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Iterated Forcing

thesis submitted in partial fulfillment of the requirements for the master of science degree

by

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1 Introduction

In his search for a natural iteration theorem for a class of $\mathcal{L}$-forcings that don't add reals, Jensen discovered the notion of subcomplete forcings and was able to prove a strong iteration theorem for them. In our master's thesis, we aim to provide an accessible introduction to iterated forcings and a full account of Jensen's iteration theorem for subcomplete Boolean algebras.
To realise this, we begin in chapter 1 by introducing complete Boolean valued models and their connection to generic extensions. Building on this, we then develop the basic theory of iterated forcings in chapter 2. Finally, in chapter 3, we combine these methods to define subcomplete Boolean algebras and prove Jensen's main iteration theorem:

Theorem. *Let $\mathcal{B} = (\mathbb{B}_i \mid i < \alpha)$ be an* RCS-*iteration such that $\mathbb{B}_0 = \{0,1\}$ is trivial and for all $i + 1 < \alpha$*

1. $\mathbb{B}_i \neq \mathbb{B}_{i+1}$,
2. $\Vdash_{\mathbb{B}_i} \check{\mathbb{B}}_{i+1}/\dot{G}_i$ *is subcomplete and*
3. $\Vdash_{\mathbb{B}_{i+1}} \operatorname{card}(\check{\mathbb{B}}_i) \leq \omega_1$.

Then every $\mathbb{B}_i$ is subcomplete.

2 Preliminaries

In this chapter we fix our notation and develop the theory of Boolean valued models and their connection to generic extensions. Any missing definitions can be found in [Jec06] and we assume that the reader is familiar with Gödel's constructible universe, basic facts about elementary substructures and non-iterated forcings. Our background theory is ZFC - including the notion of *virtual classes*, i.e. for any $\mathcal{L}_{\{\in\}}$-formula ϕ and parameters $p_1, \ldots, p_k$ we call

$$C = \{y \mid \phi(y, p_1, \ldots, p_k)\}$$

a *(virtual) class*. We say that C is *proper* iff there is no set X containing the same elements as C. We let Ord denote the class of all ordinals and for a given set X we write $\mathcal{P}(X)$ for its *powerset*, $\mathrm{card}(X)$ for its *cardinality* and $\mathrm{tc}(X)$ for its *transitive closure* - the smallest transitive set containing X as a subset. We say that X is *hereditarily of cardinality less than* θ iff $\mathrm{card}(\mathrm{tc}(\{X\}) < \theta$ and we let

$$H_\theta := \{X \mid \mathrm{card}(\mathrm{tc}\{X\}) < \theta\}$$

be the collection of all these sets. ${}^A B$ is the set of all functions $f \colon A \to B$ and given some function $f \colon A \to B$ and some $X \subseteq A$, we write $f"X := \{f(x) \mid x \in X\}$ for the *pointwise image* of X under f. If $Y \subseteq B$, we also write $f^{-1}"Y := \{x \in X \mid f(x) \in Y\}$ for the *preimage* of Y under f. By a sequence $(x_i \mid i \in I)$, we mean a function $x \colon I \to \{x_i \mid i \in I\}$ such that $x(i) = x_i$ for all $i \in I$. Given a sequence $(X_i \mid i \in I)$ of nonempty sets, we write $\prod_{i \in I} X_i$ for the set of all sequences $(x_i \mid i \in I)$ such that $x_i \in X_i$ for all $i \in I$. If $(x_i \mid i < \alpha)$ is a sequence and α is an ordinal, we let $(x_i \mid i < \alpha)^\frown x$ be the sequence $(y_i \mid i < \alpha + 1)$ where $y_i = x_i$ for $i < \alpha$ and $y_\alpha = x$.
The *cumulative hierarchy* is recursively defined by

- $V_0 := \emptyset$,
- $V_{\alpha+1} := \mathcal{P}(V_\alpha)$ for all ordinals α,
- $V_\lambda := \bigcup_{\alpha<\lambda} V_\alpha$ for limit ordinals λ and
- $V := \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$.

and $\mathrm{rk}(X)$, the *rank* of a given set X, is the least ordinal α such that $X \in V_{\alpha+1}$. Given a transitive set M and an additional (possibly empty) set A, let $\mathrm{def}_A(M)$ be the collection of $X \subseteq M$ that are definable in the structure $(M; \in\restriction M, A \cap M)$. Recall the *constructible hierarchy relative to a set* A, recursively defined by

- $L_0[A] := \emptyset$,
- $L_{\alpha+1}[A] := \mathrm{def}_A(L_\alpha[A])$ for all ordinals α,
- $L_\lambda[A] := \bigcup_{\alpha<\lambda} L_\alpha[A]$ for limit ordinals λ and
- $L[A] := \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha[A]$

and the *constructible hierarchy over a set* A, given by

- $L_0(A) := \mathrm{tc}(\{A\})$,
- $L_{\alpha+1} := \mathrm{def}_\emptyset(L_\alpha(A))$ for all ordinals α,
- $L_\lambda(A) := \bigcup_{\alpha<\lambda} L_\alpha(A)$ and
- $L(A) := \bigcup_{\alpha\in\mathrm{Ord}} L_\alpha(A)$.

In the arguments to come, we consider structures of the form

$$\mathcal{M} = (M; \in\restriction M, A \restriction M)$$

(which - by an abuse of notation - may also be denoted by $(M; \in, A)$ or simply by M, if there is no danger of confusion), where M is a transitive set (or class) such that $(M; \in)$ satisfies “a sufficienlty large fragment of ZFC” and A is a (possibly empty) set that equips $\mathcal{M}$ with a $\mathcal{L}_{\{\in,A\}}$[1]-structure in the usual way. It’s quite common to take ZFC-, i.e. ZFC minus the powerset axiom, as such a “sufficiently large fragment”. However, as Gitman, Hamkins and Johnstone demonstrated in [GHJ11], replacement is surprisingly weak in the absence of the powerset axiom and as a consequence of this, models of ZFC- may have undesirable defects. Therefore, we shall be a bit more careful and consider models of ZFC^- instead, where ZFC^- is axiomatized by extensionality, foundation, pairing, union, infinity, comprehension, the well-order principle and collection.

[1]Since there is no danger of confusion, we won’t distinguish between the actual predicates/relations and their respective symbols in our language.

2.1 Boolean Algebras

Definition 2.0.1. *A* Boolean algebra $\mathbb{B}$ *is a 6-tupel* $(B; 0, 1, +, \cdot, -)$*, where*

- *B is a nonempty set,*
- *$0 \in B$ is the* minimal (or least) element*,*
- *$1 \in B$ is the* maximal (or greatest) element*,*
- *$0 \neq 1$,*
- *$+\colon B \times B \to B$, $a + b := +(a, b)$ is the* join *of a and b,*
- *$\cdot\colon B \times B \to B$, $a \cdot b := \cdot(a, b)$ is the* meet *of a and b,*
- *$-\colon B \to B$, $-a := -(a)$ is the* complement *of a.*

such that for all $a, b, c \in B$, the following hold

$$\begin{array}{ccr} a + b = b + a, & a \cdot b = b \cdot a & \textit{(commutativity)} \\ a + (b + c) = (a + b) + c, & a \cdot (b \cdot c) = (a \cdot b) \cdot c & \textit{(associativity)} \\ a + (b \cdot c) = (a + b) \cdot (a + c), & a \cdot (b + c) = (a \cdot b) + (a \cdot c) & \textit{(distriributivity)} \\ a + (a \cdot b) = a, & a \cdot (a + b) = a & \textit{(absorption)} \\ a + (-a) = 1, & a \cdot (-a) = 0 & \textit{(complementation)} \end{array}$$

We often identify $\mathbb{B}$ with its underlying set and write $a \in \mathbb{B}$ for $a \in B$, $X \subseteq \mathbb{B}$ for $X \subseteq B$, … and whenever we don't specify symbols for the minimal/maximal element, join, meet and complement of a given Boolean algebra, we denote them by $0/1$, $+$, $\cdot$ and $-$.

If X is a nonempty set and $F \subseteq \mathcal{P}(X)$ is an algebra on X, i.e. nonempty and closed under unions, intersections and complements, then the *field of sets* $(F, 0, 1, +, \cdot, -)$ with

- $0 := \emptyset$,
- $1 := X$,
- $a + b := a \cup b$,
- $a \cdot b := a \cap b$ and
- $-a := X \setminus a$

for all $a, b \in F$ defines a Boolean algebra and in fact, Stone's Representation Theorem states that every Boolean algebra is isomorphic to a suitable field of sets. To improve the readability of Boolean expressions, let us introduce the following order of operations

$$\begin{array}{c} () \\ - \\ \cdot \\ + \end{array}$$

and omit parantheses, when there is no danger of confusion. By this convention $((-a) \cdot (-(b+c))) + d$ translates to $-a \cdot -(b+c) + d$.

Proposition 2.0.1. *Let $\mathbb{B}$ be a Boolean algebra. Then for all $a, b \in B$, we have the following identities:*

1. $a + 0 = a$ *and* $a \cdot 1 = a$,
2. $a + a = a$ *and* $a \cdot a = a$,
3. $a + 1 = 1$ *and* $a \cdot 0 = 0$,
4. $-a$ *is the unique* $x \in \mathbb{B}$ *s.t.* $a + x = 1$ *and* $a \cdot x = 0$,
5. $-0 = 1$ *and* $-1 = 0$,
6. $-(a + b) = -a \cdot -b$ *and* $-(a \cdot b) = -a + -b$,
7. $-(-a) = a$.

Proof.

