains to show that for $\overline{a} \in I$ the function $f_{\overline{a}} \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$ and $\Gamma(f_{\overline{a}})$ is closed. We begin with two claims about containing closed functions in $\widetilde{\mathrm{Rolle}(\mathcal{S})}$.

Claim 1: Let $g : U \to \mathbb{R}$ be in $\widetilde{\mathrm{Rolle}(\mathcal{S})}$ and let $\Gamma(g)$ be closed. Then $g^2 \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$ and $\Gamma(g^2)$ is closed.

Proof: Look at Remark 3.2.4. □(Claim)

Claim 2: Let $f_1 : U_1 \to \mathbb{R}$ and $f_2 : U_2 \to \mathbb{R}$ be positive ($f_1, f_2 \geq 0$) functions in $\widetilde{\mathrm{Rolle}(\mathcal{S})}$ with closed graph and define $U = \{(\overline{x}, \overline{y}_1, \overline{y}_2) \in \mathbb{R}^{n+m_1+m_2} \mid (\overline{x}, \overline{y}_1) \in U_1 \wedge (\overline{x}, \overline{y}_2) \in U_2\}$. Then the function $g : U \to \mathbb{R}$ defined by $g(\overline{x}, \overline{y}_1, \overline{y}_2) = f_1(\overline{x}, \overline{y}_1) + f_2(\overline{x}, \overline{y}_2)$ is in $\widetilde{\mathrm{Rolle}(\mathcal{S})}$ and the graph $\Gamma(g)$ is closed.

Proof:

$$\begin{aligned}
\Gamma(g) &= \{(\overline{x}, \overline{y}_1, \overline{y}_2, g(\overline{x}, \overline{y}_1, \overline{y}_2)) \mid (\overline{x}, \overline{y}_1) \in \mathrm{dom}(f_1) \\
&\qquad \wedge (\overline{x}, \overline{y}_2) \in \mathrm{dom}(f_2)\} \\
&= \big\{(\overline{x}, \overline{y}_1, \overline{y}_2, z) \mid \exists z_1, z_2\ (z = z_1 + z_2 \\
&\qquad \wedge f_1(\overline{x}, \overline{y}_1) = z_1 \wedge f_2(\overline{x}, \overline{y}_2) = z_2)\big\} \\
&= \pi\big[\{(\overline{x}, \overline{y}_1, \overline{y}_2, z, z_1, z_2) \mid z - z_1 - z_2 = 0\} \\
&\qquad \cap \{(\overline{x}, \overline{y}_1, \overline{y}_2, z, z_1, z_2) \mid (\overline{x}, \overline{y}_1, z_1) \in \Gamma(f_1)\} \\
&\qquad \cap \{(\overline{x}, \overline{y}_1, \overline{y}_2, z, z_1, z_2) \mid (\overline{x}, \overline{y}_2, z_2) \in \Gamma(f_2)\}\big]
\end{aligned}$$

And by Lemma 1.3.5 we have $\Gamma(g) \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$, since $\widetilde{\mathrm{Rolle}(\mathcal{S})}$ is an o-minimal weak structure as $\mathrm{Rolle}(\mathcal{S})$ is an o-minimal weak structure by Corollary 2.1.3.

Let $(\overline{x}_n, \overline{y}_{1,n}, \overline{y}_{2,n}) \in U$ for $n \in \mathbb{N}$ with $(\overline{x}_n, \overline{y}_{1,n}, \overline{y}_{2,n}) \to (\overline{x}, \overline{y}_1, \overline{y}_2) \in \partial U$. Define $U_2' = \{(\overline{x}, \overline{y}_1, \overline{y}_2) \in \mathbb{R}^{n+m_1+m_2} \mid (\overline{x}, \overline{y}_2) \in U_2\}$. Then $U = (U_1 \times \mathbb{R}^{m_2}) \cap U_2'$. The following claim shows $\partial U \subseteq \partial(U_1 \times \mathbb{R}^{m_2}) \cup \partial U_2'$.

Claim 3: Let $A = B \cap C$. Then $\partial A \subseteq \partial B \cup \partial C$.

Proof: Let $x \in \overline{A}$. Then exists $x_n \to x$ such that $x_n \in A$ for all n. By definition,

$x_n \in B$ and $x_n \in C$ for all n, so $x \in \overline{B} \cap \overline{C}$. Assume $x \notin int(A)$ but $x \in int(B)$ and $x \in int(C)$. Then there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \in B \cap C = A$. Contradiction. □(Claim)

As a result, we have $(\overline{x}_n, \overline{y}_{1,n}) \to (\overline{x}, \overline{y}_1) \in \partial U_1$ or $(\overline{x}_n, \overline{y}_{2,n}) \to (\overline{x}, \overline{y}_2) \in \partial U_2$. Without loss of generality, we can assume the first case. Then, since $\Gamma(f_1)$ is closed, we obtain $|f_1(\overline{x}_n, \overline{y}_{1,n})| \to \infty$. As f_1 and f_2 are positive, we obtain $|g(\overline{x}, \overline{y}_1, \overline{y}_2)| = |f_1(\overline{x}, \overline{y}_1)| + |f_2(\overline{x}, \overline{y}_2)| \to \infty$. By Remark 1.3.8 $\Gamma(g)$ is closed. This completes the proof of Claim 2. □(Claim)

By the above claims for every $\overline{a} \in I$ the function $f_{\overline{a}} \in \widetilde{\text{Rolle}(\mathcal{S})}$ and has a closed graph.

4. The equation $\bigcup_{\overline{a} \in I} \pi[Z(f_{\overline{a}})]$ holds.

Notice that

$$\begin{aligned} f_{\overline{a}}(\overline{x}, \overline{y}_0, \ldots, \overline{y}_k) = 0 \quad &\Leftrightarrow \quad \sum_{i=0}^{k} (f_i^{a_i})^2(\overline{x}, \overline{y}_i) = 0 \\ &\Leftrightarrow \quad \forall i \in \{1, \ldots, k\} f_i^{(a_i)}(\overline{x}, \overline{y}_i) = 0. \end{aligned}$$

Hence

$$\begin{aligned} \overline{x} \in \bigcup_{\overline{a} \in I} \pi_n^m[Z(f_{\overline{a}})] \quad &\Leftrightarrow \quad \exists\, \overline{a} \in I\ \exists\, \overline{y}_0, \ldots, \overline{y}_k\ \ \forall i\ f_i^{(a_i)}(\overline{x}, \overline{y}_i) = 0\} \\ &\Leftrightarrow \quad \exists a_0, \ldots, a_k\ \ \overline{x} \in \pi[Z(f_0^{(a_0)})] \wedge \ldots \\ &\qquad \wedge\ \overline{x} \in \pi[Z(f_k^{(a_k)})] \\ &\Leftrightarrow \quad \overline{x} \in \bigcup_{a_0=1}^{r_0} \pi[Z(f_0^{(a_0)})] \wedge \cdots \wedge \overline{x} \in \bigcup_{a_k=1}^{r_k} \pi[Z(f_k^{(a_k)})] \\ &\Leftrightarrow \quad \overline{x} \in A' \cap L_1 \cap \cdots \cap L_k \end{aligned}$$

□

In the rest of this chapter, we prove the DPC^N condition for Rolle leaves. We prove the claim for open and non-open cells seperately.

