The 12th Delfino Problem and universally Baire sets of reals

Philipp Doebler doebler@uni-muenster.de

Abstract

We prove a theorem of Steel that solves the 12th Delfino Problem. We build on the one hand on a lemma of Woodin on universally Baire sets and their projections in certain generic extensions in the presence of strong cardinals; on the other hand we use certain premice to find projective uniformizations of projective sets.

1 Introduction

The following is a translation of the last two chapters of the author's Diplomarbeit (Master's Thesis). Since this Diplomarbeit was mostly concerned with universally Baire sets of reals, the first part of the following deals with universally Baire sets in the presence of strong cardinals. The second part discusses the 12th Delfino Problem. For the reader's convenience we have included some definitions and some results. For details on universally Baire sets we recommend the original paper by Feng, Magidor and Woodin [FMW92].
The author would like to thank Ralf Schindler for the excellent advice provided during the work on the Diplomarbeit mentioned above. The Diplomarbeit is based on handwritten notes by Ralf Schindler. None of the material in this paper is due to the author.

We will give a definition of (λ-)universally Baire now. Note that this is not the original definition given by Feng, Magidor and Woodin but rather an equivalent one; the equivalence was proved in [FMW92]. Versions of all the results in the introduction can also be found in [FMW92].

Definition 1.1 Let $A \subseteq \omega^\omega$ and λ an infinite cardinal. We will call A λ-universally Baire if and only if there are trees T and T^* such that for every notion of forcing $\mathbb{P}$ of cardinality equal or less than λ the following three conditions hold:

1. $A = p[T]$, $\omega^\omega - A = p[T^*]$,
2. $\mathbb{P} \Vdash p[\check{T}] \cup p[\check{T}^*] = \omega^\omega$,

3. $\mathbb{P} \Vdash p[\check{T}] \cap p[\check{T^*}] = \emptyset$.

We will call A universally Baire, if it is λ-universally Baire for all infinite cardinals λ. We remark that the first condition implies the third by absoluteness of wellfoundedness; since the tree

$$S := \{(s, f, g) \in \omega^{<\omega} \times (\lambda^{<\omega})^2\,;\ (s, f) \in T \wedge (s, g) \in T^*\}.$$

searching for a branch through T and T^* is wellfounded (in V), it is wellfounded in every other model containing T and T^*.

Note that the original definition easily implies that every ω-universally Baire set has the property of Baire. The way we have formulated the definition makes this a nontrivial fact.
We can verify that a given set of reals is universally Baire by verifying the definition for the adequate Levy collapse:

Theorem 1.2 (Feng, Magidor, Woodin) *Let $A \subseteq \omega^\omega$ and λ and infinite cardinal. The following are equivalent:*

1. *There are two trees T and T^* such that:*
   (a) $A = p[T]$, $\omega^\omega - A = p[T^*]$,
   (b) $\mathrm{Col}(\omega, \lambda) \Vdash p[\check{T}] \cup p[\check{T^*}] = \omega^\omega$,
   (c) $\mathrm{Col}(\omega, \lambda) \Vdash p[\check{T}] \cap p[\check{T^*}] = \emptyset$.
2. *A is λ-universally Baire.* □

Remark 1.3 *The trees T and T^* in the preceding theorem can be chosen to be trees on $\omega \times 2^\lambda$.* □

What pointclasses are universally Baire? This question depends on large cardinal assumptions. In ZFC alone we know the following:

Proposition 1.4 *The analytic sets (and hence the coanalytic sets) are universally Baire. Without loss of generality the tree for a (lightface) Π^1_1 set is constructible, i.e. it is in L and is the appropriate Shoenfield tree.* □

This result is optimal in the following sense:

Proposition 1.5 *Every ω-universally Baire set has the Baire property and every 2^ω-universally Baire set is Lebesgue measurable. Thus in L there exists a Δ^1_2 set that is not universally Baire.* □

The presence of certain large cardinals allows us to construct trees witnessing that every member of certain pointclasses of sets of reals is (λ-)universally Baire. One example of a result of this type is the following theorem.

Theorem 1.6 (Feng, Magidor, Woodin) *The following are equivalent:*

1. *Every* $\mathbf{\Sigma}^1_2$ *set is universally Baire.*
2. *For every set* x $x^\sharp$ *exists.*[1] $\square$

Another type of axiom that influences which pointclasses are universally Baire are axioms of the type "V is Γ-absolute in relation to some class of set generic extensions of V" where Γ is a suitable subclass of the projective sets. We give an example of this type of result.

Theorem 1.7 (Feng, Magidor, Woodin) *Let* $\lambda \geq \omega$ *be an infinite cardinal. The following are equivalent:*

1. *If* $\varphi(x_1, \ldots, x_n)$ *is a* Σ^1_3 *formula with free variables ranging over* $x_1, \ldots, x_n$ *and* $a_1, \ldots, a_n \in \omega^\omega$ *then the following holds true:*

$$\varphi(a_1, \ldots, a_n) \iff \mathrm{Col}(\omega, \lambda) \Vdash \varphi(\check{a_1}, \ldots, \check{a_n}).$$

2. *Every* $\mathbf{\Delta}^1_2$ *set is* λ*-universally Baire.* $\square$

2 Strong cardinals an universally Baire sets

In this chapter we will discuss a tree construction in the presence of a strong cardinal due to Woodin. This construction yields that projections of universally Baire sets are again universally Baire in a certain generic extension. To be more precise: Let $A \subset (\omega^\omega)^2$ be universally Baire. We will construct a generic extension that contains trees witnessing that $\exists^{\mathbb{R}} A := \{x\,;\, \exists y (x, y) \in A\}$ is universally Baire in this generic extension.

2.1 Strong cardinals

Definition 2.1 Let $\lambda \geq \kappa$. A cardinal κ is λ-strong if and only if for all $X \in H_{\lambda^+}$ there is a transitive $M \subset V$ and an embedding $\pi\colon V \to M$ such that $X \in M$ and $\kappa = \mathrm{crit}(\pi)$ and $\pi(\kappa) > \lambda$. A cardinal κ is strong, if it is λ-strong for all λ.

[1] Since we are just quoting this result to give an example of a large cardinal assumption that influences which pointclasses are universally Baire, we will not go into the details about sharps.

Definition 2.2 Let $M \subset V$ be transitive. Let $\pi\colon V \to M$ be a nontrivial elementary embedding; we denote its critical point by $\mathrm{crit}(\pi) = \kappa$. Let $\lambda > \kappa$. The (κ, λ)-extender E derived from π is defined as follows:

- $E = \{E_a\,;\, a \text{ is a finite partial function } \omega \to \lambda\}$,
- $E_a \subseteq \mathcal{P}(\kappa^{\mathrm{dom}(a)})$,
- $X \in E_a \iff a \in \pi(X)$.

Every E_a is hence a κ-complete ultrafilter on $\mathcal{P}(\kappa^{\mathrm{dom}(a)})$. Let b be a finite partial function $\omega \to \lambda$ and $a \subseteq b$. Let $X \in \mathcal{P}(\kappa^{\mathrm{dom}(a)})$. We define

$$X^{ab} = \{x \in \kappa^{\mathrm{dom}(b)}\,;\, x \restriction \mathrm{dom}(a) \in X\}.$$

Then

$$X \in E_a \iff X^{ab} \in E_b.$$

The existence of an embedding π as in definition 2.1 can be traced back to the existence of an adequate extender, hence to the existence of a set (see [Jec03, 20.30]). We could thus have avoided classes in the definition of (λ-)strong cardinals, but the above definition is more natural.