1. $a + 0 = a + (a \cdot -a) = a$ and $a \cdot 1 = a \cdot (a + -a) = a$.
2. $a + a = a + (a \cdot 1) = a$ and $a \cdot a = a \cdot (a + 0) = a$.
3. $a + 1 = a + a + -a = a + -a = 1$ and $a \cdot 0 = a \cdot a \cdot -a = a \cdot -a = 0$.
4. Let $x \in \mathbb{B}$ be s.t. $a + x = 1$ and $a \cdot x = 0$. Then

$$\begin{aligned} x &= x \cdot (a + -a) \\ &= (x \cdot a) + (x \cdot -a) \\ &= 0 + x \cdot -a \\ &= (a \cdot -a) + (x \cdot -a) \\ &= (a + x) \cdot -a \\ &= 1 \cdot -a \\ &= -a \end{aligned}$$

5. Since $0 \cdot 1 = 0$ and $0 + 1 = 1$, this immediatly follows from the uniqueness of complements.

6. Observing that

$$\begin{aligned}(a+b)\cdot(-a\cdot -b) &= (a\cdot -a\cdot -b)+(b\cdot -a\cdot -b)\\ &= (0\cdot -b)+(-a\cdot 0)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}(a+b)+(-a\cdot -b) &= (a+b+-a)\cdot(a+b+-b)\\ &= (1+b)\cdot(a+1)\\ &= 1\cdot 1\\ &= 1,\end{aligned}$$

we get $-(a+b) = -a\cdot -b$ by the uniqueness of complements.
Similarly, $-(a\cdot b) = -a + -b$ follows from

$$\begin{aligned}(a\cdot b)+(-a+-b) &= (a+-a+-b)\cdot(b+-a+-b)\\ &= (1+-b)\cdot(-a+1)\\ &= 1\end{aligned}$$

and

$$\begin{aligned}(a\cdot b)\cdot(-a+-b) &= (a\cdot b\cdot -a)+(a\cdot b\cdot -b)\\ &= (0\cdot b)+(a\cdot 0)\\ &= 0.\end{aligned}$$

7.

$$\begin{aligned}-(-a)+-a &= -(-a\cdot a)\\ &= -0\\ &= 1\end{aligned}$$

and

$$\begin{aligned}-(-a)\cdot -a &= -(-a+a)\\ &= -1\\ &= 0\end{aligned}$$

and thus $a = -(-a)$, again by the uniqueness of complements.

□

Definition 2.0.2. *A* partially ordered set *is a pair* $\mathbb{P} = (P; \leq)$ *such that* P *is a nonempty set and* $\leq \subseteq P \times P$ *is a* partial order, *i.e.*

(Reflexivity) $p \leq p$,

(Antisymmetrie) $p \leq q$ *and* $q \leq p$ *implies* $p = q$ *and*

(Transitivity) $p \leq q$ *and* $q \leq r$ *implies* $p \leq r$

hold for all $p, q, r \in P$.
We say that $p, q \in P$ *are* compatible *(in symbols* $p \parallel q$*) iff there is some* $r \in P$ *with* $r \leq p$ *and* $r \leq q$. *Otherwise* p, q *are* incompatible *(in symbols* $p \perp q$*).*
Finally, $(P; \leq)$ *is* separative *iff for all* $p, q \in P$ *with* $p \nleq q$ *there is some* $r \in P$ *such that* $r \leq p$ *and* $r \perp q$.

As with Boolean algebras, we shall identify partial ordered sets with their underlying sets and also confuse partially ordered sets with their partial orders.

Definition 2.0.3. *Let* $\mathbb{B} = (B; 0, 1+, \cdot, -)$ *be a Boolean algebra. We let* $\mathbb{B}^+ := (B^+, \preceq)$, *where*

- $B^+ := B \setminus \{0\}$,
- *For all* $a, b \in B^+$: $a \preceq b :\leftrightarrow b \succeq a :\leftrightarrow a \cdot -b = 0$.

If $a \preceq b$, *we say that* a *is* stronger *than* b *or* a *is an* extension *of* b. *We also write* $a \prec b :\leftrightarrow b \succ a :\leftrightarrow a \preceq b$ *and* $a \neq b$ *in which case we say that* a *is* strictly stronger *than* b.

Proposition 2.0.2. *Let* $\mathbb{B}$ *be a Boolean algebra, let* $\mathbb{B}^+$ *be defined as above and let* $a, b \in \mathbb{B}^+$. *Then*

1. $a \preceq b$ *iff* $-b \preceq -a$ *iff* $a \cdot b = a$ *iff* $a + b = b$,
2. $\mathbb{B}^+ = (B^+; \preceq)$ *is a separative partial order with greatest element* 1 *such that* $a \perp b$ *iff* $a \cdot b = 0$,
3. $a + b$ *is the* supremum *of* $\{a, b\}$, *i.e.* $a, b \preceq a + b$ *and for all* $c \in \mathbb{B}^+$ *with* $a, b \preceq c$ *we have* $a + b \preceq c$,
4. $a \cdot b$ *is the* infimum *of* $\{a, b\}$, *i.e.* $a \cdot b \preceq a, b$ *and for* $c \in \mathbb{B}^+$ *with* $c \preceq a, b$ *we have* $c \preceq a \cdot b$.

Proof. 1. Note that $a \preceq b$ iff $a \cdot -b = 0$ iff $-b \cdot -(-a) = 0$ iff $-b \preceq -a$.
If $a \preceq b$, i.e. $a \cdot -b = 0$, then

$$\begin{aligned} a &= a \cdot (b + -b) \\ &= a \cdot b + a \cdot -b \\ &= a \cdot b \end{aligned}$$

Now $a = a \cdot b$ implies

$$\begin{aligned} a + b &= (a \cdot b) + b \\ &= b \end{aligned}$$

and if $a + b = b$, then

$$\begin{aligned} a \cdot -b &= a \cdot -(a + b) \\ &= a \cdot -a \cdot -b \\ &= 0 \cdot -b \\ &= 0 \end{aligned}$$

2. **(Reflexivity)** $a + a = a$ and thus $a \preceq a$ for all $a \in \mathbb{B}^+$.

   (Transitivity) Let $a \preceq b$ and $b \preceq c$. Then $a + c = a + (b + c) = (a + b) + c = b + c = c$ and thus $a \preceq c$.

   (Antisymmetry) Let $a \preceq b$ and $b \preceq a$. Then $a = a + b = b$.

   (Greatest Element) $a + 1 = 1$ and thus $a \preceq 1$ for all $a \in \mathbb{B}^+$.

   (Separativity) Let $a, b \in \mathbb{B}^+$. We first verify that $a \perp b$ iff $a \cdot b = 0$:
   If $a \not\perp b$, then there is some $c \in \mathbb{B}^+$ such that $c \preceq a$ and $c \preceq b$, i.e. $c = c \cdot a = c \cdot b = c \cdot a \cdot b \preceq a \cdot b$ and hence $a \cdot b \neq 0$. On the other hand, if $a \cdot b \neq 0$, then $c := a \cdot b$ satisfies $c \preceq a$ and $c \preceq b$, witnessing that $a \not\perp b$.
   This easily implies that $(\mathbb{B}^+, \preceq)$ is separative: Let $a, b \in \mathbb{B}^+$ such that $a \not\preceq b$. Then $c := a \cdot (-b)$ satisfies $c \neq 0$, $c \preceq a$ and $c \perp b$.

3. Since $a + a + b = a + b$ and $b + a + b = a + b$, we see that $a + b$ is an upper bound of $\{a, b\}$.
   If c is another upper bound, then $a + b + c = a + c = c$ and thus $a + b \preceq c$.

4. Since $-(a \cdot b) = -a + -b$, this follows immediatly from 1 and 3 .

□

While we want to remove 0 when viewing $\mathbb{B}$ as a forcing notion (, i.e. partial order), this convention can be annoying if 0 appears as a lower (or upper) bound. In these situations, we consider $\mathbb{B}$ equipped with $\preceq$ defined as above and allow a or b to be 0. Since $0 + a = a$ for all $a \in B$, 0 is then the least element of $(B; \preceq)$.

Proposition 2.0.3. *Let $\mathbb{B}$ be a Boolean algebra and let $a, b, c, d, e \in \mathbb{B}$ with $a \preceq b$ and $c \preceq d$. Then*

1. $a + c \preceq b + d$,
2. $a \cdot c \preceq b \cdot d$,
3. $a \preceq b + e$ *and*
4. $a \cdot e \preceq b$.

Proof.

1. $a + c + b + d = \underbrace{a + b}_{=b} + \underbrace{c + d}_{=d} = b + d$ and thus $a + c \preceq b + d$.
2. $a \cdot c \cdot b \cdot d = \underbrace{a \cdot b}_{=a} \cdot \underbrace{c \cdot d}_{=c} = a \cdot c$ and thus $a \cdot c \preceq b \cdot d$.
3. Let $c := 0$, $d := e$ and use 1.
4. Let $c := e$, $d := 1$ and use 2.

□

Definition 2.0.4. *Let $\mathbb{B}$ be a Boolean algebra and let $X \subseteq \mathbb{B}$.*

- $\sum X = \sum_{x \in X} x$ *denotes the* supremum *of X and*
- $\prod X = \prod_{x \in X} x$ *denotes the* infimum *of X*

in $(B; \preceq)$ whenever they exist. To avoid confusion, let us explicitly state that

- $\sum \emptyset := 0$ *and*
- $\prod \emptyset := 1$.

We call $\mathbb{B}$ a complete *Boolean algebra iff $\sum X$ and $\prod X$ exist for all $X \subseteq \mathbb{B}$.*

If we want to stress that $\sum X$ ($\prod X$) is the supremum (infimum) of X in $\mathbb{B}$, we also write $\sum_{\mathbb{B}} X$ for $\sum X$ ($\prod_{\mathbb{B}} X$ for $\prod X$).

Proposition 2.0.4. *Let $\mathbb{B}$ be a complete Boolean algebra, let $X \subseteq \mathbb{B}$ and $b \in \mathbb{B}$. Then*

1. $-\sum X = \prod -X$,
2. $-\prod X = \sum -X$,
3. $b \cdot \sum X = \sum (b \cdot X)$ *and*
4. $b + \prod X = \prod (b + X)$,

where $-X := \{-x \mid x \in X\}$, $a \cdot X = \{a \cdot x \mid x \in X\}$ *and* $a + X = \{a + x \mid x \in X\}$.

Proof. 1. For all $a \in \mathbb{B}$ we have

$$\begin{aligned} a \preceq -\sum X &\text{ iff } -a \succeq \sum X \\ &\text{ iff } -a \succeq x \text{ for all } x \in X \\ &\text{ iff } a \preceq -x \text{ for all } x \in X \\ &\text{ iff } a \preceq \prod -X \end{aligned}$$

and thus $-\sum X = \prod -X$.

2. Using 1, we obtain $-\prod X = -(-\sum -X) = \sum -X$.

3. Since $b \cdot x \preceq b \cdot \sum X$ for all $x \in X$, we have $\sum(b \cdot X) \preceq b \cdot \sum X$. Conversely $x \preceq -b + x = -b + b \cdot x$ for all $x \in X$ and thus $\sum X \preceq -b + \sum(b \cdot X)$. Multiplying both sides by b yields

$$\begin{aligned} b \cdot \sum X &\preceq b \cdot \left(-b + \sum(b \cdot X)\right) \\ &= b \cdot (-b) + b \cdot \sum(b \cdot X) \\ &= b \cdot \sum(b \cdot X) \\ &\preceq \sum(b \cdot X), \end{aligned}$$

as desired.