Lemma 5.4.2. *Let L be a Rolle leaf on some (U, ω), $U, \omega \in \mathcal{S}$. Then there exists for each N some functions $f_i : B_i \to \mathbb{R}$ in $M^N(\text{Rolle}(\mathcal{S}))$ for $i = 1, \ldots, r$ such that $B_i \subseteq \mathbb{R}^{n+2}$ and $\pi_n^{n+2}[Z(f_1) \cup \cdots \cup Z(f_r)] = L$.*

Proof. This lemma contains the difficult part of the proof. Fix $N \in \mathbb{N}$. We begin with dividing U into cells.

Let $\omega = \sum_{i=0}^{n} a_i dx_i$. Define $A_j = \{\overline{x} \mid a_j(\overline{x}) \neq 0\}$ for $j = 1, \dots, n$. For every j the set A_j is in $\mathcal{S}$, since ω and therefore a_j is in $\mathcal{S}$:

$$A_j = \pi_n^{n+1} \left[\Gamma(a_j) \cap (\{(\overline{x}, y) \mid y < 0\} \cup \{(\overline{x}, y) \mid y > 0\})\right].$$

So we can find a C^N cell decomposition compatible with $U, A_1, \dots, A_n$ by Theorem 1.2.6. By a further cell decomposition with the decomposition argument for functions in Theorem 1.2.6 b), we can assume that ω is C^N restricted to any cell, particularly that $C \cap L$ is a C^N manifold for each cell C. There are only finitely many cells, so if there are functions as wished for every cell C such that $\bigcup_{i=1}^{r} \pi[Z(f_i^C)] = C \cap L$, we can take all this functions and $\bigcup_{C \text{ cell}} \bigcup_{i=1}^{r} \pi[Z(f_i^C)] = L$.

The next step is to find DPC^N functions for intersections with open cells. Afterwards we need these functions to prove the DPC^N-statement for intersections with closed cells.

Lemma 5.4.3. *Let C be an open cell such that $L \cap C$ has all properties of a Rolle leaf on (C, ω) except connectedness and $\omega = \sum_{i=1}^{n} a_i dx_i$ is C^N on C. Then there exists open sets $B_i \subseteq \mathbb{R}^{n+1}$ and $M^N(\mathrm{Rolle}(\mathcal{S}))$-functions $f_i : B_i \to \mathbb{R}$ for $i = 1, \dots, r$ such that $\pi_n^{n+1}[Z(f_1) \cup \dots \cup Z(f_r)] = L \cap C$.*

Of course, by defining new functions $g_i(\overline{x}, y) = f_i(\overline{x})$ for $i = 1, \dots r$ we can expand the function on $B \times \mathbb{R} \subseteq \mathbb{R}^{n+2}$.

We prove the lemma in two steps; first we prove a special case, where we assume that the intersection of a Rolle leaf and a cell is connected and that the last component of ω is nowwhere zero. In the next step we reduce the general case to this special case by applying Lemma 5.4.6. Before starting the proof, we recall the inverse function theorem.

Theorem 5.4.4. *[Inverse Function Theorem] Let $M \subseteq \mathbb{R}^m$ and $N \subseteq \mathbb{R}^n$ two C^N submanifolds of the same dimension k. Let $f : M \to N$ be a C^N map and let $\overline{a} \in M$ be a regular point of f. Then there exists an open set $U \subseteq M$ and an open set $V \subseteq N$ such that there exists a C^N map $g : V \to U$ which is inverse to $f \restriction U : U \to V$.*

Proof. By Theorem 3 in [For77], Chapter 8 and the corresponding remarks the theorem holds for $M = N = \mathbb{R}^k$. It is easy to carry over this result to C^N manifolds: Look for $\overline{a} \in M$ at the open neighborhoods of $\overline{a}$ and $f(\overline{a})$ which are diffeomorphic to $\mathbb{R}^k$ and compose the corresponding maps. □

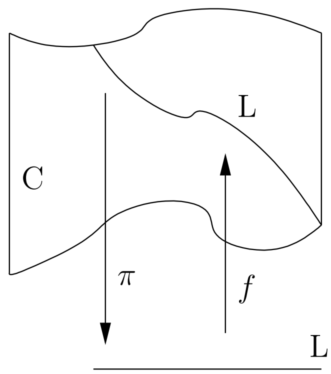

Figure 5.4: The Rolle leaf $L \cap C$ is a graph of a function.

Lemma 5.4.5. *Let C be an open cell of the open set $U \in \mathcal{S}$ and let L be a Rolle leaf on (U, ω) such that $L \cap C$ is connected and so a Rolle leaf on $(C, \omega \restriction C)$. Let $\omega(\overline{\mathrm{x}}) = \sum_{i=1}^{n} a_i(\overline{\mathrm{x}}) dx_i$ be C^N such that $a_n(\overline{\mathrm{x}}) \neq 0$ for all $\overline{\mathrm{x}} \in C$. Then:*

a) The set $L \cap C$ is the graph of a function $f : L' \to \mathbb{R}$, where $L' = \pi^n_{n-1}[L \cap C]$.

b) The set L' is open and connected and the function f is C^N.

c) There exists an open set $B \subseteq \mathbb{R}^{n+1}$ and a function $g : B \to \mathbb{R}$ which is in $M^N(\mathrm{Rolle}(\mathcal{S}))$ such that $\pi^{n+1}_n[Z(g)] = L \cap C$.

Proof. a) Define for all $(x_1, \ldots, x_{n-1}) \in L' := \pi^n_{n-1}[L \cap C]$ the map f through

$$f(x_1, \ldots, x_{n-1}) := \text{ the unique } x_n \text{ with } \overline{\mathrm{x}} \in L.$$

Claim : The function f is well-defined.

Proof: Assume that there are some different points $\overline{\mathrm{x}} = (x_1, \ldots, x_n) \in L \cap C$ and $\overline{\mathrm{x}}' = (x_1, \ldots, x_{n-1}, x_n') \in L \cap C$. Define $\gamma : [0,1] \to U$ by

$$\gamma(t) = (x_1, \ldots, x_{n-1}, (1-t)x_n + t x_n')$$

as the direct path from $\overline{\mathrm{x}}$ to $\overline{\mathrm{x}}'$. Since $C = (g, h)$ is an open cell we have $\gamma(t) \in C$ for all $t \in [0,1]$. By the Rolle leaf condition, there exists $t \in [0,1]$ such that $\gamma \perp \omega$ at $\gamma(t)$, by definition of γ it is vertical, that means that ω is horizontal. But his cannot happen, because $a_n(\overline{\mathrm{y}}) \neq 0$ for all $\overline{\mathrm{y}} \in \mathbb{R}^n$, what implies that it has a vertical component everywhere. Contradiction! □(Claim)

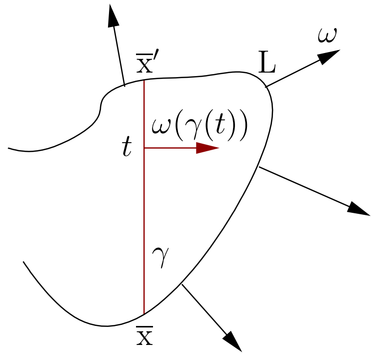

Figure 5.5: Assume $L \cap C$ is not a graph of a function; then ω must be horizontal in one point.

Obviously, $(\pi \upharpoonright L \cap C)^{-1} = f \upharpoonright L'$.

b) Since $L \cap C$ is connected $\mathrm{dom}(f) = \pi[L \cap C]$ must be connected.