2.2 Strong cardinals and tree constructions

Lemma 2.3 (Woodin) *Let κ be λ-strong and $2^{2^\kappa} < \lambda$. Let $A \subseteq (\omega^\omega)^2$ be λ-universally Baire. Let T and T^* denote the trees, that witness that $A = p[T]$ is λ-universally Baire. Let H be $\mathrm{Col}(\omega, 2^{2^\kappa})$-generic over V. Then in $V[H]$ the following holds: There are trees U and U^*, that witness the λ-universal Baireness of $\exists^{\mathbb{R}} A = p[U]$.*

Proof. By theorem 1.2 it suffices to discuss the forcing $\mathrm{Col}(\omega, \lambda)$. We set $\mathbb{P} = \mathrm{Col}(\omega, \lambda)$. We need to show, that there are trees U and U^* in $V[H]$, such that for every G $\mathbb{P}$-generic over $V[H]$

$$V[H][G] \models p[U] = \omega^\omega - p[U^*].$$

By remark 1.3 the trees T and T^* are, without loss of generality, trees on $\omega \times \omega \times 2^\lambda$. Clearly $\mathbb{P} \in H_{\lambda^+}$. Let $\pi\colon V \to M$ be an embedding with critical point κ such, that $\mathcal{P}(Col(\omega, 2^{2^\kappa})) \in M$. Note that H is $\mathrm{Col}(\omega, 2^{2^\kappa})$-generic over M as well. By [Kan03, 13.13] there is a tree U such that $p[U] = \exists^{\mathbb{R}} p[T]$. The tree U can be chosen to be a tree on $\omega \times 2^\lambda$.

Claim 1. If K is $\mathbb{P}$-generic over $V[H]$, then

$$V[H][K] \models p[U] = p[\pi(U)].$$

Proof of Claim 1. Let $x \in V[H][K] \cap \omega^\omega$. We discuss the case $x \in p[U]$, say $(x, y) \in p[T]$. Then there is $f\colon \omega \to 2^\lambda$ such that

$$\forall n (x \upharpoonright n, y \upharpoonright n, f \upharpoonright n) \in T.$$

Hence for arbitrary $n \in \omega$

$$(x \upharpoonright n, y \upharpoonright n, \pi(f \upharpoonright n)) \in \pi(T),$$

and so $x \in p[\pi(U)]$ as witnessed by $\pi''\{f \upharpoonright n\,;\, n \in \omega\} : \omega \to \pi(2^\lambda)$.
Now let $x \notin p[U]$. Hence $\forall y (x, y) \notin p[T]$. Since T and T^* are trees of a λ-universally Baire set and $V[H][K]$ is a generic extension of size $\leq \lambda$, we have

$$V[H][K] \models \forall y (x, y) \in p[T^*].$$

An argument as in the above case yields $\forall y (x, y) \in p[\pi(T^*)]$. Since

$$V \models p[T] \cap p[T^*] = \emptyset \text{ in all generic extensions of size} \leq \lambda,$$

the elementarity of π implies

$$M \models p[\pi(T)] \cap p[\pi(T^*)] = \emptyset \text{ in all generic extensions of size} \leq \pi(\lambda).$$

We reformulate this fact: The tree

$$\tilde{T} := \{(s, t, f, g) \in \omega^{<\omega} \times \omega^{<\omega} \times \pi(\lambda)^{<\omega} \times \pi(\lambda)^{<\omega}\,;\, (s, t, f) \in \pi(T) \wedge (s, t, g) \in \pi(T^*)\}$$

is well-founded. $\tilde{T}$ is the tree that searches for a"common branch" through both $\pi(T)$ and $\pi(T^*)$. Wellfoundedness is absolute, so

$$V[H][K] \models p[\pi(T)] \cap p[\pi(T^*)] = \emptyset.$$

In particular $\forall y (x, y) \notin p[\pi(T)]$, hence $x \notin p[\pi(U)]$. $\Box$(Claim 1)

We now have to look for a tree U^* in $V[H]$ such that for every K that is $\mathbb{P}$-generic over $M[H]$

$$M[H][K] \models p[\pi(U)] = \omega^\omega - p[U^*].$$

This"local" construction makes sense, as the following claim shows:

Claim 2. Let K be $\mathbb{P}$-generic over $V[H]$. Then every real in $V[H][K]$ is in a model of the form $M[H][K]$.
Proof of Claim 2. To any $x \in \omega^\omega \cap V[H][K]$ there is a name τ in $V^{\mathrm{Col}(\omega, 2^{2^\kappa}) \times \mathrm{Col}(\omega, \lambda)}$ such that $\tau^{H \times K} = x$. Since $2^{2^\kappa} \leq \lambda$, this name can be chosen to be small, i.e. $\tau \in H_{\lambda^+}$. As κ is λ-strong, there exist M and $\pi\colon V \to M$ such, that $\tau \in M$.

Hence $x \in M[H][K]$. $\square$(Claim 2)

For a fixed π we will construct a tree U^* in $V[H]$. We will then merge these trees.
We fix an embedding π; there is a $(\kappa, \pi(\kappa))$-extender E derived from π. We set $\nu_a := \pi(E_a)$, then ν_a is a measure in M. For $(s,a) \in \omega^n \times \kappa^n$ the following holds:

$$\begin{aligned}(s,a) \in \pi(U) &\iff a \in \pi(U)_s = \pi(U_s)\\ &\iff U_s \cap \kappa^n \in E_a\\ &\iff \pi(U_s \cap \kappa^n) = \pi(U_s) \cap \pi(\kappa)^n \in \nu_a.\end{aligned}$$

In $V[H]$ the set $\mathcal{P}(\mathcal{P}(\{a\,;\, a \text{ is a finite partial function } \omega \to \kappa\}))^V$ is countable, so we can enumerate the sets ν_a in $V[H]$.[2] We choose an enumeration $\langle \sigma_i \,|\, i \in \omega\rangle$ in $V[H]$, such that each ν_a appears infinitely often in the enumeration. We fix a function g such that $g\colon \omega \to \omega;\ i \mapsto \mathrm{dom}(a)$ for an a such that $\nu_a = \sigma_i$. The function g is well defined, since for all a, b such that $\nu_a = \sigma_i = \nu_b$ the measure σ_i "concentrates" on one domain, i.e. $\mathrm{dom}(a) = \mathrm{dom}(b)$ holds. Without loss of generality we can assume $g(i) \subseteq i$ for all $i < \omega$. We say σ_k *projects to* σ_i if and only if there is $a \subsetneq b$ such that $\sigma_i = \nu_a$ and $\sigma_k = \nu_b$ with the property that

$$X \in \nu_a \iff X^{ab} \in \nu_b.$$

Hence for every $i \in \omega$ there is $\pi_i\colon M \to_{\sigma_i} \mathrm{Ult}(M, \sigma_i)$. If σ_k projects to σ_i, there is an embedding $\pi_{ik}\colon \mathrm{Ult}(M, \sigma_i) \to \mathrm{Ult}(M, \sigma_k)$.
We can now (in $V[H]$) define the tree U^* on $\omega \times \pi((2^\lambda)^+)$. We put

$$(s, (\alpha_0, \dots, \alpha_{n-1})) \in U^*$$

if and only if

$$\begin{aligned}&\forall i < k < n (\pi(U_{s\restriction g(i)} \cap \kappa^n) \in \sigma_i \wedge \pi(U_{s\restriction g(k)} \cap \kappa^n) \in \sigma_k \wedge \sigma_k \text{ projects to } \sigma_i\\ &\qquad \to \pi_{ik}(\alpha_i) > \alpha_k).\end{aligned}$$

The tree U^* witnesses locally the universal Baireness of A, as the following claim shows.