4. By what we've already shown

$$\begin{aligned} -\left(b + \prod X\right) &= -b \cdot -\prod X \\ &= -b \cdot \sum -X \\ &= \sum\{-b \cdot -x \mid x \in X\} \\ &= \sum\{-(b + x) \mid x \in X\} \\ &= \sum -(b + X) \\ &= -\prod(b + X) \end{aligned}$$

and thus $b + \prod X = \prod(b + X)$.

□

Definition 2.0.5. *Let* $\mathbb{A} = (A; +_{\mathbb{A}}, \cdot_{\mathbb{A}}, 0_{\mathbb{A}}, 1_{\mathbb{A}}, -_{\mathbb{A}})$ *and* $\mathbb{B} = (B; +_{\mathbb{B}}, \cdot_{\mathbb{B}}, 0_{\mathbb{B}}, 1_{\mathbb{B}}, -_{\mathbb{B}})$ *be Boolean algebras. A map*

$$f \colon A \to B$$

is a homomorphism *from* $\mathbb{A}$ *to* $\mathbb{B}$ *(in symbols:* $f \colon \mathbb{A} \to \mathbb{B}$*) iff for all* $a, b \in A$ *the following hold*

- $f(0_{\mathbb{A}}) = 0_{\mathbb{B}}$,
- $f(1_{\mathbb{A}}) = 1_{\mathbb{B}}$,
- $f(a +_{\mathbb{A}} b) = f(a) +_{\mathbb{B}} f(b)$,
- $f(a \cdot_{\mathbb{A}} b) = f(a) \cdot_{\mathbb{B}} f(b)$,
- $f(-_{\mathbb{A}} a) = -_{\mathbb{B}} f(a)$.

If f is injective, we also call it an embedding*. An* isomorphism *is a surjective embedding.*
We call $\mathbb{A}$ a subalgebra *of $\mathbb{B}$ or say that $\mathbb{A}$ is* contained *in $\mathbb{B}$ (in symbols: $\mathbb{A} \sqsubseteq \mathbb{B}$) iff the inlusion map $\mathbb{A} \to \mathbb{B}, a \mapsto a$ is an embedding and we write $\mathbb{A} \cong \mathbb{B}$ iff there is an isomorphism $f\colon \mathbb{A} \to \mathbb{B}$.*

It is sometimes useful to know that $(\mathbb{B}; \preceq)$ contains all the information needed in order to recover the Boolean algebra $\mathbb{B}$. In fact, $0, 1, +, \cdot$ and $-$ are all first-order definable in $(\mathbb{B}; \preceq)$: $0, 1$ are the minimal/maximal elements of this order and for all $a, b, c \in \mathbb{B}$

$$a + b = c \leftrightarrow [a \preceq c \wedge b \preceq c \wedge (\forall x\colon a \preceq x \wedge b \preceq x \to c \preceq x)],$$

$$a \cdot b = c \leftrightarrow [c \preceq a \wedge c \preceq b \wedge (\forall x\colon x \preceq a \wedge x \preceq b \to x \preceq c)]$$

and

$$-a = c \leftrightarrow [a \cdot c = 0 \wedge a + c = 1].$$

This proves the following

Proposition 2.0.5. *Let $\mathbb{B} \subseteq \mathbb{C}$ be Boolean algebras such that $\preceq_{\mathbb{B}} \subseteq \preceq_{\mathbb{C}}$, $0_{\mathbb{B}} = 0_{\mathbb{C}}$ and $1_{\mathbb{B}} = 1_{\mathbb{C}}$. Then $\mathbb{B} \sqsubseteq \mathbb{C}$.*
More generally, let $\mathbb{B}, \mathbb{C}$ be Boolean algebras and let $f\colon \mathbb{B} \to \mathbb{C}$ be an injection such that $f(0_{\mathbb{B}}) = 0_{\mathbb{C}}$, $f(1_{\mathbb{B}}) = 1_{\mathbb{C}}$ and for all $a, b \in \mathbb{B}$: $a \preceq_{\mathbb{B}} b$ iff $f(a) \preceq_{\mathbb{C}} f(b)$. Then f is already an embedding.

Definition 2.0.6. *A homomorphism $f\colon \mathbb{A} \to \mathbb{B}$ between Boolean algebras $\mathbb{A}$ and $\mathbb{B}$ is* complete *iff*

- *$\mathbb{A}, \mathbb{B}$ are complete,*
- *$f(\sum_{\mathbb{A}} X) = \sum_{\mathbb{B}} f''X$ and*
- *$f(\prod_{\mathbb{A}} X) = \prod_{\mathbb{B}} f''X$*

for all $X \subseteq \mathbb{A}$.
We call $\mathbb{A}$ a complete subalgebra *of $\mathbb{B}$ or say that $\mathbb{A}$ is* completely contained *in $\mathbb{B}$ (in symbols: $\mathbb{A} \sqsubseteq_c \mathbb{B}$) iff the inclusion map $\mathbb{A} \to \mathbb{B}, a \mapsto a$ is a complete homomorphism.*

Proposition 2.0.6. *Let $\mathbb{A}, \mathbb{B}$ be complete Boolean algebras and let $f\colon \mathbb{A} \to \mathbb{B}$ be an isomorphism. Then f is complete.*

Proof. Let $X \subseteq \mathbb{A}$, let $x := \sum_{\mathbb{A}} X$ and fix $y \in \mathbb{A}$ such that $f(y) = \sum_{\mathbb{B}} f"X$. Since $\sum_{\mathbb{B}} f"X \preceq_{\mathbb{B}} f(x)$, we have $y \preceq x$. On the other hand

$$\begin{aligned} f(x \cdot y) &= f(x) \cdot f(y) \\ &= f(\textstyle\sum_{\mathbb{A}} X) \cdot \sum_{\mathbb{B}} f"X \\ &= \textstyle\sum_{\mathbb{B}} \underbrace{f(\sum_{\mathbb{A}} X) f"X}_{=f(\sum_{\mathbb{A}} X)} \\ &= f(x) \end{aligned}$$

yields $x \preceq y$. Hence $f(\sum_{\mathbb{A}} X) = \sum_{\mathbb{B}} f"X$ and by Proposition 2.0.4, we now obtain

$$\begin{aligned} f(\textstyle\prod_{\mathbb{A}} X) &= f(-\textstyle\sum_{\mathbb{A}} -X) \\ &= -f(\textstyle\sum_{\mathbb{A}} -X) \\ &= -\textstyle\sum_{\mathbb{B}} f" - X \\ &= \textstyle\prod_{\mathbb{B}} f"X. \end{aligned}$$

□

While complete Boolean algebras have their advantages when dealing with abstract forcing techniques, in practise we usually think of forcings as (separative) partially ordered sets. It is well known that starting from a (separative) partially ordered set $\mathbb{P}$, its *Boolean completion* $\mathbb{B}$ generates the same generic extensions as $\mathbb{P}$. The details of this aren't important for our purposes, but can be found in [Jec06, ch.14]. We shall however note that every separative partially ordered set admits, up to isomorphism, a unqiue Boolean completion.

Definition 2.0.7. *Let $\mathbb{P} = (P; \leq)$ be a separative partially ordered set and let $\mathbb{B}$ be a complete Boolean algebra such that*

a) $\mathbb{P} \subseteq \mathbb{B}^+$,

b) $\leq = \preceq_{\mathbb{B}} \restriction \mathbb{P}$ *and*

c) $\mathbb{P}$ *is dense in* $\mathbb{B}$*, i.e. for every* $b \in \mathbb{B}^+$ *there is some* $p \in \mathbb{P}$ *such that* $p \preceq_{\mathbb{B}} b$.

We then call $\mathbb{B}$ a (the) Boolean completion *of $\mathbb{P}$.*

Theorem 2.1. *Let $\mathbb{P} = (P; \leq)$ be a separative partially ordered set. Then there is a complete Boolean algebra $\mathbb{B}$ such that*

a) $\mathbb{P} \subseteq \mathbb{B}^+$,

b) $\leq = \preceq_{\mathbb{B}} \restriction \mathbb{P}$ *and*

c) $\mathbb{P}$ *is dense in* $\mathbb{B}$.

Moreover, $\mathbb{B}$ is unique up to isomorphism. More precisely: Let $\mathbb{C}$ be another complete Boolean algebra such that $\mathbb{P} \subseteq \mathbb{C}^+$, $\leq = \preceq_{\mathbb{C}} \restriction \mathbb{P}$ and such that $\mathbb{P}$ is dense in $\mathbb{C}$. Then $\mathbb{B} \cong \mathbb{C}$.

Proof. [Jec06, Theorem 14.10]. □

Lemma 2.1.1. *If $f\colon \mathbb{A} \to \mathbb{B}$ is a homomorphism and $\mathbb{A}$, $\mathbb{B}$ are complete Boolean algebras satisfying*

1. $f(\sum_{\mathbb{A}} X) \preceq_{\mathbb{B}} \sum_{\mathbb{B}} f''X$ *for all* $X \subseteq \mathbb{A}$ *or*
2. $f(\prod_{\mathbb{A}} X) \succeq_{\mathbb{B}} \prod_{\mathbb{B}} f''X$ *for all* $X \subseteq \mathbb{A}$,

then f is complete.

Proof. First note that $\sum_{\mathbb{B}} f''X \preceq_{\mathbb{B}} f(\sum_{\mathbb{A}} X)$ and $\prod_{\mathbb{B}} f''X \succeq_{\mathbb{B}} f(\prod_{\mathbb{A}} X)$ always hold true for $X \subseteq \mathbb{A}$ (regardless of whether 1. or 2. are satisfied).
To see this, let $b \prec_{\mathbb{B}} \sum_{\mathbb{B}} f''X$. Then there is some $x \in X$ with $b \prec_{\mathbb{B}} f(x) \preceq_{\mathbb{B}} f(\sum_{\mathbb{A}} X)$.
On the other hand, if $b \succ_{\mathbb{B}} \prod_{\mathbb{B}} f''X$, then there is some $x \in X$ with $b \succ_{\mathbb{B}} f(x) \succeq_{\mathbb{B}} f(\prod_{\mathbb{A}} X)$. We may thus replace $\preceq_{\mathbb{B}}$ in 1. and $\succeq_{\mathbb{B}}$ in 2. with equality.

1. Let $X \subseteq \mathbb{A}$. Then
$$\begin{aligned} f(\prod\nolimits_{\mathbb{A}} X) &= f(-\sum\nolimits_{\mathbb{A}} -X) \\ &= -\sum\nolimits_{\mathbb{B}} f'' - X \\ &= -\sum\nolimits_{\mathbb{B}} -f''X \\ &= \prod\nolimits_{\mathbb{B}} f''X. \end{aligned}$$
2. Likewise we obtain
$$\begin{aligned} f(\sum\nolimits_{\mathbb{A}} X) &= f(-\prod\nolimits_{\mathbb{A}} -X) \\ &= -\prod\nolimits_{\mathbb{B}} f'' - X \\ &= -\prod\nolimits_{\mathbb{B}} -f''X \\ &= \sum\nolimits_{\mathbb{B}} f''X \end{aligned}$$
for all $X \subseteq \mathbb{A}$.