Take any point $\overline{x} \in L' = \pi[L \cap C]$. Then there exists a point $\overline{z} \in L \cap C$ such that $\overline{x} = \pi(\overline{z})$. It is $T_{\overline{x}}L = \mathbb{R}^{n-1}$ and $d\pi$ is surjective, so $\overline{x}$ is a regular point of $\pi_L : L \cap C \to \mathbb{R}^{n-1}$. The leaf $L \cap C$ is a manifold of dimension $n-1$, so π is a map between to manifolds of the same dimension. Because L is a C^N submanifold of $\mathbb{R}^n$ on C, the projection π_L is also C^N. By the Inverse Function Theorem 5.4.4, there exist an open neighborhood $U \subseteq L$ of $\overline{z}$, an open neighborhood $V \subseteq \mathbb{R}^{n-1}$ of $\overline{x}$ and a C^N function $g : V \to U$ such that $\pi \circ g = id_V$ and $g \circ \pi = id_U$. Obviously, $g = f \upharpoonright V$. So f is C^N in $\overline{x}$. Particularly, we have $V \subseteq L'$, so L' is open.

c) Note that C is an open C^N cell in $\mathcal{S}$, so by Lemma 1.3.9 there exists a closed set $D \in \mathcal{S}_{n+1}$ such that $\pi[D] = C$ and D is a graph of a C^N function $d : C \to \mathbb{R}$.

Define $B := ((L' \times \mathbb{R}) \cap C) \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$, which is an open set by b) and define $g : B \to \mathbb{R}$ by

$$g(\overline{x}, x', y) = (f(\overline{x}) - x')^2 + (d(\overline{x}, x') - y)^2 + (d(\overline{x}, f(\overline{x})) - y)^2.$$

Note that

$$\begin{aligned} g(\overline{x}, x', y) = 0 \quad & \Leftrightarrow \quad f(\overline{x}) - x' = 0 \;\wedge\; d(\overline{x}, x') - y = 0 \\ & \qquad \wedge\, d(\overline{x}, f(\overline{x})) - y = 0 \\ & \Leftrightarrow \quad y = d(\overline{x}, x') = d(\overline{x}, f(\overline{x})) \\ & \qquad \wedge x' = f(\overline{x}) \\ & \overset{\Gamma(f)=L, \Gamma(d)=D}{\Leftrightarrow} \quad (\overline{x}, x') \in L \;\wedge\; (\overline{x}, x', y) \in D \end{aligned}$$

Hence

$$(\overline{x}, x') \in \pi_n^{n+1}(Z(g)) \quad \Leftrightarrow \quad \exists y \in \mathbb{R}\ (\overline{x}, x') \in L\ \wedge\ (\overline{x}, x', y) \in D$$
$$\overset{\pi(D)=C}{\Leftrightarrow} \quad (\overline{x}, x') \in L \cap C.$$

Since $L' = \pi(L \cap C) \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$ and $C \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$ and because $\widetilde{\mathrm{Rolle}(\mathcal{S})}$ is an o-minimal weak structure by the last section, we can apply Lemma 1.3.5, a) and replace variables in semi-algebraic sets by values of functions in a set in $\widetilde{\mathrm{Rolle}(\mathcal{S})}$. A closer look on the graph of g makes clear that we obtain $\Gamma(g) \in \widetilde{\mathrm{Rolle}(\mathcal{S})}$ by this method.

As a combination of C^N functions g is a C^N function itself.

Lastly, we show that $\Gamma(g)$ is closed. Let $(\overline{x}_n, x'_n, y_n) \to (\overline{x}, x', y) \in \partial B$. Recall Claim 3 in the proof of 5.4.1. So for $B = (L' \times \mathbb{R}^2) \cap (C \cap \mathbb{R})$ holds $\partial B \subseteq \partial(L' \times \mathbb{R}^2) \cap \partial(C \cap \mathbb{R})$, i.e. $(\overline{x}, x', y) \in \partial(L' \times \mathbb{R}^2) \cup \partial(C \cap \mathbb{R})$. Hence $(\overline{x}, x') \in \partial C$ or $\overline{x} \in \partial L'$. In the first case we have $(\overline{x}_n, x'_n) \to (\overline{x}, x') \in \partial C$. Since the graph of d is closed, we obtain $d(\overline{x}_n, x'_n) \to \infty$. Obviously, $g \geq 0$, so $g(\overline{x}_n, x'_n, y_n) \to \infty$. In the second case, notice that $(\overline{x}_n, f(\overline{x}_n)) \in L$ for all N and $\overline{x}_n \to \overline{x} \in \partial L'$, i.e. $(\overline{x}_n, f(\overline{x}_n)) \to (\overline{x}, z) \in \partial L$ (or $z = \infty$, then we are ready). Since $L \cap C$ is closed in C, we obtain $(\overline{x}_n, f(\overline{x}_n)) \to (\overline{x}, z) \in \partial C$. Again we can use that the graph of d is closed and obtain $d(\overline{x}_n, f(\overline{x}_n)) \to \infty$, i.e. $g(\overline{x}_n, x'_n, y_n) \to \infty$. By Remark 1.3.8 $\Gamma(g)$ is closed.

□

The next step is to show that we can reduce the case of a Rolle leaf L intersected with an open cell C to the special case we examined in the above lemma. We show that we can assume that $L \cap C$ is a connected set and that $a_n(\overline{x}) \neq 0$ for all $\overline{x} \in C$.

Lemma 5.4.6. *Let $C \subseteq \mathbb{R}^n$ be an open cell in the above cell decomposition, let L be a Rolle leaf on U and $\omega = \sum_{i=1}^n a_i dx_i$, where ω is C^N on C and $U, \omega \in \mathcal{S}$. Then the DPC^N condition holds for $L \cap C$ with $m = n+1$.*

Proof. Since L is a Rolle leaf and C is an open cell, $L \cap C$ is a manifold. It has all properties of a Rolle leaf on (C, ω) except connectedness.

1. We can assume $L \cap C$ is connected. Let $C \in \mathcal{S}_n$ be an open C^N cell. $C \cap L$ has only finitely many connected components, because it is in $\mathrm{Rolle}(\mathcal{S})$, which is an o-minimal weak structure. Each connected component is a Rolle leaf. So if we can find functions f_i as desired for each connected component C_i, which are C^N, all these functions will fulfill the DPC^N-condition, because $\bigcup_i \pi(Z(f_i)) = \bigcup_i C_i$. So it is enough to show the claim for $L \cap C$ connected. The other properties of a Rolle leaf are kept in the intersection obviously.

2. We can assume that $a_n(\overline{x}) \neq 0$ for all $\overline{x} \in C$.

The cell decomposition is compatible with the $A_j = \{\overline{x} \mid a_j(\overline{x}) \neq 0\}$ for $j = 1, \ldots, n$, that is for every j we have for all $\overline{x} \in C$ that $a_j(\overline{x}) = 0$ or for all $\overline{x} \in C$ is $a_j(\overline{x}) \neq 0$. Furthermore, for all $\overline{x} \in C$ we have $\omega(\overline{x}) \neq 0$ by the assumption $S(\omega) = \emptyset$. So there exists a maximal $j_0 \in \{1, \ldots, n\}$ such that for all $\overline{x} \in C$ we have $a_{j_0}(\overline{x}) \neq 0$.