Claim 3. If K is $\mathbb{P}$-generic over $M[H]$, then

$$M[H][K] \models p[\pi(U)] = \omega^\omega - p[U^*].$$

Proof of Claim 3. We assume, there was $x \in p[\pi(U)] \cap p[U^*]$ and work towards a contradiction. Say $(x, f) \in [\pi(U)]$ and $(x, \vec{\alpha}) \in [U^*]$ where $\vec{\alpha} = \langle \alpha_i \,|\, i \in \omega\rangle$. For

[2] Note that the following can hold: $\mathrm{dom}(a) = \mathrm{dom}(b)$ and $a \neq b$ but $\nu_a = \nu_b$.

all n the following holds: $\pi(U_{x\restriction n}) \cap \pi(\kappa)^n \in \nu_{f\restriction n}$. If $m < n$ it is evident that $\nu_{f\restriction n}$ projects to $\nu_{f\restriction m}$. We set

$$\tilde{M} := \operatorname{dirlim}_n \operatorname{Ult}(M; \nu_{f\restriction n}).$$

We fix a sequence $(i_n)_{n\in\omega}$ such that $\nu_{f\restriction n} = \sigma_{i_n}$. By ρ_n we denote the canonical embedding from $\operatorname{Ult}(M; \nu_{f\restriction n})$ to $\tilde{M}$. Now

$$\begin{aligned} & \pi_{i_n i_{n+1}}(\alpha_{i_n}) > \alpha_{i_{n+1}} \\ \iff & \rho_{n+1}(\pi_{i_n i_{n+1}}(\alpha_{i_n})) > \rho_{n+1}(\alpha_{i_{n+1}}) \\ \iff & \rho_n(\alpha_{i_n}) > \rho_{n+1}(\alpha_{i_{n+1}}). \end{aligned}$$

Hence the sequence $\langle \rho_n(\alpha_{i_n}) \,|\, n \in \omega \rangle$ witnesses that $\tilde{M}$ is not well-founded. But $\tilde{M}$ can embedded into $\operatorname{Ult}(M; \pi(E))$ by the universal property of the direct limit. This is a contradiction to the wellfoundedness[3] of $\operatorname{Ult}(M; \pi(E))$.

Let now $x \notin p[\pi(U)]$, so the tree $\pi(U)_x$ is well-founded. We have to show $x \in p[U^*]$. For $n < \omega$ and $\vec{\gamma} \in \pi(\kappa)^n$ we define

$$\begin{aligned} f_n(\vec{\gamma}) &= \|\vec{\gamma}\|_{\pi(U)_x} \\ &= \text{ the rank of } \vec{\gamma} \text{ in the tree order of } \pi(U)_x. \end{aligned}$$

Every σ_i is a measure in M. We extend each σ_i to a measure $\tilde{\sigma_i}$ in $M[H][K]$, by setting

$$\tilde{\sigma_i} := \{X \,;\, \exists X' \in \sigma_i \; X' \subseteq X\}^{M[H][K]}.$$

By this procedure we can extend the canonical embeddings $\pi_i \colon M \to \operatorname{Ult}(M; \sigma_i)$ to embeddings $\tilde{\pi_i}$. Analogously we proceed with π_{ik} for any relevant i, k.

We can now put

$$\begin{aligned} \alpha_k &= [f_{g(k)}]_{\tilde{\sigma_k}} \\ &= \text{ the equivalence class of } f_{g(k)} \text{ in } \operatorname{Ult}(M[H][K]; \tilde{\sigma_k}). \end{aligned}$$

Let $\vec{\alpha} = \langle \alpha_i \,|\, i \in \omega \rangle$. We have to show $(x, \vec{\alpha}) \in [U^*]$. Let $\pi(U_{s\restriction g(i)}) \in \sigma_i$, $\pi(U_{s\restriction g(k)}) \in \sigma_k$ and let σ_k project to σ_i. Note that $g(i) \subsetneq g(k)$. Then we have

$$\begin{aligned} \pi_{ik}(\alpha_i) &= \tilde{\pi_{ik}}(\alpha_i) \\ &= \tilde{\pi_{ik}}([\vec{\gamma} \mapsto \|\vec{\gamma}\|_{\pi(U)_x}]_{\tilde{\sigma_i}}) \\ &= [\vec{\varepsilon} \mapsto \|\vec{\varepsilon} \restriction g(i)\|_{\pi(U)_x}]_{\tilde{\sigma_k}} \\ &> [\vec{\varepsilon} \mapsto \|\vec{\varepsilon}\|_{\pi(U)_x}]_{\tilde{\sigma_k}} \\ &= \alpha_k. \end{aligned}$$

$\square$(Claim 3)

[3] See remarks after [Jec03, 20.28].

We are now ready to merge the trees constructed so far. Let K be $\mathbb{P}$-generic over $V[H]$. By Claims 2 and 3 there is for every $y = \tau^{H\times K} \in V[H][K] \cap \omega^\omega$ an embedding $\pi\colon V \to M$ and a tree $U^*_{\tau,\pi}$ that witnesses in the following sense the λ-universal Baireness:

- $y \in M[H][K]$
- $M[H][K] \models p[\pi(U)] = \omega^\omega - p[U^*_{\tau,\pi}]$.

We now define disjoint trees $U^+_{\tau,\pi}$ by replacing each $(s, f^*) \in U^*_{\tau,\pi}$ by (s, f^+) where

$$\mathrm{dom}(f^+) = \mathrm{dom}(f^*) \wedge [\forall n \in \mathrm{dom}(f^*)\ (f^*(n) = \alpha \iff f^+(n) = (\alpha, \tau))].$$

In $V[H]$ we put

$$U^* = \bigcup\{U^+_{\tau,\pi}\,;\ \tau^H \text{ is a } \mathrm{Col}(\omega, 2^{2^\kappa})\times\mathrm{Col}(\omega,\lambda)\text{-nice name for a real and } \tau \in H_{\lambda^+}\}.$$

Clearly

$$p[U^*] = \bigcup_{\tau,\pi} p[U^+_{\tau,\pi}] = \bigcup_{\tau,\pi} p[U^*_{\tau,\pi}].$$

Let K be $\mathbb{P}$-generic over $V[H]$ and $x \in \omega^\omega \cap V[H][K]$. We discuss the case $x \in p[U]$, then by Claim 3 x is in none of the $p[U^*_{\tau,\pi}]$, hence x is in none of the $p[U^+_{\tau,\pi}]$. So x is not in $p[U^*]$ either.
Let now $x \notin p[U]$. By claims 2 and 3 there is $U^*_{\tau,\pi}$ such that $x \in p[U^*_{\tau,\pi}]$. So $x \in p[U^*]$. Hence

$$V[H]^{\mathbb{P}} \models p[U] = \omega^\omega - p[U^*].$$

□

3 The 12th Delfino Problem

Lemma 2.3 allows us to analyse the consistency strength of the following statement: "Every projective set is Lebesgue measurable, has the Baire property and has a projective uniformization". More precisely we will give an (optimal) upper bound for the consistency strength. By a projective uniformization of a set $A \subseteq (\omega^\omega)^2$ we understand a projective set F such that

$$F \subseteq A \wedge \forall x(\exists y((x,y) \in A) \iff \exists! y((x,y) \in F))).$$

For the sake of brevity we define:

Definition 3.1 $\triangle :\equiv$ "Every projective set is Lebesgue measurable, has the Baire property and every projective set in $(\omega^\omega)^2$ has a projective uniformization".

Now we define the #12 hypothesis, a large cardinal axiom. The name #12 hypothesis stems from the 12th Delfino Problem. We will explicate the 12th Delfino Problem below.

Definition 3.2 We say that the #12 hypothesis holds if and only if there are cardinals κ_i, $i \in \omega$, $\kappa_i < \kappa_{i+1}$ and $\lim \kappa_i = \lambda$ such that

$$\forall i \in \omega \ \kappa_i \text{ is } \lambda\text{-strong}.$$

The #12 hypothesis is an upper bound for $\triangle$, i.e.:

Theorem 3.3 (Steel, 1997) *If there is a model of "*ZFC*+ #12 hypothesis", there is a model of "*$\mathsf{ZFC}+\triangle$*".*

In the course of this chapter we will prove theorem 3.3. This theorem is closely related to the 12th Delfino Problem. The 12th Delfino Problem was formulated by Woodin in [Woo82] and is the following question:

Does $\triangle$ imply projective determinacy?