□

It is possible to have two complete Boolean algebras $\mathbb{A} \sqsubseteq \mathbb{B}$ such that $\mathbb{A}$ is *not* completely contained in $\mathbb{B}$, i.e. there is some $X \subseteq A$ with $\sum_{\mathbb{A}} X \succ \sum_{\mathbb{B}} X$.

Definition 2.1.1. *Let $\mathbb{B}$ be a complete Boolean algebra and let $\mathbb{B} \sqsubseteq \mathbb{C}$ be another (not necessarily complete) Boolean algebra. We define the* canonical projection *by*

$$h_{\mathbb{B},\mathbb{C}}\colon \mathbb{C} \to \mathbb{B}, c \mapsto \prod\{b \in \mathbb{B} \mid c \preceq b\}.$$

In general, the canonical projection is not a homomorphism between Boolean algebras (not even in the finite case), but it still admits some desirable properties that make it quite useful when dealing with forcing iterations:

Proposition 2.1.1. *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$ be complete Boolean algebras and let $h_{\mathbb{B},\mathbb{C}}$ be the canonical projection. Then for all $b \in \mathbb{B}$, all $c, d \in \mathbb{C}$ and all $X \subseteq \mathbb{C}$*

1. $h_{\mathbb{B},\mathbb{C}}(b) = b$,
2. $h_{\mathbb{B},\mathbb{C}}(b \cdot c) = b \cdot h_{\mathbb{B},\mathbb{C}}(c)$
3. $h_{\mathbb{B},\mathbb{C}}(c) = 0$ *iff* $c = 0$ *and*
4. $h_{\mathbb{B},\mathbb{C}}(\sum_{\mathbb{C}} X) = \sum_{\mathbb{B}} h_{\mathbb{B},\mathbb{C}}{}''X$.

As will be immediate from our proof, the assumption that $\mathbb{C}$ is complete and $\mathbb{B}$ is completely contained in $\mathbb{C}$ is only needed in order to conclude item 4. So even in cases where $\mathbb{B} \sqsubseteq \mathbb{C}$ and $\mathbb{B}$ is complete, items 1.-3. hold true.

Proof. 1.
$$h_{\mathbb{B},\mathbb{C}}(b) = \prod\{b' \in \mathbb{B} \mid b \preceq b\} = b.$$

2. For all $b' \in \mathbb{B}$ we note that $b \cdot c \preceq b' \to b \cdot c \preceq b \cdot b' \preceq b'$ and thus

$$\begin{aligned} h_{\mathbb{B},\mathbb{C}}(b \cdot c) &= \prod\{b' \in \mathbb{B} \mid b \cdot c \preceq b'\} \\ &= \prod\{b \cdot b' \in \mathbb{B} \mid b \cdot c \preceq b'\} \\ &= b \cdot \prod\{b' \in \mathbb{B} \mid b \cdot c \preceq b'\} \\ &= b \cdot h_{\mathbb{B},\mathbb{C}}(c). \end{aligned}$$

3. Clearly $h_{\mathbb{B},\mathbb{C}}(0) = 0$. Conversely

$$\begin{aligned} h_{\mathbb{B},\mathbb{C}}(c) &= \prod\{b' \in \mathbb{B} \mid c \preceq b'\} \\ &\succeq \prod\{c' \in \mathbb{C} \mid c \preceq c'\} \\ &= c, \end{aligned}$$

hence $h_{\mathbb{B},\mathbb{C}}(c) \neq 0$ whenever $c \neq 0$.

4. Note that for $b' \in \mathbb{B}$

$$\begin{aligned}\textstyle\sum_{\mathbb{C}} X \preceq b' &\leftrightarrow \forall x \in X\colon x \preceq b' \\ &\leftrightarrow \forall x \in X\colon h_{\mathbb{B},\mathbb{C}}(x) \preceq b' \\ &\leftrightarrow \textstyle\sum_{\mathbb{B}} h_{\mathbb{B},\mathbb{C}}\text{''}X \preceq b'\end{aligned}$$

and therefore

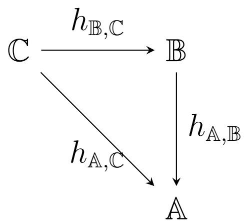
$$\begin{aligned}h_{\mathbb{B},\mathbb{C}}\left(\textstyle\sum_{\mathbb{C}} X\right) &= \textstyle\prod\{b' \in \mathbb{B} \mid \sum_{\mathbb{C}} X \preceq b'\} \\ &= \textstyle\prod\{b' \in \mathbb{C} \mid \sum_{\mathbb{B}} h_{\mathbb{B},\mathbb{C}}\text{''}X \preceq b'\} \\ &= \textstyle\sum_{\mathbb{B}} h_{\mathbb{B},\mathbb{C}}\text{''}X.\end{aligned}$$

□

Proposition 2.1.2. *Let $\mathbb{A} \sqsubseteq_c \mathbb{B} \sqsubseteq_c \mathbb{C}$ and let $h_{\mathbb{A},\mathbb{B}}, h_{\mathbb{B},\mathbb{C}}$ and $h_{\mathbb{B},\mathbb{C}}$ be the associated canonical projections. Then*

$$h_{\mathbb{A},\mathbb{C}} = h_{\mathbb{A},\mathbb{B}} \circ h_{\mathbb{C},\mathbb{B}}.$$

resulting in the following commutative diagram



Proof. For all $c \in \mathbb{C}$, we have

$$\begin{aligned}h_{\mathbb{A},\mathbb{C}}(c) &= \textstyle\prod\{a \in \mathbb{A} \mid c \preceq a\} \\ &\preceq \textstyle\prod\{a \in \mathbb{A} \mid \prod\{b \in \mathbb{B} \mid c \preceq b\} \preceq a\} \\ &= h_{\mathbb{A},\mathbb{B}}(\textstyle\prod\{b \in \mathbb{B} \mid c \preceq b\}) \\ &= h_{\mathbb{A},\mathbb{B}} \circ h_{\mathbb{B},\mathbb{C}}(c).\end{aligned}$$

On the other hand, for all $a' \in A$ with

$$a' \prec h_{\mathbb{A},\mathbb{B}} \circ h_{\mathbb{B},\mathbb{C}}(c) = \textstyle\prod\{a \in \mathbb{A} \mid \prod\{b \in \mathbb{B} \mid c \preceq b\} \preceq a\},$$

we have

$$\begin{aligned}a' &\prec \textstyle\prod\{b \in \mathbb{B} \mid c \preceq b\} \\ &\preceq \textstyle\prod\{a \in \mathbb{A} \mid c \preceq\} \\ &= h_{\mathbb{A},\mathbb{C}}(c).\end{aligned}$$

□

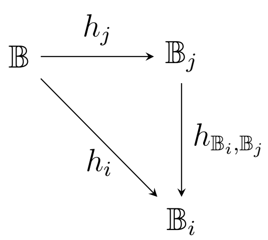
Proposition 2.1.3. *Let* $(\mathbb{B}_i \mid i < \alpha)$ *be a sequence of complete Boolean algebras such that* $\mathbb{B}_i \sqsubseteq_c \mathbb{B}_j$ *for all* $i \leq j < \alpha$*. Then* $\mathbb{B} := (B; 0_{\mathbb{B}}, 1_{\mathbb{B}}, +_{\mathbb{B}}, \cdot_{\mathbb{B}}, -_{\mathbb{B}})$ *with*

- $B := \bigcup_{i<\alpha} \mathbb{B}_i$*,*
- $0_{\mathbb{B}} := 0_{\mathbb{B}_0}$*,* $1_{\mathbb{B}} := 1_{\mathbb{B}_0}$*,*
- $+_{\mathbb{B}} \colon B \times B \to B, (x,y) \mapsto x +_{\mathbb{B}_i} y$ *for some* $i < \alpha$ *with* $x, y \in \mathbb{B}_i$*,*
- $\cdot_{\mathbb{B}} \colon B \times B \to B, (x,y) \mapsto x \cdot_{\mathbb{B}_i} y$ *for some* $i < \alpha$ *with* $x, y \in \mathbb{B}_i$*,*
- $-_{\mathbb{B}} \colon B \to B, x \mapsto -_{\mathbb{B}_i} x$ *for some* $i < \alpha$ *with* $x \in \mathbb{B}_i$

is a well-defined Boolean algebra and the canonical projections

$$h_i \colon \mathbb{B} \to \mathbb{B}_i, x \mapsto \prod\{b \in \mathbb{B}_i \mid x \preceq b\}$$

are well-defined and satisfy $h_{\mathbb{B}_i,\mathbb{B}_j} \circ h_j = h_i$ *for all* $i \leq j < \alpha$*. We thus have the following commutative diagram*



Proof. In order to see that $\mathbb{B}$ is a Boolean algebra, it suffices to note that its operations are well-defined. This in turn is an immediate consequence of that fact that $\mathbb{B}_i \sqsubseteq \mathbb{B}_j$ for all $i \leq j < \alpha$.
To see that h_i is well-defined, fix some $x \in \mathbb{B}$ and let $j < \alpha$ be minimal such that $x \in \mathbb{B}_j$. If $j \leq i$, then $h_i(x) = x$ and otherwise $h_i(x) = \prod\{b \in \mathbb{B}_i \mid x \preceq b\} = h_{\mathbb{B}_i,\mathbb{B}_j}(x)$. This is independent of j, because for all $j \leq k < \alpha$:

$$\begin{aligned} h_{\mathbb{B}_i,\mathbb{B}_k}(x) &= h_{\mathbb{B}_i,\mathbb{B}_j} \circ h_{\mathbb{B}_j,\mathbb{B}_k}(x) \\ &\overset{h_{\mathbb{B}_j,\mathbb{B}_k} \restriction \mathbb{B}_j = \mathrm{id}}{=} h_{\mathbb{B}_i,\mathbb{B}_j}(x). \end{aligned}$$

This also yields $h_i = h_{\mathbb{B}_i,\mathbb{B}_j} \circ h_j$ for all $i \leq j < \alpha$. □

2.2 Filters and Quotients

Definition 2.1.2. *Let $\mathbb{B}$ be a Boolean algebra. A nonempty subset $F \subseteq \mathbb{B}$ is a* filter *iff*

- $0 \notin F$,
- $a \in F$ *and* $a \preceq b \in \mathbb{B} \to b \in F$,
- $a, b \in F \to a \cdot b \in F$.