If $j_0 = n$ we are finished. So assume $j_0 \neq n$. Our aim is now to construct a cell which is diffeomorphic to C, where still $a_j \neq 0$ holds and where it is possible to change jth component to the last coordinate. This is achieved if we have a cell of the form $C_{j_0} \times (0,1)^{n-j_0}$, where C_{j_0} is an open cell in $\mathbb{R}^{j_0}$.

The cell C is open, so it is constructed in the following inductive way out of open cells: The first cell $C_1 = (a, b)$ is an interval in $\mathbb{R}$, the j-th cell is constructed by to functions $f_j, g_j : C_{j-1} \to \mathbb{R}$ where $C_j = (f_j, g_j)$. Note that $f_j = -\infty$ and $g_j = \infty$ are allowed. At last $C = C_n = (f_n, g_n)$.

Claim : Define a sequence (D_j) of open cells in $\mathbb{R}^n$ by $D_n = C$ and $D_j := C_j \times (0,1)^{n-j}$ for $j = j_0, \ldots n$. Then there exist diffeomorphisms $\phi_j : D_{j-1} \to D_j$ and 1-forms $\omega^{j-1} = \sum_{i=1}^n a_i^{j-1}(\overline{x}) dx_i = \phi_j^* \omega^j$ beginning from the 1-form $\omega^n = \omega = \sum_{i=1}^n a_i dx_i$ such that the following holds:

For all $j_0 < j \leq n$ we retain $\forall \overline{x} \in D_j \ a_{j_0}(\overline{x}) \neq 0 \ \wedge \ a_i^j(\overline{x}) = 0$ for $j_0 < i \leq j$.

Proof: We do this by induction, beginning with $j = n$ and ending with $j = j_0$.

Induction beginning: It is clear that $D_n = C$ has the wished form and it is not necessary to construct the homomorphism. By assumption we have for all $\overline{x} \in C$ that $a_i^n(\overline{x}) = 0$ for $j_0 < i \leq n$ and $a_{j_0}(\overline{x}) \neq 0$.

Induction hypothesis: *Let $j \in \{n, \ldots, j_0 + 1\}$ and assume that D_j and ω^j are already constructed. For all $\overline{x} \in D_j$ we have $a_{j_0}^j(\overline{x}) = 0$ and $a_i^j(\overline{x}) = 0$ for $j_0 < i \leq j$.*

Induction step: Now examine $j \mapsto j - 1$. Recall $D_{j-1} := C_{j-1} \times (0,1)^{n-j+1}$ and $C_j = (f_j, g_j)$, where $f_j, g_j : C_{j-1} \to \mathbb{R}$.

Let $\psi_{a,b} : (0,1) \to (a,b)$ for $a \in \mathbb{R} \cup \{-\infty\}$ and $b \in \mathbb{R} \cup \{\infty\}$ be defined in the following way:

- If $a, b \in \mathbb{R}$, let $\psi_{a,b}(x) := (b-a)x + a$.
- If $a \in \mathbb{R}, b = \infty$, let $\psi_{a,b}(x) := a - 1 + \frac{1}{1-x}$.
- If $a = -\infty, b \in \mathbb{R}$, let $\psi_{a,b}(x) := b + 1 - \frac{1}{x}$.
- If $a = -\infty, b = \infty$, let $\psi_{a,b}(x) := \frac{1}{1-x^2} - \frac{1}{x^2}$.

Some easy calculations show that $\psi'_{a,b}(x) > 0$ for all $x \in (0,1)$ and so the map is strictly monoton and C^N as the inverse map is and so a diffeomorphism between $(0,1)$ and (a,b). Note that the map $\psi(a,b,x) = \psi_{a,b}(x)$ is also C^N in a if $a > -\infty$ and in b if $b < \infty$. Furthermore, its graph is a semialgebraic set and hence in $\mathcal{S}$.

Define $\phi_j : D_{j-1} \to D_j$ for $\overline{\mathrm{x}} = (x_1, \dots, x_{j-1})$ and $\overline{\mathrm{y}} = (x_{j+1}, \dots, x_n)$ through

$$\phi_j(\overline{\mathrm{x}}, x_j, \overline{\mathrm{y}}) = (\overline{\mathrm{x}}, \psi_{f_j(\overline{\mathrm{x}}), g_j(\overline{\mathrm{x}})}(x_j), \overline{\mathrm{y}}).$$

It is obvious, that ϕ_j is well-defined, since

$$(\overline{\mathrm{x}}, x_{j+1}, \overline{\mathrm{y}}) \in D_{j-1} = C_{j-1} \times (0,1)^{n-j+1} \;\Rightarrow\; (\overline{\mathrm{x}}, \psi_{f_j(\overline{\mathrm{x}}), g_j(\overline{\mathrm{x}})}(x_{j+1})) \in C_j$$

by the definition of $\psi_{a,b}$. Additionally, ϕ_j is constructed from the diffeomorphisms id and ψ and the C^N functions f_j and g_j, so it is again a C^N map. It is easy to see that it is bijective. By the inverse function theorem we obtain that the inverse map is again C^N.

Notice that $\phi_j^* \omega^j(\overline{\mathrm{x}}) = \sum_{i=1}^n a_i^{j-1}(\overline{\mathrm{x}}) dx_i$, where

$$a_i^{j-1}(\overline{\mathrm{x}}) = \sum_{k=1}^n a_k \circ \phi_j(\overline{\mathrm{x}}) \frac{\partial(\phi_j)_k}{\partial x_i}.$$

(This is a simple calculation and needs only the definition of the pullback.) Now verify that for $\overline{\mathrm{x}} \in D_{j-1}$ we have $a_{j_0}^j(\overline{\mathrm{x}}) \neq 0$ and $a_i^j(\overline{\mathrm{x}}) = 0$ for all $j_0 < i \leq j-1$.

It is $(\phi_j)_k = id$ for $k \neq j$, so for $i \neq j$ it is

$$\begin{aligned} a_i^{j-1}(\overline{\mathrm{x}}) &= a_i^j \circ \phi_j(\overline{\mathrm{x}}) \frac{\partial(id)_i}{\partial x_i} + a_j^j \circ \phi_j(\overline{\mathrm{x}}) \frac{\partial(\phi_j)_j}{\partial x_i} \\ &= a_i^j \circ \phi_j(\overline{\mathrm{x}}) + a_j^j \circ \phi_j(\overline{\mathrm{x}}) \frac{\partial \psi(x_i, f_j(\overline{\mathrm{x}}), g_j(\overline{\mathrm{x}}))}{\partial x_i} \end{aligned}$$

By induction hypothesis and since $j > j_0$ we have $a_j^j(\overline{\mathrm{x}}) = 0$ for $\overline{\mathrm{x}} \in D_j$. Hence $a_i^{j-1}(\overline{\mathrm{x}}) = a_i^j \circ \phi_j(\overline{\mathrm{x}})$ and this is zero, if $j_0 < i \leq j-1$, again by induction hypothesis.

If $i = j_0$ we have $a_{j_0}^{j-1}(\overline{\mathrm{x}}) = a_{j_0}^j \circ \phi_j(\overline{\mathrm{x}}) \neq 0$ for all $\overline{\mathrm{x}} \in D_{j-1}$, since $a_{j_0}^j(\overline{\mathrm{x}}) \neq 0$ for all $\overline{\mathrm{x}} \in D_j$. □(Claim)

So after defining this sequences, we define $\phi := \phi_n \circ \cdots \circ \phi_{j_0+1}$ and obtain a diffeomorphism $\phi : D_{j_0} \to C$. The claim implies that $\phi^* \omega(\overline{\mathrm{x}}) = \sum_{i=0}^n b_i(\overline{\mathrm{x}}) dx_i$, where $b_{j_0}(\overline{\mathrm{x}}) \neq 0$ for all $\overline{\mathrm{x}} \in D_{j_0}$.