Another way of formulating this would be: If one considers $\triangle$ as the natural consequences[4] of projective determinacy, do these natural consequences yield projective determinacy? Compare the list of Delfino Problems in [KMS88]. Steels answer to this question is hence no, since the #12 hypothesis far weaker as projective determinacy, for projective determinacy implies (by a result of Woodin) the existence of a model with one Woodin cardinal. But below one Woodin cardinal δ there are inaccessibly many δ-strong cardinals, so in particular countably many. If the #12 hypothesis would yield a model of projective determinacy, we had arrived at contradiction to Gödel's incompleteness theorem. Later we will prove more directly building on a result of Martin that projective determinacy is not implied by the #12 hypothesis.

Proposition 3.4 *Let $\kappa_0 < \kappa_1 < \cdots$ with limit λ be witnesses to the #12 hypothesis. Let G be* $\mathrm{Col}(\omega, \lambda)$*-generic over V. In $V[G]$ every projective set is Lebesgue measurable and has the Baire property.*

Proof. We fix a sequence $\langle G_i \,|\, i \in \omega \rangle$ such that

- G_0 is $\mathrm{Col}(\omega, 2^{2^{\kappa_0}})$-generic over V,
- G_{i+1} is $\mathrm{Col}(\omega, 2^{2^{\kappa_{i+1}}})$-generic over $V[G_0, \ldots, G_i]$.

[4]We see $\triangle$ as the natural consequences of projective determinacy since projective determinacy decides certain regularity properties of the projective sets, in particular $\triangle$ is a consequence of projective determinacy. See [Kec95]

Note that all G_i can be chosen in $V[G]$, since in $V[G]$ there are only countably many dense subsets of $\text{Col}(\omega, 2^{2^{\kappa_i}})$ in $V[G_0, \dots, G_{i-1}]$, because λ is a strong limit. Inductively we construct for $i \geq 1$ trees S_i, T_i with the following properties

- $p[T_i]$ is a universal $\mathbf{\Pi^1_i}$ set in every generic extension of size equal or less than λ,
- $S_i, T_i \in V[G_0, \dots, G_{i-2}]$ for $i \geq 2$, else $S_i, T_i \in V$,
- S_i, T_i witness that $A = p[T_i]$ is λ-universally Baire in $V[G_0, \dots, G_{i-2}]$ for $i \geq 2$ and in V respectively.

Let $A \subseteq (\omega^\omega)^m$ be a universal $\mathbf{\Pi^1_1}$ set. By proposition 1.4 every coanalytic set is universally Baire, where the γ-universal Baireness is witnessed by the Shoenfield tree on $\omega^m \times 2^\gamma$ for each γ. The Shoenfield tree T_1 for A on $\omega^m \times 2^\lambda$ can even be chosen to be in L. By proposition 1.4 there is a "complementary" tree S_1.
Let $A = p[T_n] \subseteq (\omega^\omega)^{m+2}$ be a universal $\mathbf{\Pi^1_n}$ set and $n \geq 1$. We construct T_{n+1} and S_{n+1} now. We set

$$\bar{B} = \{(\vec{x}, z)\,;\, \exists y (\vec{x}, y, z) \in A\}.$$

By lemma 2.3 there are trees T_{n+1} and S_{n+1} such that $\bar{B} = p[S_{n+1}]$ and $(\omega^\omega)^{m+1} - p[S_{n+1}] = p[T_{n+1}]$ in every size $\leq \lambda$ generic extension of $V[G_0, \dots, G_{n-1}]$. We have to check that $p[T_{n+1}]$ is a universal $\mathbf{\Pi^1_{n+1}}$ set in every generic extension of size $\leq \lambda$. We first discuss an easier case: let $\bar{C} \subseteq (\omega^\omega)^m$ be a $\mathbf{\Pi^1_{n+1}}$ set in $V[G_0, \dots, G_{n-1}]$. Then

$$(\omega^\omega)^m - \bar{C} = \exists^{\mathbb{R}} C$$

for a $\mathbf{\Pi^1_n}$ set C. Then there is a real z such that

$$\begin{aligned}(\omega^\omega)^m - \bar{C} &= \{\vec{x}\,;\, \exists y (\vec{x}, y) \in C\} \\ &= \{\vec{x}\,;\, \exists y (\vec{x}, y, z) \in A\} \\ &= \{\vec{x}\,;\, (\vec{x}, z) \in \bar{B}]\} \\ &= \{\vec{x}\,;\, (\vec{x}, z) \notin p[T_{n+1}]\}.\end{aligned}$$

This implies

$$\bar{C} = \{\vec{x}\,;\, (\vec{x}, z) \in p[T_{n+1}]\}.$$

If C is a set in a generic extension of $V[G_0, \dots, G_{n-1}]$ of size $\leq \lambda$ we find the real parameter z by our inductive hypothesis (the point is, that z is a real in the generic extension). By the product lemma the forcing extension $V[G]$ is (in particular) a forcing extension of $V[G_0, \dots, G_i]$ of size $\leq \lambda$ for every $i \in \omega$. If H is a further generic object to a forcing of size $\leq \lambda$ in $V[G]$, then $V[G][H]$ is a forcing extension of $V[G_0, \dots, G_i]$ of size $\leq \lambda$ for every $i \in \omega$ too. The universal $\mathbf{\Pi^1_n}$ sets constructed above are universal in $V[G]$. In particular we can construct trees out of the trees for these universal sets for every projective set: If $A \subseteq \omega^\omega$ is a $\mathbf{\Pi^1_n}$ set, say $A = \{x\,;\,(x,y) \in p[T_n]\}$ for a fixed $y \in \omega^\omega$, we set

$$T = \{(s,f) \in \omega^{<\omega} \times (2^\lambda)^{<\omega}\,;\,\exists t \subset y (s,t,f) \in T_n\}.$$

Let S be analogously defined. Clearly $p[T] = \omega^\omega - p[S]$ in every generic extension of size $\leq \lambda$ and $p[T] = A$.
Hence every projective set in $V[G]$ is λ-universally Baire, so in particular by proposition 1.5 it has the property of Baire.
The λ-universal Baireness of every projective set in $V[G]$ is too weak, to infer the Lebesgue measurability directly from proposition 1.5. We thus resort to methods of Solovay. We remind the reader of the following (compare [Jec03, 26.4, 26.6]):

Definition and Remark 3.5 A set $A \subset \omega^\omega$ is *Solovay* over a transitive model N of set theory, if there is a formula φ and there are parameters $\vec{s} \in N$ such that

$$x \in A \iff N[x] \models \varphi(x, \vec{s}).$$

If A, N, $\vec{s}$ are all as above, then we have:

1. If the set $\bigcup\{B\,;\,B$ is a Borel null set with code in $N\}$ is null, the set A is Lebesgue measurable.
2. In particular: If $N \cap \omega^\omega$ is countable, then A is Lebesgue measurable, since there are only countably many Borel codes for null sets.

Starting with a projective set we will construct a transitive N such that $V[G]$ believes that $N \cap \omega^\omega$ is countable. Let A be a $\mathbf{\Pi^1_n}$ set, say $A \in \Pi^1_n(\{y\})$ for $y \in V[G] \cap \omega^\omega$; without loss of generality we assume:

$$x \in A \iff (x,y) \in p[T_n].$$

Let $\dot{y} \in V$ be a name for y. Without loss of generality we assume $\dot{y} \in H_{\lambda^+}$. The ordinal κ_{n-1} is λ-strong in $V[G_0,\ldots,G_{n-2}]$. So there is an extender E with critical point κ_{n-1} and

$$\pi\colon V[G_0,\ldots,G_{n-2}] \to_E M[G_0,\ldots,G_{n-2}] =: \tilde{M}$$

such that $\dot{y} \in \tilde{M}$. Without loss of generality assume that M is the ultrapower of V by E, we can arrange this in this fashion by [Jec03, 20.30]. Since y is in a generic extension of $\tilde{M}$, there is, by [Jec03, 15.42], a forcing $\mathbb{Q}$ such that y is $\mathbb{Q}$-generic over $\tilde{M}$. So the model $\tilde{M}[y]$ is well defined.