If additionally $a \in F$ or $-a \in F$ for all $a \in \mathbb{B}$, then F is an ultrafilter.

Notation 2.1.1. *Motivated by the field of sets $(\mathcal{P}(X); \emptyset, X, \cup, \cap, ^{\complement})$ for nonempty sets X, we introduce the following notation: Let $\mathbb{B}$ be a Boolean algebra and let $a, b \in \mathbb{B}$. Then*

$$\Delta(a, b) := a\Delta b := a \cdot -b + b \cdot -a$$

denotes the symmetric difference *of a and b, where $a\Delta b$ is used only when the order of operations is unambiguous.*

Proposition 2.1.4. *Let $\mathbb{B}$ be a Boolean algebra and let $F \subseteq \mathbb{B}$ be a filter. For all $a, b \in \mathbb{B}$ let*

$$a \sim_F b :\leftrightarrow -(a\Delta b) \in F.$$

This defines an equivalence relation and for all $a, b, c, d \in \mathbb{B}$, $f \in F$ with $a \sim_F b$ and $c \sim_F d$ the following hold

1. $a + c \sim_F b + d$,
2. $a \cdot c \sim_F b \cdot d$,
3. $-a \sim_F -b$ *and*
4. $f \sim_F 1$.

We write $a/F := \{b \in \mathbb{B} \mid a \sim_F b\}$ for the equivalence class of a and $B/F := \{a/F \mid a \in \mathbb{B}\}$.

Proof. Let $a, b, c \in \mathbb{B}$. We first show, that $\sim_F$ is an equivalence relation:

(Reflexivity) $-(a\Delta a) = -(a \cdot -a + a \cdot -a) = 1 \in F$ and thus $a \sim_F a$.

(Symmetry) $a\Delta b = a \cdot -b + b \cdot -a = b\Delta a$ and thus $a \sim_F b$ iff $b \sim_F a$.

(Transitivity) Let $a \sim_F b$ and $b \sim_F c$, i.e. $-(a\Delta b), -(b\Delta c) \in F$. We claim that $-(a\Delta b) \cdot -(a\Delta c) \leq -(a\Delta c)$ or equivalently $a\Delta c \leq (a\Delta b) + (b\Delta c)$. In fact

$$\begin{aligned} a\Delta c &= a \cdot -c + c \cdot -a \\ &= a \cdot -c \cdot (b + -b) + c \cdot -a \cdot (b + -b) \\ &= \underbrace{a \cdot -c \cdot b}_{②} + \underbrace{a \cdot -c \cdot -b}_{①} + \underbrace{c \cdot -a \cdot -b}_{②} + \underbrace{c \cdot -a \cdot b}_{①} \\ &\preceq \underbrace{(a\Delta b)}_{①} + \underbrace{(b\Delta c)}_{②}. \end{aligned}$$

Now $-(a\Delta b) \cdot -(b\Delta c) \preceq -(a\Delta c)$ yields $-(a\Delta c) \in F$ and thus $a \sim_F c$.

The remaining identities are proved similarly: Let $a \sim_F b$ and $c \sim_F d$. Then

1.

$$\begin{aligned} \Delta(a + c, b + d) &= (a + c) \cdot -(b + d) + (b + d) \cdot -(a + c) \\ &= \underbrace{a \cdot -b \cdot -d}_{①} + \underbrace{c \cdot -b \cdot -d}_{②} + \underbrace{b \cdot -a \cdot -c}_{①} + \underbrace{d \cdot -a \cdot -c}_{②} \\ &\preceq \underbrace{(a\Delta b)}_{①} + \underbrace{c\Delta d}_{②} \end{aligned}$$

Thus $-(a\Delta b) \cdot -(c\Delta d) \preceq -\Delta(a + c, b + d)$ and consequently $a + c \sim_F b + d$.

2.

$$\begin{aligned} \Delta(a \cdot c, b \cdot d) &= a \cdot c \cdot -(b \cdot d) + b \cdot d \cdot -(a \cdot c) \\ &= \underbrace{a \cdot c \cdot -b}_{①} + \underbrace{a \cdot c \cdot -d}_{②} + \underbrace{b \cdot d \cdot -a}_{①} + \underbrace{b \cdot d \cdot -c}_{②} \\ &\preceq \underbrace{(a\Delta b)}_{①} + \underbrace{(c\Delta d)}_{②} \end{aligned}$$

yielding $a \cdot c \sim_F b \cdot d$.

3.

$$\begin{aligned} \Delta(-a, -b) &= -a \cdot -(-b) + -b \cdot -(-a) \\ &= -a \cdot b + -b \cdot a \\ &= \Delta(a, b) \end{aligned}$$

and therefore $-a \sim_F -b$.

4. For $f \in F$, we have

$$\begin{aligned} -\Delta(f, 1) &= -(f \cdot 0 + (-f)) \\ &= -(-f) \\ &= f \in F, \end{aligned}$$

i.e. $f \sim_F 1$.

□

Proposition 2.1.5. *Let $\mathbb{B}$ be a Boolean algebra and let $F \subseteq \mathbb{B}$ be a filter. For all $a, b \in \mathbb{B}$ we have*

$$a/F \preceq b/F \text{ iff } -a + b = -(a \cdot -b) \in F.$$

Proof. We have

$$\begin{aligned} a/F \preceq b/F &\leftrightarrow a \cdot (-b)/F = 0/F \\ &\leftrightarrow -\Delta(a \cdot -b, 0) \in F \\ &\leftrightarrow -(a \cdot (-b) \cdot 1 + 0 \cdot -(a \cdot (-b))) \in F \\ &\leftrightarrow -(a \cdot (-b)) = -a + b \in F. \end{aligned}$$

□

Definition 2.1.3. *Let $\mathbb{B} = (B; 0, 1, +, \cdot, -)$ be a Boolean algebra and let $F \subseteq \mathbb{B}$ be a filter. We define $\mathbb{B}/F := (B/F; 0', 1', +', \cdot', -')$ by letting*

- $0' := 0/F$,
- $1' := 1/F$,
- $+' \colon B/F \times B/F \to B/F, (a/F, b/F) \mapsto (a + b)/F$,
- $\cdot' \colon B/F \times B/F \to B/F, (a/F, b/F) \mapsto (a \cdot b)/F$ *and*
- $-' \colon B/F \to B/F, a/F \mapsto (-a)/F$.

By Proposition 2.1.4 this yields a well-defined Boolean algebra, the quotient *of $\mathbb{B}$ by F. The canonical homomorphism*

$$\mathbb{B} \to \mathbb{B}/F, \; a \mapsto a/F$$

is called quotient map.

Definition 2.1.4. *Given* $\mathbb{B} \sqsubseteq \mathbb{C}$ *and a filter* $G \subseteq \mathbb{B}$*, the* upward closure of G in $\mathbb{C}$ *defined by*

$$G_{\uparrow} := \{c \in \mathbb{C} \mid \exists b \in G \colon b \preceq c\}$$

is the $\subseteq$*-smallest filter on* $\mathbb{C}$ *containing* G *and we will often write* ${}^{\mathbb{C}}\!/_{G}$ *instead of* ${}^{\mathbb{C}}\!/_{G_{\uparrow}}$ *in these cases.*

Notation 2.1.2. *Unless stated otherwise,* $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{A}_i$ *and* $\mathbb{B}_i$*, for some index* i*, will from now on always denote complete Boolean algebras.*

2.3 Boolean-Valued Models and Generic Extensions

Definition 2.1.5. *A* $\mathbb{B}$-valued model *is a tripel* $\mathcal{U} = (U; \| = \|, \| \in \|)$*, where*

- U *is a nonempty set or class,*
- $\| = \| \colon U \to \mathbb{B}, (x, y) \mapsto \|x = y\|$ *is the Boolean evaluation of* $x = y$ *and*
- $\| \in \| \colon U \to \mathbb{B}, (x, y) \mapsto \|x \in y\|$ *is the Boolean evaluation of* $x \in y$*,*

such that for all $v, w, x, y, z \in U$

1. $\|x = x\| = 1$,
2. $\|x = y\| = \|y = x\|$,
3. $\|x = y\| \cdot \|y = z\| \preceq \|x = z\|$ *and*
4. $\|x \in y\| \cdot \|v = x\| \cdot \|w = y\| \preceq \|v \in w\|$.

We say that $\mathcal{U}$ *is an* $\mathbb{B}$*-valued* identity *model iff it additionally satisfies*

5. $\|x = y\| = 1 \to x = y$

and we also refer to $\mathcal{U}$ *as a Boolean valued (identity) model, if we don't want to specify the underlying Boolean algebra.*
Given the Boolean evaluations of atomic formulas, we can then define $\|\phi(x_1, \ldots, x_n)\|$ *for general* $\mathcal{L}_{\{\in\}}$*-formulas and* $x_1, \ldots, x_n \in U$ *by induction on their complexity as*

follows:

$$\begin{aligned}
\|\neg\psi(x_1,\dots,x_n)\| &:= -\|\psi(x_1,\dots,x_n)\| \\
\|(\psi_1 \wedge \psi_2)(x_1,\dots,x_n)\| &:= \|\psi_1(x_1,\dots,x_n)\| \cdot \|\psi_2(x_1,\dots,x_n)\| \\
\|(\psi_1 \vee \psi_2)(x_1,\dots,x_n)\| &:= \|\psi_1(x_1,\dots,x_n)\| + \|\psi_2(x_1,\dots,x_n)\| \\
\|(\psi_1 \to \psi_2)(x_1,\dots,x_n)\| &:= \|(\neg\psi_1 \vee \psi_2)(x_1,\dots,x_n)\| \\
\|(\psi_1 \leftrightarrow \psi_2)(x_1,\dots,x_n)\| &:= \|((\psi_1 \to \psi_2) \wedge (\psi_2 \to \psi_1))(x_1,\dots,x_n)\| \\
\|\exists v\psi(v,x_1,\dots,x_n)\| &:= \textstyle\sum_{x\in U} \|\psi(x,x_1,\dots,x_n)\| \\
\|\forall v\psi(v,x_1,\dots,x_n)\| &:= \textstyle\prod_{x\in U} \|\psi(x,x_1,\dots,x_n)\|.
\end{aligned}$$

We say that $\phi(x_1,\dots,x_n)$ is valid in $\mathcal{U}$ iff $\|\phi(x_1,\dots,x_n)\| = 1$.
Finally, we say that $\mathcal{U}$ is full *iff for any $\mathcal{L}_{\{\in\}}$formula ϕ and all $x_1,\dots,x_n$ there is an $x \in U$ s.t.*

$$\|\exists v\phi(v,x_1,\dots,x_n)\| = \|\phi(x,x_1,\dots,x_n)\|.$$

The definition of a Boolean valued models guarantees that the axioms of predicate calculus are valid and applying the rule of interference to a valid sentence again results in a valid sentence. Therefore, given a Boolean valued model $\mathcal{U}$ in which every axiom of ZFC is valid and a set theoretical formula ϕ with $\|\phi\| \neq 0$, we may conclude the consistency of ϕ relative to ZFC: If ϕ were not consistent relative to ZFC, then ZFC proves $\neg\phi$ and thus $\|\neg\phi\| = -\|\phi\| = 1$ yields $\|\phi\| = 0$.