Define now

$$C' := \left\{ (\overline{\mathrm{x}}, \overline{\mathrm{y}}, z) \in \mathbb{R}^{j_0-1} \times (0,1)^{n-j_0} \times \mathbb{R} \mid (\overline{\mathrm{x}}, z) \in C_{j_0} \right\}.$$

Obviously, C' is an open C^N cell in $\mathbb{R}^n$, simply notice that $C_{j_0-1} \times (0,1)^{n-j_0}$ is an open cell in $\mathbb{R}^{n-1}$ and define $f, g : C_{j_0-1} \times (0,1)^{n-j_0} \to \mathbb{R}$ through $f(\overline{\mathrm{x}}, \overline{\mathrm{y}}) = f_{j_0}(\overline{\mathrm{x}})$

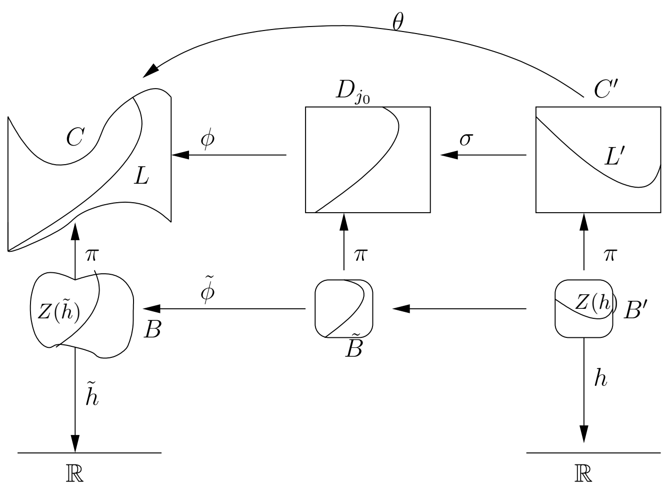

Figure 5.6: Transformation of cell and Rolle leaf.

and $g(\overline{x}, \overline{y}) = g_{j_0}(\overline{x})$. In fact, it is only a permutation of coordinates that puts the coordinate x_{j_0} onto the end. Call this permutation $\sigma : C' \to D_{j_0}$. Then $\sigma^*(\phi^*\omega) = \sum_{i=1}^n c_i(\overline{x})dx_i$, where $c_n(\overline{x}) = b_{j_0}(\overline{x}) \neq 0$ for all $\overline{x} \in C'$, since the permutation swaps also the coordinate functions in $\phi^*\omega$.

By Corollary 5.2.5 $\phi^{-1}(L)$ is a Rolle leaf on $(D_{j_0}, \phi^*\omega)$. The map σ is as permutation a diffeomorphism and by notating $\theta = \phi \circ \sigma$ we obtain also by Corollary 5.2.5 a Rolle leaf $L' := \theta^{-1}(L \cap C)$ on $(C', \tilde{\omega})$, where C' is a cell and $\tilde{\omega} = \theta^*\omega = \sum_{i=1}^n c_i(\overline{x})dx_i$ such that $c_n(\overline{x}) \neq 0$ for all $\overline{x} \in C'$.

After having constructed this new cell, we transform the functions given by assumption. Let $h_1, \ldots, h_r \in \widetilde{\text{Rolle}}(\mathcal{S})$ be some functions satisfying the DPC^N condition for L'. These functions exist by Lemma 5.4.5 and they are defined on some open sets $B'_1, \ldots, B'_r \subseteq \mathbb{R}^{n+1}$, their graphs are closed and we have $\pi_n^{n+1}[Z(h_1) \cup \cdots \cup Z(h_r)] = L'$. Define now $B_i := \{(\theta(\overline{x}), y) \mid (\overline{x}, y) \in B'_i\} \subseteq \mathbb{R}^{n+1}$ and $\tilde{h}_i : B_i \to \mathbb{R}$ by $\tilde{h}_i(\overline{x}, y) := h_i(\theta^{-1}(\overline{x}), y)$ for $i = 1, \ldots, r$.

$$
\begin{aligned}
& \overline{x} \in \pi_n^{n+1}[Z(\tilde{h}_1) \cup \cdots \cup Z(\tilde{h}_r)] \\
\Leftrightarrow\ & \exists y \exists i\ (\overline{x}, y) \in Z(\tilde{h}_i) \\
\Leftrightarrow\ & \exists y \exists i\ (\overline{x}, y) \in B_i \ \wedge\ \tilde{h}_i(\overline{x}, y) = 0 \\
\Leftrightarrow\ & \exists y \exists i \exists \overline{x}'\ \overline{x} = \theta(\overline{x}') \ \wedge\ (\overline{x}', y) \in B'_i \ \wedge\ h_i(\theta^{-1}(\overline{x}), y) = 0 \\
\Leftrightarrow\ & \exists y \exists i \exists \overline{x}'\ \overline{x}' = \theta^{-1}(\overline{x}) \ \wedge\ (\overline{x}', y) \in B'_i \ \wedge\ h_i(\overline{x}', y) = 0 \\
\Leftrightarrow\ & \exists y \exists i\ (\theta^{-1}(\overline{x}), y) \in Z(h_i)
\end{aligned}
$$

$$\begin{aligned}
&\Leftrightarrow \quad \theta^{-1}(\overline{\mathrm{x}}) \in \pi_n^{n+1}[Z(h_1) \cup \cdots \cup Z(h_r)] \\
&\Leftrightarrow \quad \theta^{-1}(\overline{\mathrm{x}}) \in L' \\
&\Leftrightarrow \quad \overline{\mathrm{x}} \in L \cap C
\end{aligned}$$

Thus we find functions $\tilde{\mathrm{h}}_1, \ldots, \tilde{\mathrm{h}}_r$ satisfying the DPC^N condition for $L \cap C$, if we can show that $\tilde{\mathrm{h}}_i \in M^N(\mathrm{Rolle}(\mathcal{S}))$ for $i = 1, \ldots, r$. Since $h_1, \ldots, h_r, \phi, \sigma \in \mathrm{Rolle}(\mathcal{S})$ and C^N, so are the compositions $\tilde{\mathrm{h}}_1, \ldots, \tilde{\mathrm{h}}_r$. It is obvious, that $B_1, \ldots, B_r$ are open sets, since σ and ϕ are diffeomorphisms and so their inverse function is continuous.