Claim 1. The reals of $N := \tilde{M}[y]$ are countable in $V[G]$.
Proof of Claim 1. We will proof this for the case $n = 1$. So

$$\pi\colon V \to_E \mathrm{Ult}(V,E) = \tilde{M}$$

for an extender E with $\mathrm{crit}(E) = \kappa := \kappa_0$. By definition $\pi(\kappa) > \lambda$. We now show $\pi(\kappa) < \lambda^+$. The elements of $\tilde{M}$ can be represented by equivalence classes of the form $[a, f]$ for $a \in \lambda^{<\omega}$ and $f\colon \kappa^{|a|} \to V$. By the properties of these equivalence classes the following holds:

$$\begin{aligned}\pi(\kappa) &= \{[a, f]\,;\, [a, f] \in \pi(\kappa)\}\\ &= \{[a, f]\,;\, f[X] \subseteq \kappa \text{ for some } X \in E_a\}\\ &= \{[a, f]\,;\, f\colon \kappa^{|a|} \to \kappa\}.\end{aligned}$$

Now we can calculate the cardinality (in V) of $\pi(\kappa)$:

$$\lambda \leq \mathrm{Card}(\{[a, f]\,;\, f\colon \kappa^{|a|} \to \kappa\}) \leq \mathrm{Card}(\lambda^{<\omega}) \cdot \kappa^\kappa = \lambda \cdot 2^\kappa = \lambda,$$

where the last equality holds since λ is a strong limit. Then we have

$$\tilde{M} \models \lambda < \pi(\kappa) < \lambda^+ \wedge \pi(\kappa) \text{ is a strong limit},$$

and hence

$$\tilde{M} \models \mathrm{Card}(\mathrm{Col}(\omega, \lambda)) < \pi(\kappa).$$

Thus it follows that there are less than $\pi(\kappa)$ many nice names for reals for $\mathrm{Col}(\omega, \lambda)$ in $\tilde{M}$, in particular less than $\pi(\kappa)$ many nice names for reals for $\mathbb{Q}$. Then there are less than $\pi(\kappa)$ many reals in $N = \tilde{M}[y]$. Note that the proof works in $V[G]$ as well. We have $\pi(\kappa) < \lambda^+ = \omega_1^{V[G]}$, hence the reals of N are countable in $V[G]$.
If $n > 1$ we can use the same proof as above, since collapsing finitely many $\kappa_i < \kappa_{n-1}$ does not change the relevant properties of the ultrapowers involved.
□(Claim 1)

It remains to show that A is Solovay over N.

Claim 2. $x \in A \iff \tilde{M}[y][x] \models (x, y) \in p[\pi(T_n)]$.
Proof of Claim 2. Note that models of the form $\tilde{M}[y][x]$ are well defined by [Jec03, 15.42] and the fact, that there is always a smallest transitive model containing $\tilde{M}[y]$ and x. The proof of this claim is nearly the same as the proof of claim 1 in the proof of lemma 2.3. Let $x \in A$, say

$$\forall n(x \restriction n, y \restriction n, f \restriction n) \in T_n.$$

Then the following holds too:

$$\forall n(x \restriction n, y \restriction n, \pi(f \restriction n)) \in \pi(T_n).$$

So $(x, y) \in p[\pi(T_n)]$.
Let now $x \notin A$. Then $(x, y) \in p[S_n]$. By an analogous argument as above $(x, y) \in p[\pi(S_n)]$. Since

$$V[G_0, \ldots, G_{n-2}] \models p[T_n] \text{ is } \lambda\text{-universally Baire, witnessed by } T_n \text{ and } S_n.$$

implies

$$\tilde{M} \models p[\pi(T_n)] \text{ is } \pi(\lambda)\text{-universally Baire, witnessed by } \pi(T_n) \text{ and } \pi(S_n),$$

in particular

$$\tilde{M} \models p[\pi(T_n)] \cap p[\pi(S_n)] = \emptyset \text{ in all generic extensions of size } \leq \lambda.$$

The absoluteness of wellfoundedness implies:

$$V[G] \models p[\pi(T_n)] \cap p[\pi(S_n)] = \emptyset.$$

So

$$\tilde{M}[y][x] \models (x, y) \notin p[\pi(T_n)].$$

□(Claim 2)

This finishes the proof of the proposition. □

In the remainder of this chapter we will prove theorem 3.3 and we will explain, why this theorem solves the 12th Delfino Problem. It is an open problem whether all projective sets in the model $V[G]$ just discussed have a projective uniformization. To proof theorem 3.3, we must replace V by an adequate core model. We use the minimal, fully iterable, fine structural inner model $L[E]$, that satisifes the #12 hypothesis. The construction of such a model would go beyond the scope of this paper. We can not give proof of all properties of this model we need. For a construction of $L[E]$-type models see [Sch02], [Steb], [Stea] and [MS94]. From now on until the end of this paper let $V = L[E]$. As in the preceding proposition we fix a filter G that is $\mathrm{Col}(\omega, \lambda)$-generic over $L[E]$.
An important property of $L[E]$ is that it is the core model (in a precise sense) of all its set sized generic extensions. Therefore we write $L[E] = K$ too. We will not further elaborate this property of K, since we will not work out any details of proofs that rely on this fact.

Remark 3.6 *In $K[G]$ there is a non determined projective set, i.e. $\neg(\mathbf{\Delta^1_2}$ determinacy) holds in $K[G]$, as we will see later. This is why theorem 3.3 is a solution to the 12th Delfino Problem.*

So we must show that in $L[E][G] = V[G]$ all projective sets in $(\omega^\omega)^2$ have a projective uniformization. We fix a projective set $A \subset (\omega^\omega)^2, A \in \mathbf{\Pi^1_n}$, say $A \in \Pi^1_n(\{z_0\})$ for a $z_0 \in \omega^\omega$. Let T_n, S_n on $\omega^3 \times 2^\lambda$ be as in the proof of proposition 3.4, then $T_n, S_n \in V[G_0, \ldots G_{n-2}]$[5]. Without loss of generality let $A = \{(x, y)\,;\, (x, y, z_0) \in p[T_n]\}$.

[5]By theorem 1.6 we could assume $T_2, S_2 \in V$ and $T_n, S_n \in V[G_0, \ldots, G_{n-3}]$ for larger n. This is a nice fact when calculating the complexity of the uniformization. But since we have neither proved theorem 1.6 nor will we calculate the complexity of the uniformization, $T_2, S_2 \notin V$ will not do any harm.