To simplify our notation, we shall assume that all of our Boolean valued models are in fact identity models. Since this can always be achieved by (externally) factoring a given Boolean valued model $(U; \| = \|, \| \in \|)$ via the equivalence relation given by $x \sim y :\leftrightarrow \|x = y\| = 1$, this is completely harmless.
The following Lemma provides a useful method to convert full Boolean valued models into conventional, 2 valued, models.

Lemma 2.1.2. *Let F be an ultrafilter on $\mathbb{B}$ and let $\mathcal{U} = (U; \| = \|, \| \in \|)$ be a full $\mathbb{B}$-valued model. Define an equivalence relation*

$$x \equiv y :\leftrightarrow \|x = y\| \in F$$

for all $x, y \in U$ and let $[x] := \{y \in U \mid x \equiv y\}$ denote the equivalence class of x. [2] *Let $U/_{\equiv} := \{[x] \mid x \in U\}$. Then*

$$[x] \ E \ [y] :\leftrightarrow \|x \in y\| \in F$$

[2] In case that $[x]$ is a proper class, we use Scott's trick and replace $[x]$ by $[x] \cap V_\alpha$, where α is least such that $[x] \cap V_\alpha \neq \emptyset$.

yields a well-defined binary relation on ${}^{U}\!/_{\equiv}$*, such that for all* $\mathcal{L}_{\{\in\}}$*-formulas* ϕ *and* $x_1, \ldots, x_n \in U$

$$\left({}^{U}\!/_{\equiv}; E\right) \models \phi([x_1], \ldots, [x_n]) \text{ if and only if } \|\phi(x_1, \ldots, x_n)\| \in F.$$

We also write $\mathcal{U}/F := \left({}^{U}\!/_{\equiv}; E\right)$ *for this model.*

Proof. We first verify that $\equiv$ is an equivalence relation. Let $x, y, z \in U$.

(Reflexivity) $\|x = x\| = 1 \in F$ yields $x \equiv x$.

(Symmetry) Since $\|x = y\| = \|y = x\|$, we have $x \equiv y$ iff $y \equiv x$.

(Transitivity) Suppose that $x \equiv y$ and $y \equiv z$. Then $\|x = y\|, \|y = z\| \in F$ implies $\|x = y\| \cdot \|y = z\| \in F$. Since $\|x = y\| \cdot \|y = z\| \preceq \|x = z\|$, this yields $\|x = z\| \in F$, i.e. $x \equiv z$.

Next, we check that E is well-defined. Suppose that $[x]E[y]$, $x \equiv x'$ and $y \equiv y'$. Then $\|x \in y\|, \|x = x'\|, \|y = y'\| \in F$ and thus $\|x \in y\| \cdot \|x = x'\| \cdot \|y = y'\| \in F$. Since $\|x' \in y'\| \succeq \|x \in y\| \cdot \|x = x'\| \cdot \|y = y'\|$, this implies $\|x' \in y; \| \in F$ and thus $[x']E[y']$.
Finally, let's prove

$$\left({}^{U}\!/_{\equiv}; E\right) \models \phi([x_1], \ldots, [x_n]) \text{ if and only if } \|\phi(x_1, \ldots, x_n)\| \in F.$$

for all $\mathcal{L}_{\{\in\}}$ formulae ϕ and $x_1, \ldots, x_n \in {}^{U}\!/_{\equiv}$ by induction on the complexity of ϕ.
If ϕ is an atomic formula, this follows immediatetly from the definition of $\equiv$ and E.
Suppose that $\phi = \neg\psi$ and the claim holds for ψ. Then, since F is an ultrafilter,

$$\begin{aligned} \left({}^{U}\!/_{\equiv}; E\right) \models \phi([x_1], \ldots, [x_n]) &\text{ iff } \left({}^{U}\!/_{\equiv}; E\right) \models \neg\psi([x_1], \ldots, [x_n]) \\ &\text{ iff } \left({}^{U}\!/_{\equiv}; E\right) \not\models \psi([x_1], \ldots, [x_n]) \\ &\text{ iff } \|\psi(x_1, \ldots, x_n)\| \notin F \\ &\text{ iff } -\|\psi(x_1, \ldots, x_n)\| = \|\neg\psi(x_1, \ldots, x_n)\| \in F \\ &\text{ iff } \|\phi(x_1, \ldots, x_n)\| \in F. \end{aligned}$$

In case that $\phi = \psi \wedge \chi$ and the claim holds for ψ and ψ, we have

$$\begin{aligned} \left({}^{U}\!/_{\equiv}; E\right) \models \phi([x_1], \ldots, [x_n]) &\text{ iff } \left({}^{U}\!/_{\equiv}; E\right) \models \psi([x_1], \ldots, [x_n]) \wedge \left({}^{U}\!/_{\equiv}; E\right) \models \chi([x_1], \ldots, [x_n]) \\ &\text{ iff } \|\psi(x_1, \ldots, x_n)\| \in F \wedge \|\chi(x_1, \ldots, x_n)\| \in F \\ &\text{ iff } \|\psi(x_1, \ldots, x_n)\| \cdot \|\chi(x_1, \ldots, x_n)\| = \|\phi(x_1, \ldots, x_n)\| \in F. \end{aligned}$$

The only remaining case is $\phi = \exists v\psi$, where ψ satisfies the claim. Since $\mathcal{U}$ is full, we may fix some $x \in U$ with $\|\exists v\psi(v, x_1, \ldots, x_n)\| = \|\psi(x, x_1, \ldots, x_n)\|$. Since $\|\psi(y, x_1, \ldots, x_n)\| \preceq \|\exists v\psi(v, x_1, \ldots, x_n)\|$, we now have

$$\begin{aligned}\left({}^{U}\!/_{\equiv}; E\right) \models \phi([x_1], \ldots, [x_n]) &\text{ iff } \exists v \in U \left({}^{U}\!/_{\equiv}; E\right) \models \psi([v], [x_1], \ldots, [x_n]) \\ &\text{ iff } \exists v \in U \|\psi(v, x_1, \ldots, x_n)\| \in F \\ &\text{ iff } \|\psi(x, x_1, \ldots, x_n\| = \|\exists v\psi(v, x_1, \ldots, x_n)\| \in F.\end{aligned}$$

□

Definition 2.1.6. *We define a class $V^{\mathbb{B}}$ of $\mathbb{B}$-names $\dot{x}$ by*

1. $V_0^{\mathbb{B}} := \emptyset$,
2. $V_{\alpha+1}^{\mathbb{B}} := \{\dot{x} \colon \operatorname{dom}(\dot{x}) \to \mathbb{B} \mid \operatorname{dom}(\dot{x}) \subseteq V_\alpha^{\mathbb{B}}\}$,
3. $V_\lambda^{\mathbb{B}} := \bigcup_{\beta<\lambda} V_\beta^{\mathbb{B}}$ *for limit ordinals* λ *and*
4. $V^{\mathbb{B}} := \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha^{\mathbb{B}}$.

For each $\dot{x} \in V^{\mathbb{B}}$ we define its rank as $\rho(\dot{x}) := \min\{\alpha \in \mathrm{Ord} \mid \dot{x} \in V_{\alpha+1}^{\mathbb{B}}\}$ and for $\dot{x}, \dot{y} \in V^{\mathbb{B}}$ we define

1. $\|\dot{x} \in \dot{y}\| := \sum_{\dot{z} \in \operatorname{dom}(\dot{y})}(\|\dot{x} = \dot{z}\| \cdot \dot{y}(\dot{z}))$,
2. $\|\dot{x} \subseteq \dot{y}\| := \prod_{\dot{z} \in \operatorname{dom}(\dot{x})}(\dot{x}(\dot{z}) \Rightarrow \|\dot{z} \in \dot{y}\|)$ *and*
3. $\|\dot{x} = \dot{y}\| := \|\dot{x} \subseteq \dot{y}\| \cdot \|\dot{y} \subseteq \dot{x}\|$

by induction on $(\rho(\dot{x}), \rho(\dot{y}))$ under the canonical well-ordering of Ord × Ord, *where $a \Rightarrow b := -a + b$ for all $a, b, \in \mathbb{B}$.*
We call $(V^{\mathbb{B}}; \| = \|, \| \in \|)$ the maximal $\mathbb{B}$-valued model *and often identify it with its underlying universe $V^{\mathbb{B}}$.*

Theorem 2.2. *$V^{\mathbb{B}}$ is a full $\mathbb{B}$-valued model in which every axiom of* ZFC *is valid.*

Proof. [Jec06, p. 209 ff]. □

Proposition 2.2.1. *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$ be complete Boolean algebras. Then there is an injection*

$$i \colon V^{\mathbb{B}} \to V^{\mathbb{C}}$$

such that

$$\|\phi(\dot{x}_1, \ldots, \dot{x}_n)\|_{\mathbb{B}} = \|\phi(i(\dot{x}_1), \ldots, i(\dot{x}_n))\|_{\mathbb{C}}$$

for all $\dot{x}_1, \ldots, \dot{x}_n \in V^{\mathbb{B}}$ and all Δ_1-formulae ϕ.