At last, we have to show that the graph of $\tilde{\mathrm{h}}_i$ is closed. For an easier notation examine only one function and write $\tilde{\mathrm{h}} : B \to \mathbb{R}$. Let $\tilde{B} = \{(\sigma(\overline{\mathrm{x}}), y) \mid (\overline{\mathrm{x}}, y) \in B'\}$. Let $\tilde{\phi}$ be the extension of ϕ from $\tilde{B}$ to B, defined by $\tilde{\phi}(\overline{\mathrm{x}}, y) = (\phi(\overline{\mathrm{x}}), y)$. Now we use Remark 1.3.8 and let $\overline{\mathrm{z}}_n = (\overline{\mathrm{x}}_n, y) \in B$ such that $\overline{\mathrm{z}}_n \to \overline{\mathrm{z}} = (\overline{\mathrm{x}}, y) \in \partial B$. Write $\overline{\mathrm{x}}_n = (\overline{\mathrm{v}}_n, \overline{\mathrm{w}}_n) \in \mathbb{R}^{j_0} \times \mathbb{R}^{n-j_0}$ and the same for $\overline{\mathrm{x}} = (\overline{\mathrm{v}}, \overline{\mathrm{w}})$. Then by the definitions of ϕ and $\tilde{\phi}$ we have $\tilde{\phi}^{-1}(\overline{\mathrm{v}}_n, \overline{\mathrm{w}}_n, y_n) = (\overline{\mathrm{v}}_n, \overline{\mathrm{w}}'_n, y_n)$, where $\overline{\mathrm{w}}'_n \in (0,1)^{n-j_0}$. So the sequence $\overline{\mathrm{w}}'_n$ is bounded, hence there exists a convergent subsequence. Thus we can assume that $\overline{\mathrm{w}}'_n \to \overline{\mathrm{w}}'$. We obtain $\overline{\mathrm{z}}'_n := (\overline{\mathrm{v}}_n, \overline{\mathrm{w}}'_n, y_n) \to (\overline{\mathrm{v}}, \overline{\mathrm{w}}', y) =: \overline{\mathrm{z}}'$. Now $\overline{\mathrm{z}}' \in \partial\tilde{B}$: Since $\overline{\mathrm{z}}'_n \in \tilde{B}$ has the limit $\overline{\mathrm{z}} \in \overline{\tilde{B}}$ and if $\overline{\mathrm{z}}' \in \tilde{B}$ we would obtain that $\tilde{\phi}(\overline{\mathrm{z}}')$ is defined and $\tilde{\phi}(\overline{\mathrm{z}}') = (\overline{\mathrm{v}}, \overline{\mathrm{w}}, y) = \overline{\mathrm{z}} \in B$. This is a contradiction to the assumption that $\overline{\mathrm{z}} \in \partial B$ and B is open. As a permutation, σ is defined on whole $\mathbb{R}^n$ and preserves convergence. So we obtain that $(\theta^{-1}(\overline{\mathrm{x}}_n), y_n) = (\sigma^{-1}(\overline{\mathrm{v}}_n, \overline{\mathrm{w}}'_n), y_n) \to (\sigma^{-1}(\overline{\mathrm{v}}, \overline{\mathrm{w}}'), y) =: \overline{\mathrm{z}}''$. The same argument as above justifies $\overline{\mathrm{z}}'' \in \partial B'$. Because h has a closed graph by assumption, the limit of $|h(\theta^{-1}(\overline{\mathrm{x}}_n), y_n)|$ is ∞. However, $\tilde{\mathrm{h}}(\overline{\mathrm{x}}_n, y_n)$ is defined by $h(\theta^{-1}(\overline{\mathrm{x}}_n), y_n)$ and so $\tilde{\mathrm{h}}$ has a closed graph and the proof is finished. □

Not all cells in the cell decomposition are open, so we have to examine also cells with lower dimension. For this case we will use the following lemma of Speisegger. Recall that we assumed $S(\omega) = \emptyset$.

Lemma 5.4.7. *[Lemma (1.4) in [Spe99]] Let $U \subseteq \mathbb{R}^n$ be open and let ω be a 1-form of class C^1 on U. Let L be an integral manifold of $\omega = 0$ with $dim(L) = n-1$. Let $D \subseteq U$ be a connected manifold of dimension at most $n-1$ such that $T_{\overline{\mathrm{x}}}D \subseteq ker(\omega(x))$ for all $\overline{\mathrm{x}} \in D$ and $D \cap L$ is closed in D. Then either $D \cap L = \emptyset$ or $D \subseteq L$.*

Proof. This differs slightly from the version of Lemma (1.4) in [Spe99], but a quick look at the beginning of the proof shows that the condition that there exists $K \subseteq \mathbb{R}^n$ closed with $L = K \cap U$ is only used to proof $D \cap L$ is closed in D, so we can replace it in the lemma.

□

The following lemma verifies the DPC^N condition for Rolle leaves intersected with closed cells. In the proof we distinguish the two cases, where the Rolle leaf is tangent to the cell and where it is transverse to the cell. In the first case we can apply Speisegger's Lemma, in the second case we reduce to Rolle leaves on open cells of lower dimension.

Lemma 5.4.8. *Let C be a cell with dimension $k < n$; then there exist finitely many functions $f_i : B_i \to \mathbb{R}$ in $M^N(\text{Rolle}(\mathcal{S}))$ $(i = 1, \dots, r)$ satisfying DPC^N for $L \cap C$ with $m = n + 2$, i.e. $\bigcup_{i=1}^r \pi_n^{n+2}[Z(f_i)] = L \cap C$ and $B_i \subseteq \mathbb{R}^{n+2}$.*

Proof. The proof works by induction on k. If $k = 0$, C is only a point, let $C = \{\overline{a}\} = \{(a_1, \dots, a_n)\}$. Hence $L \cap C = \emptyset$, which is a trivial case, or $L \cap C = \{\overline{a}\}$. This is the zero set of the function $f(x_1, \dots, x_n) = (x_1 - a_1)^2 + \cdots + (x_n - a_n)^2$, where $f : \mathbb{R}^n \to \mathbb{R}$. The function f is obviously C^N and in Rolle($\mathcal{S}$) and has as totally defined function a closed graph.

Now let C be a cell with dimension k less than n.

Claim 1: The tangential space TC is in $\mathcal{S}$.

Proof: Define

$$\begin{aligned} NC &:= \{(\overline{x}, \overline{v}) \mid \overline{x} \in C \ \wedge \ \langle \overline{x}, \overline{v} \rangle = 0\} \\ &= \{(\overline{x}, \overline{v}) \mid \overline{x} \in C \ \wedge \ \forall \varepsilon > 0 \ \exists \delta > 0 \ (\forall \overline{y} \in C |\overline{x} - \overline{y}| < \delta \Rightarrow \langle \overline{y}, \overline{v} \rangle < \varepsilon)\}. \end{aligned}$$

If we can prove, that NC is in $\mathcal{S}$, we notice that $T_{\overline{x}}C = \{\overline{u} \mid \exists \overline{v}(\overline{x}, \overline{v}) \in NC \wedge \langle \overline{v}, \overline{u} \rangle = 0\}$ (which is obviously in $\mathcal{S}$ if NC is). Since the scalar product is a polynomial and we can take intersections and projections this is a set in $\mathcal{S}$. To show $NC \in \mathcal{S}$, define

$$\begin{aligned} H &:= \left\{(\overline{x}, \overline{v}, \varepsilon) \mid \overline{x} \in C \wedge \exists \delta > 0 (\forall \overline{y} \in C |\overline{x} - \overline{y}| < \delta \Rightarrow \langle \overline{y}, \overline{v} \rangle < \varepsilon)\right\} \\ &= \pi \left[\{(\overline{x}, \overline{v}, \varepsilon, \delta) \mid \{\overline{y} \in C \mid |\overline{x} - \overline{y}| < \delta\} \subseteq \{\overline{y} \in C \mid \langle \overline{y}, \overline{v} \rangle < \varepsilon\} \right] \end{aligned}$$

Since $A \subseteq B \Leftrightarrow A \cap B^C = \emptyset$, the set H is in $\mathcal{S}$. By Lemma (3.4) in [vdD98] also the topological closure $\overline{H}$ is in $\mathcal{S}$. Now

$$\begin{aligned} NC &= \{(\overline{x}, \overline{v}) \mid (\overline{x}, \overline{v}, 0) \in \overline{H}\} \\ &= \{(\overline{x}, \overline{v}) \mid \exists z((\overline{x}, \overline{v}, z) \in \overline{H} \wedge z = 0)\} \in \mathcal{S} \end{aligned}$$

$\square$(Claim)

Define

$$X := \{\overline{x} \in C \mid T_{\overline{x}}C \subseteq ker(\omega(\overline{x}))\}.$$

By the claim and since $\omega \in \mathcal{S}$ we obtain $X \in \mathcal{S}$. Examine not the two sets X and $C - X$.