Lemma 3.7 *The construction of T_n, S_n can be modified to yield: In every generic extension of $V[G_0, \dots G_{n-2}]$ of size $< \kappa_{n-1}$ the following holds*

- $p[T_n \restriction \kappa_{n-1}]$ *is the universal* $\mathbf{\Pi^1_n}$ *set in* $(\omega^\omega)^3$,
- $p[T_n \restriction \kappa_{n-1}] = (\omega^\omega)^3 - p[S_n \restriction \kappa_{n-1}]$,

where $T_n \restriction \kappa_{n-1} := \{(s, t, u, f) \in T_n \,; \operatorname{ran}(f) \subset \kappa_{n-1}\}$ *and* $S_n \restriction \kappa_{n-1} := \{(s, t, u, f) \in S_n \,; \operatorname{ran}(f) \subset \kappa_{n-1}\}$

Proof. The point is to change the basis of the induction. Since κ_{n-1} is in particular a strong limit, the Shoenfield tree of a $\mathbf{\Pi^1_1}$ set on $(\omega)^3 \times \kappa_{n-1}$ witnesses the γ-universal Baireness for every $\gamma < \kappa_{n-1}$. As in the proof of propostion 3.4 we have at the basis of the induction

$$p[T_1 \restriction \kappa_{n-1}] = (\omega^\omega)^3 - p[S_1 \restriction \kappa_{n-1}].$$

If we choose a universal $\mathbf{\Pi^1_1}$ set $p[T_1 \restriction \kappa_{n-1}]$ at the basis of the induction, a construction as in the proof of 3.4 yields the desired conclusion. □

We fix $x \in \omega^\omega$. We will find, in a projective and uniform way, a real $F(x)$ such that

$$\exists y(x, y) \in A \implies \exists y((x, y) \in A \wedge y = F(x)),$$

So

$$\exists y(x, y, z_0) \in p[T_n] \implies (x, F(x), z_0) \in p[T_n].$$

We find F by analysing certain premice. These premice correspond to reals by coding. Thus we will be able to identify projective sets with sets of these mice. This will yield a projective uniformization in the end. To define the relevant class of premice we need the following notion:

Definition 3.8 A pair $((M_0, M_1), \kappa)$ is a phalanx, if and only if M_0, M_1 are premice and the following conditions are met:

- $\kappa \in M_0 \cap M_1$ is a cardinal in M_1;
- $E^{M_0} \restriction \kappa = E^{M_1} \restriction \kappa$;
- if $\kappa' < \kappa$, then κ' is a cardinal in M_0 if and only if κ' is a cardinal in M_1.

We call κ the exchange point of the phalanx. If it is clear which exchange point is intended we write, for the sake of brevity, (M_0, M_1) instead of $((M_0, M_1), \kappa)$. There is a notion of the iterability of a phalanx and of coiterations of two phalanges. Since these are the kind of details we are not really concerned here with, we refer to [Zem02, 9.1] and [Ste96] for these notions.

Remark 3.9 *In our situation iterations of phalanges are not anymore linear, since the iterations of premice are not either. Hence we need iteration trees. If we use a Jensen style indexing, these trees are almost linear: see [Sch02]. We suppress all details of these iteration trees in the context of this paper.*

Definition 3.10 A premice M is called (z_0, x)-*good*, if all of the following conditions are met:

1. $M \rhd \mathcal{J}^K_{\kappa^{+K}_{n-1}}$.
2. The phalanx $((K, M), \kappa^{+K}_{n-1})$ is iterable.
3. M has a topextender F^M with critical point κ_{n-1}.
4. If $\pi\colon M \to_{F^M} \tilde{M}$ and $\tilde{\pi}\colon M[G_0, \ldots, G_{n-2}] \to \tilde{M}[G_0, \ldots, G_{n-2}]$ is the canonical extension of π, then
$$\exists y(x, y, z_0) \in p[\tilde{\pi}(T_n \restriction \kappa_{n-1})].$$
Note that $T_n \restriction \kappa^{+K}_{n-1} \in \mathcal{J}^K_{\kappa^{+K}_{n-1}} \in M$.
5. M is minimal (in respect to the canonical prewellordering of premice) among all mice satisfying 1. through 4.

Remark 3.11 *In $V[G]$ there are (z_0, x)-good premice which are countable, as we will see below. The fifth condition makes sense, since the second condition implies the iterability of the premouse. We will not prove this fact here.*

Definition 3.12 Let M_0 be a premice and $\mathcal{T}$ an iteration tree on M_0. If $0\mathcal{T}\alpha$, we say that M_α is an iterate of M_0. If
$$[0, \alpha]_{\mathcal{T}} \cap D^{\mathcal{T}} = \emptyset,$$
i.e. there are no drops on the branch through $\mathcal{T}$ from 0 to α, then we say that M_α is a simple iterate of M_0. Note that there is an elementary embedding $i_{0,\alpha}\colon M_0 \to M_\alpha$ in this case.

We will use the following lemma without proof, since such a proof would go beyond the scope of this paper.

Lemma 3.13 *If M, M' are (z_0, x)-good premice, then M, M' coiterate simply above κ^{+K}_{n-1}.*

We examine now, which iterates of a (z_0, x)-good premice are (z_0, x)-good.

Lemma 3.14 *Simple iterates above κ_{n-1}^{+K} of a (z_0, x)-good premouse are (z_0, x)-good.*

Proof. Let M be (z_0, x)-good and M^* a simple iterate of M above κ_{n-1}^{+K}. Let $i\colon M \to M^*$ denote the embedding. We verify the definition step by step.

1. Since the iteration is above κ_{n-1}^{+K}, clearly $M^* \triangleright \mathcal{J}^K_{\kappa_{n-1}^{+K}}$ holds.

2. The phalanx (K, M^*) is an iterate of the phalanx (K, M), so it is iterable, since any iteration of (K, M^*) is an iteration of (K, M).

3. Clearly $i(\kappa_{n-1}) = \kappa_{n-1}$. It may be the case that $ht(M)$ and $ht(M^*)$ differ, but the image of M's topextender F^{M^*} has critical point κ_{n-1} too.

4. In an abuse of notation we write π for ultrapower map of the ultrapower of M^* by F^{M^*} too. As in the case for M, the map π induces a map

$$\tilde{\pi}\colon M^*[G_0, \dots, G_{n-2}] \to \tilde{M}^*[G_0, \dots, G_{n-2}].$$

Since F^{M^*} is the image of M's topextender, we get another map

$$\tilde{i}\colon M[G_0, \dots, G_{n-2}] \to M^*[G_0, \dots, G_{n-2}].$$

By the shift lemma (see [Ste96]) there is a map j and the following commutative diagram:

$$\begin{array}{ccc} M & \xrightarrow{\pi} & \tilde{M} \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ M^* & \xrightarrow{\pi} & \tilde{M}^* \end{array}$$

Then there is a commutative diagram with the extended maps $\tilde{i}, \tilde{j}$ and $\tilde{\pi}$:

$$\begin{array}{ccc} M[G_0, \dots, G_{n-2}] & \xrightarrow{\tilde{\pi}} & \tilde{M}[G_0, \dots, G_{n-2}] \\ {\scriptstyle \tilde{i}}\downarrow & & \downarrow{\scriptstyle \tilde{j}} \\ M^*[G_0, \dots, G_{n-2}] & \xrightarrow{\tilde{\pi}} & \tilde{M}^*[G_0, \dots, G_{n-2}]. \end{array}$$

Then clearly the following holds on the M^*-side:

$$\exists y (x, y, z_0) \in p[\tilde{\pi}(T_n \restriction \kappa_{n-1})].$$

5. Since M^* is a simple iterate of M, both are in the same equivalence class in respect to the canonical prewellordering of mice.

□

The following lemma will yield in the end that the uniformization F we are looking for is projective.

Lemma 3.15 *The set of all reals that code (z_0, x)-good premice is projective.*

Proof. We verify that we can formulate the definition in a projective way step by step.

1. $\mathcal{J}^K_{\kappa_{n-1}^{+K}}$ is countable in $V[G]$, so we can code it into a real. We can code structures of the form $M||\delta$ by reals too.

2. The iterability of a phalanx (K, M) is a projective statement in the codes by results of Hauser, see [Hau95, 3.4].

3. Clearly the following statement is projective in the codes: "M is a countable premouse with a topextender with critical point κ_{n-1}". Note that κ_{n-1} is countable in $V[G]$.