Proof. By induction on $\rho(\dot{x})$, we define

$$i(\dot{x})\colon \{i(\dot{y}) \mid \dot{y} \in \operatorname{dom}(\dot{x})\} \to \mathbb{C},\ i(\dot{y}) \mapsto \|\dot{y} \in \dot{x}\|_{\mathbb{B}}.$$

Let $\dot{x}, \dot{y} \in V^{\mathbb{B}}$. We prove $\|i(\dot{x}) \in i(\dot{y})\|_{\mathbb{C}} = \|\dot{x} \in \dot{y}\|_{\mathbb{B}}$ and $\|i(\dot{x}) = i(\dot{y})\|_{\mathbb{C}} = \|\dot{x} = \dot{y}\|_{\mathbb{B}}$ by induction on $\rho(\dot{x})$ and $\rho(\dot{y})$. We have

$$\begin{aligned}\|i(\dot{x}) \in i(\dot{y})\|_{\mathbb{C}} &= \sum\{\|i(\dot{x}) = \dot{c}\|_{\mathbb{C}} \cdot i(\dot{y})(\dot{c}) \mid \dot{c} \in \operatorname{dom}(i(\dot{y}))\} \\ &= \sum\{\underbrace{\|i(\dot{x}) = i(\dot{z})\|_{\mathbb{C}}}_{=\|\dot{x}=\dot{z}\|_{\mathbb{B}}} \cdot \underbrace{i(\dot{y})(i(\dot{z}))}_{=\|\dot{z}\in\dot{y}\|_{\mathbb{B}}} \mid \dot{z} \in \operatorname{dom}(\dot{y})\} \\ &= \sum\{\|\dot{x} = \dot{z}\|_{\mathbb{B}} \cdot \|\dot{z} \in \dot{y}\|_{\mathbb{B}} \mid z \in \operatorname{dom}(\dot{y})\} \\ &= \|\dot{x} \in \dot{y}\|_{\mathbb{B}}\end{aligned}$$

and

$$\begin{aligned}\|i(\dot{x}) \subseteq i(\dot{y})\|_{\mathbb{C}} &= \prod\{i(\dot{x})(\dot{c}) \Rightarrow \|\dot{c} \in i(\dot{y})\|_{\mathbb{C}} \mid \dot{c} \in \operatorname{dom}(i(\dot{x}))\} \\ &= \prod\{\underbrace{i(\dot{x})(i(\dot{z}))}_{=\|\dot{z}\in\dot{x}\|_{\mathbb{B}}} \Rightarrow \underbrace{\|i(\dot{z}) \in i(\dot{y})\|_{\mathbb{C}}}_{=\|\dot{z}\in\dot{y}\|_{\mathbb{B}}} \mid \dot{z} \in \operatorname{dom}(\dot{x})\} \\ &= \prod\{\|\dot{z} \in \dot{x}\|_{\mathbb{B}} \Rightarrow \|\dot{z} \in \dot{y}\|_{\mathbb{B}} \mid \dot{z} \in \operatorname{dom}(\dot{x})\} \\ &= \|\dot{x} \subseteq \dot{y}\|_{\mathbb{B}}.\end{aligned}$$

Since $\|i(\dot{x}) = i(\dot{y})\|_{\mathbb{C}} = \|i(\dot{x}) \subseteq i(\dot{y})\|_{\mathbb{C}} \cdot \|i(\dot{y}) \subseteq i(\dot{x})\|_{\mathbb{C}}$, this also yields

$$\|i(\dot{x}) = i(\dot{y})\|_{\mathbb{C}} = \|\dot{x} = \dot{y}\|_{\mathbb{B}}.$$

We proceed to prove

$$\|\phi(\dot{x}_1, \ldots, \dot{x}_n)\|_{\mathbb{B}} = \|\phi(i(\dot{x}_1), \ldots, i(\dot{x}_n))\|_{\mathbb{C}}$$

for all $\dot{x}_1, \ldots, \dot{x}_n \in V^{\mathbb{B}}$ and all Δ_1-formulae ϕ by induction on the complexity of ϕ. We already handled the atomic case that $\phi \equiv \dot{x} \in \dot{y}$ or $\phi \equiv \dot{x} = \dot{y}$ and the inductions steps for $\phi \equiv \neg\psi$ and $\phi = \psi \wedge \chi$ are immediate. Suppose now that the claim holds for ϕ and let $\dot{x}_1, \ldots, \dot{x}_n \in V^{\mathbb{B}}$. Then

$$\begin{aligned}\|\exists \dot{x} \in \dot{x}_1\colon \phi(\dot{x}, \dot{x}_1, \ldots, \dot{x}_n)\| &= \sum\{\|\phi(\dot{x}, \dot{x}_1, \ldots, \dot{x}_n)\|_{\mathbb{B}} \mid \dot{x} \in \operatorname{dom}(\dot{x}_1)\} \\ &= \sum\{\|\phi(i(\dot{x}), i(\dot{x}_1), \ldots, i(\dot{x}_n))\|_{\mathbb{C}} \mid \dot{x} \in \operatorname{dom}(\dot{x}_1)\} \\ &= \sum\{\|\phi(\dot{c}, i(\dot{x}_1), \ldots, i(\dot{x}_n))\|_{\mathbb{C}} \mid \dot{c} \in \operatorname{dom}(i(\dot{x}_1))\} \\ &= \|\exists \dot{x} \in i(\dot{x}_1)\colon \phi(\dot{x}, i(\dot{x}_1), \ldots, i(\dot{x}_n))\|_{\mathbb{C}}.\end{aligned}$$

If $\phi(\dot{x}_1, \dots, \dot{x}_n) = \exists \dot{x} \colon \psi(\dot{x}, \dot{x}_1, \dots, \dot{x}_n)$ and the claim holds for ψ, then (by the fullness of $V^{\mathbb{B}}$) there is some $\dot{x}_0 \in V^{\mathbb{B}}$ such that

$$\|\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} = \|\psi(\dot{x}_0, \dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}}$$

and thus

$$\begin{aligned} \|\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} &= \|\psi(\dot{x}_0, \dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} \\ &= \|\psi(i(\dot{x}_0), i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}} \\ &\preceq \|\phi(i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}}. \end{aligned}$$

In particular, if ϕ is Δ_1 then there are Σ_0-formulae ψ, χ such that $\phi = \exists \dot{x} \colon \psi$ and $\neg\phi = \exists \dot{x} \colon \chi$ which (by the above) yields

$$\|\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} \preceq \|\phi(i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}},$$

as well as

$$\begin{aligned} -\|\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} &= \|\neg\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} \\ &\preceq \|\neg\phi(i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}} \\ &= -\|\phi(i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}} \end{aligned}$$

and hence $\|\phi(\dot{x}_1, \dots, \dot{x}_n)\|_{\mathbb{B}} = \|\phi(i(\dot{x}_1), \dots, i(\dot{x}_n))\|_{\mathbb{C}}$, as desired. □

From now on, we shall identify $V^{\mathbb{B}}$ with its "isomorphic copy" in $V^{\mathbb{C}}$. More precisely: We shall assume that for all $\mathbb{B} \sqsubseteq_c \mathbb{C}$, the identity map satisfies the claim of Proposition 2.2.1, i.e. for all $x_1, \dots, x_n \in V^{\mathbb{B}}$ and for all Δ_1-formulae ϕ we have

$$\|\phi(x_1, \dots, x_n)\|_{\mathbb{B}} = \|\phi(x_1, \dots, x_n)\|_{\mathbb{C}}.$$

Since we don't care about the actual presentation of the elements of Boolean valued models, this is harmless and avoids unecessary notational obstacles - especially in the context of general forcing iterations.

Definition 2.2.1. *By induction on its $\in$-rank, we define a* canonical name $\check{x} \in V^{\mathbb{B}}$ *for each $x \in V$ as follows*

1. $\check{\emptyset} := \emptyset$ *and*

2. $\check{x} \colon \{\check{y} \mid y \in x\} \to \mathbb{B},\ \check{y} \mapsto 1.$

Furthermore, we let

$$\dot{G} \colon \{\check{a} \mid a \in \mathbb{B}\} \to \mathbb{B},\ \check{a} \mapsto a$$

and call it the canonical generic name *(for $\mathbb{B}$) or* the canonical $\mathbb{B}$-generic name*.*

It is sometimes useful to add V as a predicate to $V^{\mathbb{B}}$ by letting

$$\|\dot{x} \in \check{V}\| := \sum\{\|\dot{x} = \check{x}\| \mid x \in V\}$$

for all $\dot{x} \in V^{\mathbb{B}}$.
We are now ready to link maximal $\mathbb{B}$-valued models to generic extensions of V.

Definition 2.2.2. *A nonenmpty subset $A \subseteq \mathbb{B}$ is an* antichain *iff it is an antichain in $\mathbb{B}^+$ that doesn't contain 1, i.e. $0, 1 \notin A$ and for all distinct $a, b \in A$ we have $a \perp b$ (i.e. $a \cdot b = 0$). A is* maximal *iff there is no antichain A' such that $A \subsetneq A' \subseteq \mathbb{B}$.*
A subset $D \subseteq \mathbb{B}$ is dense *iff it is dense in $\mathbb{B}^+$, i.e. $0 \notin D$ and for all $b \in \mathbb{B}^+$ there is some $d \in D$ with $d \preceq b$.*

Proposition 2.2.2. *Let $D \subseteq \mathbb{B}$ be dense and let $W \subseteq D$ be an antichain. Then there is a maximal antichain $W \subseteq W' \subseteq D$.*
On the other hand, if $W' \subseteq \mathbb{B}$ is a maximal antichain, then

$$W'_{\downarrow} := \{b \in \mathbb{B}^+ \mid \exists w \in W' \colon b \preceq w\}$$

is dense and $W' \subseteq W'_{\downarrow}$.

Proof. Let

$$\mathcal{W} := \{X \subseteq \mathbb{B} \mid X \subseteq D \text{ is an antichain}\}.$$

By the Hausdorff maximal principle there is a $\subseteq$-maximal element $W' \in \mathcal{W}$.
Suppose that W' is not a maximal antichain in $\mathbb{B}$. Then there is some $b \in \mathbb{B}^+ \setminus W'$ such that for all $w \in W'$: $w \cdot b = 0$. As D is dense, we may fix $d \in D$ with $d \preceq b$. $W' \subsetneq W' \cup \{d\} \subseteq D$ is an antichain in D that properly contains W'. (Contradiction!)
Now let W' be a maximal antichain and let $W'_{\downarrow}$ be defined as above. Given $b \in \mathbb{B}^+ \setminus W'$ there is some $w \in W'$ such that $b \cdot w \neq 0$. Since $b \cdot w \preceq b, w$, the density of $W'_{\downarrow}$ follows. □

Definition 2.2.3. *Let $\mathbb{B}$ be a complete Boolean algebra and let $G \subseteq \mathbb{B}$ be an ultrafilter. We say that G is* $\mathbb{B}$-generic *iff $G \cap D \neq \emptyset$ for all dense $D \subseteq \mathbb{B}$.*
More generally: Let M be a transitive (class) model of ZFC^- *and let $\mathbb{B} \in M$ be such that $M \models \mathbb{B}$ is a complete Boolean algebra. We say that G is $\mathbb{B}$-generic over M iff $G \subseteq \mathbb{B}$ is an ultrafilter such that $G \cap D \neq \emptyset$ for all $D \in M$ that are dense in $\mathbb{B}$.* [3]

[3] Since M is transitive and "being dense" is a Σ_0-property, we have for any $D \in M$ that $D \subseteq \mathbb{B}$ is dense iff $M \models D \subseteq \mathbb{B}$ is dense.