Claim 2: There exists a C^N function $f_X : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\pi[Z(f_X)] = X \cap L$.

Proof: Take a cell decomposition compatible with X. Let $E \subseteq C$ be a cell of this cell decomposition. If $E \cap X = \emptyset$, especially $E \cap L = \emptyset$. Now examine the case $E \subseteq X$. The set E is a manifold of dimension less than n, the integral manifold $L \cap C$ of $\omega = 0$ has dimension $n-1$ and since L is closed, $L \cap E$ is closed in E, so for all $\overline{x} \in E$ we have $T_{\overline{x}}E \subseteq T_{\overline{x}}C \subseteq \ker(\omega(x))$. Thus we can apply Lemma 5.4.7 and obtain that $E \cap L = \emptyset$ or $E \subseteq L$. Now we can write $X \cap L = \bigcup\{E \subseteq X \mid E \text{ cell } \wedge E \subseteq L\}$. So $X \cap L$ is a finite union of cells which are in $\mathcal{S}$, so it is in $\mathcal{S}$ with (WS1). As $\mathcal{S}$ satisfies DC^N for all N, by Lemma 4.4 there exists actually a C^N function $f : \mathbb{R}^n \to \mathbb{R}$ such that $X \cap L = Z(f)$. Define $f_X : \mathbb{R}^{n+2} \to \mathbb{R}$ by $f_X(\overline{x}, y) = f(\overline{x})$. Then $\pi_n^{n+2}[Z(f_X)] = X \cap L$. □(Claim)

So examine $C - X$. Note that $X = X(C)$ depends on the cell C. Do another cell decomposition which divides $C - X$ into cells. Then for all cells $C' \subseteq C - X$ with the same dimension as C for all $\overline{x} \in C'$ holds $T_{\overline{x}}C' = T_{\overline{x}}C$ since C' is open in C, i.e. $X(C) \cap C' = X(C')$. If C' is a cell with lower dimension, the tangent space can be of smaller dimension than C is and so for some $\overline{x} \in C'$ it may be that $T_{\overline{x}}C' \subseteq \ker(\omega(\overline{x}))$. For these cells with lower dimension than k we get some functions satisfying the DPC^N condition for $C' \cap L$ with $m = n+2$ by the induction hypothesis. Hence, it is enough to examine the cells C' with dimension k, where for all $\overline{x} \in C'$ we have $T_{\overline{x}}C' \nsubseteq \ker(\omega(\overline{x}))$. Write C for such a cell, for an easier notation.

By Lemma (2.7) in [vdD98], Chapter 3, each cell is homeomorphic to an open cell under a coordinate projection $\sigma : C \to C'$, while C' is an open cell in $\mathbb{R}^k$. This coordinate projection is C^N, since C is a C^N cell. By the inverse function theorem σ is a C^N diffeomorphism. Particularly, C is a graph of a C^N function, whose domain is open.

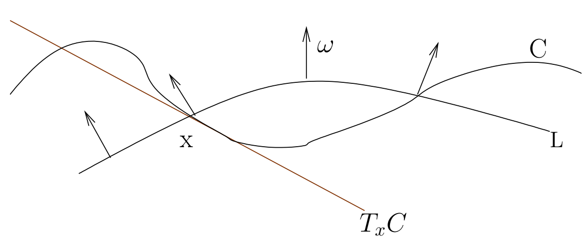

Figure 5.7: Not all $\overline{x} \in L \cap C$ must be in X.

For every $\overline{x} \in C$ we have $T_{\overline{x}}C + N_{\overline{x}}C = \mathbb{R}^n$ and $T_{\overline{x}}C \perp N_{\overline{x}}C$. By assumption $\omega(\overline{x}) = \sum_{i=1}^n a_i(\overline{x})dx_i$ with $a_i : C \to \mathbb{R}$ is a C^N function. Hence we can write $\overline{a}(\overline{x}) = (a_1(\overline{x}), \ldots, a_n(\overline{x}) \in \mathbb{R}^n$ and so let $\overline{a}(\overline{x}) = \overline{t}(\overline{x}) + \overline{n}(\overline{x})$ where $\overline{t}(\overline{x}) \in T_{\overline{x}}C$ and $\overline{n}(\overline{x}) \in N_{\overline{x}}C$ such that $\overline{t}(\overline{x}) \perp \overline{n}(\overline{x})$. These functions $\overline{t}$ and $\overline{n}$ are C^N as $\overline{a}$ is C^N and C is a C^N cell, which implies $\overline{x} \mapsto T_{\overline{x}}$ is C^N and as $\overline{t}$ and $\overline{n}$ are constructed by a linear separation into direction components. Define $\tau(\overline{x}) := \sum_{i=1}^n t_i(\overline{x})dx_i$.Then $\tau(\overline{x})$ is a 1-form and C^N, since it depends linearly on $\omega(\overline{x})$.

Define $\tilde{\omega} = (\sigma^{-1})^*\tau$.

Claim 3: The set $\sigma(L \cap C)$ is a Rolle leaf on $(C', \tilde{\omega})$.

Proof: **1. The set $\sigma(L \cap C)$ is a C^N manifold.**

Look at $T_{\overline{x}}L + T_{\overline{x}}C = \ker(\omega(\overline{x})) + T_{\overline{x}}C = \mathbb{R}^n = T_{\overline{x}}\mathbb{R}^n$, since we assumed that $T_{\overline{x}}C \nsubseteq \ker(\omega(\overline{x}))$ and $\dim(\ker(\omega(\overline{x}))) = n - 1$. So L and C are in general position, i.e. the inclusion map $\iota : C \to \mathbb{R}^n$ is transverse to L.

Since L is a submanifold of $\mathbb{R}^n$ and $\iota^{-1}(L) = L \cap C$, we obtain by Theorem 5.2.3 that $L \cap C$ is a submanifold of C with co-dimension 1, so of dimension $k - 1$. Furthermore σ is a diffeomorphism, thus $\sigma(L \cap C)$ is still a manifold of dimension $k - 1$, particularly a submanifold of $\mathbb{R}^k$.

As ω is C^N on C, it is also C^N on $L \cap C$, so $L \cap C = \ker(\omega) \cap C$ is a C^N manifold.