4. The filters $G_0, \ldots, G_{n-2}$ are countable in $V[G]$. So after coding we can identify them with reals. The ultrapower of a countable (z_0, x)-good M by its topextender is countable. Since we can use the codes of $G_0, \ldots, G_{n-2}$ as real parameters, we can code the map $\tilde{\pi}$ in definition 3.10 by a real. In particular we can code $\tilde{\pi}(T_n \restriction \kappa_{n-1})$ by a real. Thus

$$\exists y (x, y, z_0) \in p[\tilde{\pi}(T_n \restriction \kappa_{n-1})].$$

is projective in the codes.

5. Coiterations of countable premice are basically countable sequences of countable well founded models, so we can code coiterations by reals. Hence we can formulate the minimality in respect to the canonical prewellordering of premice of some M in a projective way in the codes. Thus we can state projectively that M is minimal among all premice satisfying conditions 1. through 4.

□

The following lemma allows us to analyse membership in A by looking at (z_0, x)-good premice.

Lemma 3.16

$$\begin{aligned}\exists y(x,y,z_0)\in p[T_n] &\iff \exists y(x,y)\in A\\ &\iff \textit{There is a countable } (z_0,x)\textit{-good } M.\end{aligned}$$

Proof. The first equivalence clearly holds by our choice of z_0. We proceed as in the proof of lemma 2.3: Let $y\in\omega^\omega$ such that $(x,y,z_0)\in p[T_n]$. Then there is an initial segment M of K with a topextender F^M with critical point κ_{n-1} such that the tuple (x,y,z_0) is in a generic extension of size $<\tilde{\pi}(\kappa_{n-1})$ of $\tilde{M}[G_0,\dots G_{n-2}]$. The initial segment M is iterable by basic properties of K. Without loss of generality we can choose M to be countable in $V[G]$. The phalanx (K,M) is iterable. We have to show

$$(x,y,z_0)\in p[\tilde{\pi}(T_n\restriction\kappa_{n-1})],$$

then M witnesses the existence of a (z_0,x)-good premouse. We assume

$$(x,y,z_0)\notin p[\tilde{\pi}(T_n\restriction\kappa_{n-1})]$$

and work towards a contradiction. Since $T_n\restriction\kappa_{n-1}$ and $S_n\restriction\kappa_{n-1}$ are trees to a γ-universally Baire set, for all $\gamma<\kappa_{n-1}$, we have

$$(x,y,z_0)\in p[\tilde{\pi}(S_n\restriction\kappa_{n-1})].$$

We can form the ultrapower of $V[G_0,\dots,G_{n-2}]$ by F^M. For the ultrapower map we write in an abuse of notation $\tilde{\pi}$. Then

$$(x,y,z_0)\in p[\tilde{\pi}(S_n\restriction\kappa_{n-1})]\subseteq p[\tilde{\pi}(S_n)].$$

Since $(x,y,z_0)\in p[T_n]$ and reals are not moved by $\tilde{\pi}$ we have $(x,y,z_0)\in p[\tilde{\pi}(T_n)]$. But then

$$p[\tilde{\pi}(T_n)]\cap p[\tilde{\pi}(S_n)]\neq\emptyset,$$

which implies

$$p[T_n]\cap p[S_n]\neq\emptyset.$$

Contradiction!
Let M be a (z_0,x)-good premouse with topextender F^M. As in the proof of the converse direction we extend $\tilde{\pi}$ to a map with domain $V[G_0,\dots,G_{n-2}]$. Let $y\in\omega^\omega$ such that

$$(x,y,z_0)\in p[\tilde{\pi}(T_n\restriction\kappa_{n-1})]\subset p[\tilde{\pi}(T_n)].$$

If $(x,y,z_0)\notin p[T_n]$ would hold true, then $(x,y,z_0)\in p[S_n]$ and so $(x,y,z_0)\in p[\tilde{\pi}(S_n)]$. So this would yield the same contradiction as above! □

For any given (z_0, x)-good M and any $y, \vec{\alpha} \in M$ such that

$$(x, y, z_0, \vec{\alpha}) \in [\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})]$$

there is a leftmost branch $(y, \vec{\alpha})$. This branch is not necessarily a honest leftmost branch in the sense of [Kan03].

Definition 3.17 We write $(y, \vec{\alpha})^M$ for the leftmost branch in M witnessing $\exists y(x, y, z_0) \in p[\tilde{\pi}(T_n \upharpoonright \kappa_{n-1})]$.

Definition 3.18 We define for any (z_0, x)-good N

$$B(N) := \{N^* \, ; \, N^* \text{ is a simple iterate above } \kappa_{n-1}^{+K} \text{ of } N\}.$$

By lemma 3.14 the set $B(N)$ contains only (z_0, x)-good premice. A (z_0, x)-good M will be called stable if and only if for all $M^* \in B(M)$ with iteration map $i : M \to M^*$ the following holds

$$y \leq_{lex} y^* \wedge (\forall n \in \omega) i(\alpha(n)) \leq \alpha^*(n),$$

where $(y, \vec{\alpha}) = (y, \vec{\alpha})^M$ and $(y^*, \vec{\alpha}^*) = (y, \vec{\alpha})^{M^*}$.

Lemma 3.19 *If there is a (z_0, x)-good M, then there is a stable (z_0, x)-good M. If M and M' are stable (z_0, x)-good premice, then $y = y'$, where $(y, \vec{\alpha}) = (y, \vec{\alpha})^M$ and $(y', \vec{\alpha}') = (y, \vec{\alpha})^{M'}$.*

This lemma finishes the proof of theorem 3.3; we can now define the uniformization F of A by setting

$$(x, y) \in F \iff x \in A \wedge \exists M \exists \vec{\alpha}(M \text{ is stable } (z_0, x)\text{-good} \wedge (y, \vec{\alpha}) = (y, \vec{\alpha})^M).$$

"M is a (z_0, x)-good premouse" is a projective statement in the codes. Note that we mentioned above, that statements about countable iterations of countable premice are projective in the codes. By the preceding remarks and lemmata F is a projective set. So we have found a projective uniformization of A. It remains to proof lemma 3.19.

Proof. Let M be (z_0, x)-good. We choose $M_1 \in B(M)$ such that for all $M_1^* \in B(M_1)$ the following holds true: If $(y, \vec{\alpha}) = (y, \vec{\alpha})^{M_1}$ and $(y^*, \vec{\alpha}^*) = (y, \vec{\alpha})^{M_1^*}$, then $y(0) \leq y^*(0)$. Here $y(0)$ is minimal among all (z_0, x)-good N, since if Q denotes the result of a coiteration of M with a (z_0, x)-good N, then $Q \in B(M)$ by lemma 3.13.
We hone this procedure to minimise $\vec{\alpha}(0)$. Let $(y_1, \vec{\alpha}_1) = (y, \vec{\alpha})^{M_1}$. We choose, if it exists, $M_{1,1} \in B(M_1)$ such that

$$\alpha_{1,1}(0) < i_{M_1, M_{1,1}}(\alpha_1(0)),$$

where $i_{M_1,M_{1,1}}$ denotes the iteration map and $(y_{1,1}, \vec{\alpha}_{1,1}) = (y, \vec{\alpha})^{M_{1,1}}$. We proceed inductively to construct, if it exists, a (maybe finite) sequence of models $M_{1,n+1}$ such that $M_{1,n+1} \in B(M_{1,n})$ and $\alpha_{1,n+1}(0) < i_{M_{1,n},M_{1,n+1}}(\alpha_{1,n}(0))$, where $(y_{1,n}, \vec{\alpha}_{1,n}) = (y, \vec{\alpha})^{M_{1,n}}$ and $i_{M_{1,n},M_{1,n+1}}$ is the according iteration map.