2. The manifold $\sigma(L \cap C)$ is a leaf on $(C', \tilde{\omega})$.

First we prove the following statement:

$$T_{\overline{x}}(L \cap C) = \ker(\tau(\overline{x}))$$

Let $\overline{v} \in T_{\overline{x}}(L \cap C)$, that is $\overline{v} = c'(0)$, where $c : [-\varepsilon, \varepsilon] \to L \cap C$ is a C^1 curve and $c(0) = \overline{y}$. On the one hand, $\overline{v} \in T_{\overline{x}}C$ and so $n(\overline{x})(\overline{v}) = 0$, i.e. $\omega(\overline{x})(\overline{v}) = \tau(\overline{x})(\overline{v})$. On the other hand, $\overline{v} \in T_{\overline{x}}L = \ker(\omega(\overline{x}))$. Hence $0 = \omega(\overline{x})(\overline{v}) = \tau(\overline{x})(\overline{v})$.

Similar to the proof of Lemma 5.2.4 we obtain

$$T_{\overline{y}}\sigma(L \cap C) = \ker\left(((\sigma^{-1})^*\tau)(\overline{x})\right) = \ker(\tilde{\omega}(\overline{x})).$$

3. The leaf $\sigma(L \cap C)$ is a Rolle leaf on $(C', \tilde{\omega})$

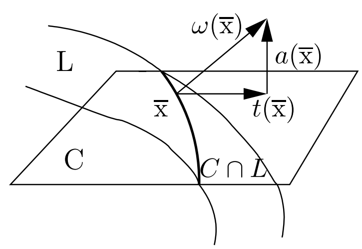

Figure 5.8: Splitting $\omega(\overline{x})$ into one component on C and one orthographic to it

First notice that σ is a diffeomorphism, so $L \cap C$ is closed in C and hence $\sigma(L \cap C)$ is closed in C'.

Now only the Rolle leaf condition is left to check. So let $\gamma : [0,1] \to C'$ be a graph such that $\gamma(0), \gamma(1) \in \sigma(L \cap C)$. Look at the graph $\delta := \sigma^{-1} \circ \gamma : [0,1] \to C$ with $\delta(0), \delta(1) \in L \cap C$, particularly in L. Since $C \subseteq U$, while U is the set where the Rolle leaf L is defined on, we obtain that there exists a $t \in [0,1]$ such that $\omega(\delta(t))(\delta'(t)) = 0$. Since $\Gamma(\delta) \in C$, we have that $\delta'(t) \in T_{\delta(t)}C$, hence $n(\delta(t))(\delta'(t)) = 0$. So $0 = \omega(\delta(t))(\delta'(t)) = \tau(\delta(t))(\delta'(t)) = \tau(\sigma^{-1} \circ \gamma(0))((\sigma^{-1} \circ \gamma)'(0))$. Similar to the proof of Lemma 5.2.4 we obtain that $\tilde{\omega}(\gamma(t))(\gamma'(t)) = 0$. □(Claim)

Next, we apply Lemma 5.4.3 for a Rolle leaf on an open cell on the Rolle leaf $\sigma(L \cap C)$, thus there exists functions $f_i \in M^N(\mathrm{Rolle}(\mathcal{S}))$ such that they are defined on $B_i \subseteq \mathbb{R}^{k+1}$ for $i = 1 \dots r$ and $\sigma(L \cap C) = \pi_k^{k+1}[Z(f_1) \cup \dots Z(f_r)]$.

Additionally $C \in \mathcal{S}$, so by Lemma 1.3.9 there exists a closed set $D \in \mathcal{S}_{n+1}$ such that $\pi_n^{n+1}[D] = C$. Since $D \in \mathcal{S}$ and closed, we can apply Lemma 4.4 and obtain a C^N function $g : \mathbb{R}^{n+1} \to \mathbb{R}$ in $\mathcal{S}$ such that $Z(g) = D$ and hence $\pi_n^{n+1}[Z(g)] = C$.

For an easier notation, we assume that σ is the projection on the first k coordinates. Note that

$$L \cap C = (\sigma(L \cap C) \times \mathbb{R}^{n-k}) \cap C.$$

Define for $i = 1, \dots, r$

$$B_i' := \{(\overline{\mathrm{x}}, \overline{\mathrm{y}}, z, z') \mid (\overline{\mathrm{x}}, z) \in B_i \ \wedge\ \overline{\mathrm{y}} \in \mathbb{R}^{n-k} \ \wedge z' \in \mathbb{R}\} \subseteq \mathbb{R}^{n+2}$$

and $g_i : B_i' \to \mathbb{R}$ by

$$(\overline{\mathrm{x}}, \overline{\mathrm{y}}, z) \mapsto (f_i(\overline{\mathrm{x}}, z))^2 + (g(\overline{\mathrm{x}}, \overline{\mathrm{y}}, z'))^2.$$

Then by the Claims 1 and 2 in the proof of Lemma 5.4.1 the functions $g_i \in M^N(\mathrm{Rolle}(\mathcal{S}))$ for $i = 1, \dots, r$. Thus look at

$$\begin{aligned} & (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in \pi_n^{n+2}[Z(g_1) \cup \dots \cup Z(g_r)] \\ \Leftrightarrow\ & \exists i \exists z, z' \ (\overline{\mathrm{x}}, \overline{\mathrm{y}}, z, z') \in B_i' \ \wedge\ f_i(\overline{\mathrm{x}}, z) = 0 \ \wedge g(\overline{\mathrm{x}}, \overline{\mathrm{y}}, z') = 0 \\ \Leftrightarrow\ & \overline{\mathrm{x}} \in \pi_k^{k+1}[Z(f_1) \cup \dots \cup Z(f_r)] \ \wedge\ (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in \pi_n^{n+1}[Z(g)] \\ \Leftrightarrow\ & (\overline{\mathrm{x}}, \overline{\mathrm{y}}) \in (\sigma(L \cap C) \times \mathbb{R}^{n-k}) \cap C = L \cap C \end{aligned}$$

So the functions $g_1, \dots, g_r$ satisfy the DPC^N condition for $L \cap C$ with $m = n+2$. □

At last, we only have to collect all functions we defined for the different cells intersected with L and then we have a projection of a finite union of zero sets of $M^N(\mathrm{Rolle}(\mathcal{S}))$-functions satisfying the DPC^N condition for the Rolle leaf L. This finishes the proof of the lemma. □

This completes the proof of Theorem 5.1.4, so we have shown that the o-minimal weak structure Rolle($\mathcal{S}$) satisfies DC^N for all N and by the theorem of the complement it is an o-minimal structure.

6 Results and Perspective

This thesis dealt with o-minimal structures. We improved Wilkie's theorem of the complement and thus provided a method to prove that certain interesting constructions are in fact o-minimal structures. In Wilkie's theorem, a central requirement is that the function approximating a set in an o-minimal weak structure must be smooth. Karpinski and Macintyre's vary this claim by assuming the existence of total C^N functions. We have shown that this condition can be weakened - there only have to be finitely many functions which are not defined in the whole space, but only on some open set and which have a closed graph.

This generalization allows us to apply the theorem to the Pfaffian closure of an o-minimal structure. Particularly we can use it for the Pfaffian closure of arbitrary expansions of the real field. For example, we could show that $\mathbb{R}_{an,\exp}$ is still an o-minimal structure. However, we can also expand an o-minimal structure in $\overline{\mathbb{R}}$ by arbitrary Pfaffian functions.

An interesting question to the theorem of the complement is whether the assumptions on the functions in the DPC^N condition can be further weakened, for example it will be enough to examine continuous functions, as Wilkie suggests in [Wil99].

Another question with respect to the expansion of o-minimal structures on $\mathbb{R}$ may be, if it is possible to intersect with other manifold then these, which are Rolle leaves.

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