Claim 1. The sequence of the $M_{1,n}$ is finite.
Proof of Claim 1. We assume that this was not the case. Then the direct limit of the $M_{1,n}$ is an iterate of M_1. But since $\alpha_{1,n+1}(0) < i_{M_{1,n},M_{1,n+1}}(\alpha_{1,n}(0))$ holds true for all n, this direct limit is not well founded, contradicting the iterability of M_1. □(Claim 1)

We set M_2 to be the last model in the sequence of the $M_{1,n}$ or M_1 itself if the sequence is empty. This construction obviously yields for all $N \in B(M_2)$

$$i(\alpha_2(0)) \leq \alpha_N(0),$$

where $i\colon M_2 \to N$ denotes the iteration map, $(y_2, \vec{\alpha}_2) = (y, \vec{\alpha})^{M_2}$ and $(y_N, \vec{\alpha}_N) = (y, \vec{\alpha})^N$. We repeat these to steps to produce a sequence $(M_i)_{i\in\omega}$: If $i = 2j+1$, we minimize $y(j)$; if $i = 2j$, we minimize $\vec{\alpha}(j)$. In each case we use the procedures described above for M_1 and M_2 respectively. Since for every $n \leq m \in \omega$ the model M_m is a simple iterate of M_n, the set of the M_i is a directed system. Let M_ω denote the direct limit of the M_i. Then M_ω is a simple iterate above κ_{n-1}^{+K} of M, so it is (z_0, x)-good. Clearly M_ω is stable. □

This lemma completes the proof of theorem 3.3. We will now look again at the model $K[G]$ and analyse why it is not a model of projective determinacy. We will see that there is a $\mathbf{\Delta^1_2}$ set in $K[G]$ that is not determined.
We will say a few words about games and strategies. We see strategies in games of the type $G(\omega, A)$ as (partial) functions $\sigma\colon \omega^{<\omega} \to \omega$. See [Kec95, 20.A]. These strategies can obviously be coded by reals. If σ is a real, that codes a strategy for II and if $x \in \omega^\omega$ is the real that I plays, then $\sigma * x$ denotes the real II plays by using the strategy σ on x. If τ codes a strategy for I, the we write $x * \tau$ for the real that I plays by using τ on an x played by II. Compare [Kec95, 39.1].

Theorem 3.20 *There is a non determined $\mathbf{\Delta^1_2}$ set in $K[G]$.*

We noted above that by an unpublished result of Woodin we already know consistency strength wise that $\mathbf{\Delta^1_2}$ determinacy can not hold in $K[G]$, but the proof below is more elementary and gives some insight into the nature of that non determined set. The proof shows that a non determined $\mathbf{\Delta^1_2}$ set in K induces (by the same definition) a non determined $\mathbf{\Delta^1_2}$ set in $K[G]$. First of all we need a general proposition on determinacy and regularity properties of sets of reals.

Proposition 3.21 *If $\mathbf{\Delta^1_{2n}}$ determinacy holds, then all $\mathbf{\Sigma^1_{2n+1}}$ sets are Lebesgue measurable, have the Baire property and have a perfect subset or are countable.*

Proof. By a result of Martin $\mathbf{\Delta^1_{2n}}$ determinacy implies $\mathbf{\Pi^1_{2n}}$ determinacy. For a proof of this result see [KS85, 5.1]. By a result of Kechris and Martin (about unfolded games) this implies already the desired conclusion. For a proof of this result see [Kan03, 27.14]. □

We proof theorem 3.20 now.

Proof. By lemma 2.3 every projective set of reals in $K[G]$ is λ-universally Baire. In particular all $\mathbf{\Delta^1_2}$ sets are λ-universally Baire. By theorem 1.7 we know

$$K \prec_{\mathbf{\Sigma^1_3}} K[G].$$

In K there is a $\mathbf{\Sigma^1_3}$ wellordering of the reals. Clearly this wellordering is not measurable. So by proposition 3.21 there is a non determined $\mathbf{\Delta^1_2}$ set A in K. Let φ and ψ be $\mathbf{\Sigma^1_2}$ formulas and $y \in \omega^\omega \cap K$ a parameter with

$$K \models x \in A \iff \varphi(x,y) \iff \neg\psi(x,y).$$

The statement

$$\forall x(\varphi(x,y) \iff \neg\psi(x,y))$$

is $\mathbf{\Pi^1_3}$. So

$$K[G] \models \forall x(\varphi(x,y) \iff \neg\psi(x,y)).$$

Claim 1. The non determinacy of A is a $\mathbf{\Pi^1_3}$ statement.
Proof of Claim 1. Since A is not determined for every strategy τ of player I there is an $x \in \omega^\omega$ such that $\tau * x \notin A$ and analogously for II. The following statement is $\mathbf{\Pi^1_3}$ and expresses the non determinacy of A in K:

$$(\forall\tau\exists x_1\psi(x_1 * \tau, y)) \wedge (\forall\sigma\exists x_2\varphi(\sigma * x_2, y)).$$

□(Claim 1)

So the following holds true in $K[G]$

$$(\forall\tau\exists x_1\psi(x_1 * \tau, y)) \wedge (\forall\sigma\exists x_2\varphi(\sigma * x_2, y)) \wedge \forall x(\varphi(x,y) \iff \neg\psi(x,y)).$$

The set $\bar{A} = \{x \in \omega^\omega \cap K[G]\,;\, \varphi(x,y)\}$ is then complementary to $\{x \in \omega^\omega \cap K[G]\,;\, \psi(x,y)\}$ in $K[G]$ and not determined. □

To sum it up:

$$K[G] \models \triangle + \neg(\mathbf{\Delta^1_2} \text{ determinacy}).$$

This completes our examination of the 12th Delfino Problem. We remark that our upper bound for $\triangle$ is the best possible.

Remark 3.22 *With core model theory one can show that the #12 hypothesis and $\triangle$ are equiconsistent, see [Sch02, 9.1].*

References

[FMW92] Qi Feng, Menachem Magidor, and Hugh Woodin. Universally Baire sets of reals. In *Set theory of the continuum (Berkeley, CA, 1989)*, volume 26 of *Math. Sci. Res. Inst. Publ.*, pages 203–242. Springer, New York, 1992.

[Hau95] Kai Hauser. The consistency strength of projective absoluteness. *Ann. Pure Appl. Logic*, 74(3):245–295, 1995.

[Jec03] Thomas Jech. *Set theory.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[Kan03] Akihiro Kanamori. *The higher infinite.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003.

[Kec95] Alexander S. Kechris. *Classical descriptive set theory*, volume 156 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 1995.

[KMS88] Alexander S. Kechris, Donald A. Martin, and John R. Steel. *Cabal Seminar 81-85: Proceedings, Caltech-UCLA Logic Seminar, 1981-85*, volume 1333 of *Springer Lecture Notes in Math.* Springer-Verlag, New York, 1988.

[KS85] Alexander S. Kechris and Robert M. Solovay. On the relative consistency strength of determinacy hypotheses. *Trans. Amer. Math. Soc.*, 290(1):179–211, 1985.

[MS94] William J. Mitchell and John R. Steel. *Fine structure and iteration trees*, volume 3 of *Lecture Notes in Logic.* Springer-Verlag, Berlin, 1994.

[Sch02] Ralf-Dieter Schindler. The core model for almost linear iterations. *Ann. Pure Appl. Logic*, 116(1-3):205–272, 2002.

[Stea] John R. Steel. A coarse-structural local K^c-construction. in preparation.

[Steb] John R. Steel. Local K^c-constructions. unpublished. See http://math.berkeley.edu/∼steel/.

[Ste96] John R. Steel. *The core model iterability problem*, volume 8 of *Lecture Notes in Logic.* Springer-Verlag, Berlin, 1996.

[Woo82] W. Hugh Woodin. On the consistency strength of projective uniformization. In *Proceedings of the Herbrand symposium (Marseilles, 1981)*, volume 107 of *Stud. Logic Found. Math.*, pages 365–384, Amsterdam, 1982. North-Holland.

[Zem02] Martin Zeman. *Inner models and large cardinals*, volume 5 of *de Gruyter Series in Logic and its Applications*. Walter de Gruyter & Co., Berlin, 2002.