The following Lemma provides alternative characterizations for generic ultrafilters which are easier to handle in certain arguments.

Lemma 2.2.1. *For an ultrafilter $G \subseteq \mathbb{B}$, the following are equivalent:*

1. *G is $\mathbb{B}$-generic.*
2. *$G \cap W \neq \emptyset$ for all maximal antichains $W \subseteq \mathbb{B}$,*
3. *G is $\mathbb{B}$-complete, i.e. for all $X \subseteq \mathbb{B}$ with $\sum X \in G$ we have $X \cap G \neq \emptyset$,*
4. *G is $\mathbb{B}$-closed, i.e. for all $Y \subseteq G$ we have $\prod Y \in G$.*

Proof. 1. $\leftrightarrow$ 2. This is an immediate consequence of Proposition 2.2.2.

1. $\rightarrow$ 3. Let $X \subseteq \mathbb{B}$ be such that $\sum X \in G$ and let D be the set of all $b \in \mathbb{B}^+$ such that either there is some $x \in X$ with $b \preceq x$ or such that $b \cdot x = 0$ for all $x \in X$. If $a \in \mathbb{B}^+$ is such that for some $x \in X$ we have $a \cdot x \neq 0$, then $a \cdot x \preceq a$ is in D and otherwise a itself is in D. Therefore D is dense and we may fix $a \in D \cap G$. We claim that there is an $x \in X$ with $a \preceq x$ (witnessing $X \cap G \neq \emptyset$ as G is upward closed):
We have $a, \sum X \in G$ and therefore $a \cdot \sum X = \sum\{a \cdot x \mid x \in X\} \in G$. As $0 \neq G$, there is some $x \in X$ with $a \cdot x \neq 0$. By the definition of D this yields $a \preceq x$ as desired.

3. $\rightarrow$ 1. Since $1 \in G$ it suffices to prove that $\sum D = 1$ for all dense $D \subseteq \mathbb{B}$:
Let $b \in \mathbb{B} \setminus \{1\}$. Then $-b \neq 0$ and thus there is some $d \in D$ with $d \preceq -b$, i.e. $d \cdot b = 0$. As $D \subseteq \mathbb{B}^+$ we have $d \neq d \cdot b = 0$ and therefore $d \not\preceq b$. Thus b is no upper bound for D and the claim follows.

3. $\rightarrow$ 4. If $Y \subseteq G$, then $-Y \cap G = \emptyset$ and thus $\sum -Y \notin G$. Therefore $-\sum -Y = \inf Y \in G$.

4. $\rightarrow$ 3. If $X \cap G = \emptyset$, then $-X \subseteq G$ and therefore $\prod -X = -\sum X \in G$. This implies $\sum X \notin G$.

□

Definition 2.2.4. *Let $G \subseteq \mathbb{B}$ be a $\mathbb{B}$-generic filter. By induction on $\rho(\dot{x})$, we define the G-interpretation $\dot{x}^G$ of $\dot{x}$ by*

$$\dot{x}^G := \{\dot{y}^G \mid \dot{x}(\dot{y}) \in G\}$$

and we define the generic extension of V by G as

$$V[G] := \{\dot{x}^G \mid \dot{x} \in V^{\mathbb{B}}\}.$$

We also say that $V[G]$ is the forcing extension *of V by G.*

Theorem 2.3. *Let $G \subseteq \mathbb{B}$ be $\mathbb{B}$-generic. For all $\mathcal{L}_{\{\in\}}$-formulas ϕ and $\dot{x}_1, \ldots, \dot{x}_n \in V^{\mathbb{B}}$, we have*

$$V[G] \models \phi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \text{ if and only if } \|\phi(\dot{x}_1, \ldots, \dot{x}_n)\| \in G.$$

Furthermore, $\check{x}^G = x \in V[G]$ for all $x \in V$, $\dot{G}^G = G \in V[G]$, $V \cap \mathrm{Ord} = V[G] \cap \mathrm{Ord}$ and $V[G]$ is the smallest transitive class model M of ZFC *s.t. $V \cup \{G\} \subseteq M$.*

Proof. [Jec06, p. 216 ff.] □

In fact, when defining the equivalence relation $\dot{x} \equiv \dot{y} \leftrightarrow \|\dot{x} = \dot{y}\|$ and the membership relation $[\dot{x}]E[\dot{y}] \leftrightarrow \|\dot{x} \in \dot{y}\| \in G$ for the induced equivalence classes just like we did in Lemma 2.1.2, we have a natural isomorphism between $(V[G]; \in)$ and $({V^{\mathbb{B}}}/{\equiv}; E)$, given by

$$\pi \colon V[G] \to {V^{\mathbb{B}}}/{\equiv}, \dot{x}^G \mapsto [\dot{x}].$$

Proof. Work in $V[G]$:
For all $\dot{x}, \dot{y} \in V^B$

$$\begin{aligned} \dot{x}^G = \dot{y}^G &\leftrightarrow \|\dot{x} = \dot{y}\| \in G \\ &\leftrightarrow [\dot{x}] = [\dot{y}], \end{aligned}$$

hence π is a well-defined bijection.
Furthermore, for all $\dot{x}, \dot{y} \in V^{\mathbb{B}}$

$$\begin{aligned} \dot{x}^G \in \dot{y}^G &\leftrightarrow \|\dot{x} \in \dot{y}\| \in G \\ &\leftrightarrow [\dot{x}]E[\dot{y}], \end{aligned}$$

verifying that $\pi \colon (V[G]; \in) \to ({V^{\mathbb{B}}}/{\equiv}; E)$ is an isomorphism. □

A careful analysis of Jech's proof yields that given any transitive (class) model N of ZFC^-, any $\mathbb{B} \in N$ such that $N \models \mathbb{B}$ is a complete Boolean algebra and any ultrafilter G that is $\mathbb{B}$-generic over N, we may still build $N^{\mathbb{B}}$ and the associated the generic extension $N[G]$ in the same way as above, by defining $\mathbb{B}$-names and Boolean evaluations in N. This results in the least transitive (class) model of ZFC^- such that $N \cup \{G\} \subseteq N[G]$ and we still have that for all $\mathcal{L}_{\{\in\}}$-formulas ϕ and all $\dot{x}_1, \ldots, \dot{x}_n \in N^{\mathbb{B}}$:

$$N[G] \models \phi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \text{ if and only if } \|\phi(\dot{x}_1, \ldots, \dot{x}_n)\| \in G.$$

Furthermore, if N is countable, there are only countably many dense sets $D \subseteq \mathbb{B}$ such that $D \in N$. Let $\mathcal{D} = \{D_n \mid 0 < n < \omega\}$ be an enumeration of all of them

and let $b \in \mathbb{B}^+$. We may now recursively define a sequence $(b_n \mid n < \omega)$ by letting $b_0 := b$ and choosing $b_{n+1} \in D_{n+1}$ such that $b_{n+1} \preceq_{\mathbb{B}} b_n$. Using Zorn's Lemma, we may now (in V) form an ultrafilter $G \subseteq \mathbb{B}^+$ such that $\{b_n \mid n < \omega\} \subseteq G$. Now $b \in G$ and, since $b_n \in G \cap D_n$ for all $n < \omega$, G is in fact $\mathbb{B}$-generic over N. So, for any $b \in \mathbb{B}^+$ there is some $\mathbb{B}$-generic ultrafilter G over N in V such that $b \in G$. This will be important in the definition of subcomplete Boolean algebras. More details can be found in [Kun11, ch.14].
Since $V^{\mathbb{B}}$ is a Boolean valued model in which every axiom of ZFC is valid, we may now prove the relative consistency of some statement ϕ (that may have parameters $\dot{x}_1, \ldots, \dot{x}_k \in V^B$) relative to ZFC by verifying that $\|\phi\| \neq 0$. However, as we've seen in some of the arguments above, calculating the Boolean value for a given statement can be quite cumbersome and we'd much rather be able to work with the transitive class model $V[G]$.
Suppose that $\|\phi(\dot{x}_1, \ldots, \dot{x}_k)\| \neq 0$ and there is some $\mathbb{B}$-generic filter G such that $\|\phi(\dot{x}_1, \ldots, \dot{x}_k)\| \in G$. By Lemma 2.1.2 and the isomorphism above, we'd then have $({}^{V^{\mathbb{B}}}\!/\!_{\equiv}; E) \models \phi([\dot{x}_1], \ldots, [\dot{x}_k])$ and $(V[G]; \in) \models \phi(\dot{x}_1^G, \ldots, \dot{x}_k^G)$. Conversely $(V[G]; \in) \models \phi(\dot{x}_1^G, \ldots, \dot{x}_k^G)$ iff $({}^{V^{\mathbb{B}}}\!/\!_{\equiv}; E) \models \phi([\dot{x}_1], \ldots, [\dot{x}_k])$ iff $\|\phi(\dot{x}_1, \ldots, \dot{x}_k)\| \in G$. Thus, if for every $b \in \mathbb{B}^+$ there were a $\mathbb{B}$-generic filter G such that $b \in G$, then $\phi(\dot{x}_1, \ldots, \dot{x}_k)$ were consistent relative to ZFC iff $\|\phi(\dot{x}_1, \ldots, \dot{x}_k)\| \neq 0$ iff there is some $\mathbb{B}$-generic filter G such that $(V[G]; \in) \models \phi(\dot{x}_1^G, \ldots, \dot{x}_k^G)$. The problem is, that generic filters over V only exist in trivial cases. In fact, for any generic filter $G \in V$, we have $V = V[G]$ as an immediate consequence of the Theorem above. There are different approaches as how to overcome this issue and we'll outline two of them below. But first let us stress that in both arguments to come, the requirements on $\dot{G}$ only refer to canonical names for subsets of $\mathbb{B}$ and thus only to those subsets that are elements of V. This distinction is of importance for several reasons. For example $V^{\mathbb{B}}$ may see new subsets of $\mathbb{B}$ that don't have suprema/infima or new dense sets, that are provably disjoint from $\dot{G}$. So, while $\dot{G}$ looks like a $\mathbb{B}$-generic filter over V, we don't have in general, that it is generic with respect to all subset of $\mathbb{B}$ in $V^{\mathbb{B}}$.

1. Suppose that $\mathbb{B}$ only has countably many dense sets and let $(D_n \mid n < \omega)$ be an enumeration of all of them. As we've already seen, in this case there is for any $b \in \mathbb{B}^+$ some $\mathbb{B}$-generic ultrafilter G with $b \in G$. If $\mathbb{B}$ has uncountably many dense sets, say κ many, the argument above yields that the statement "there is an ultrafilter $\dot{G}$ such that $\check{b} \in \dot{G}$ and $\check{D} \cap \dot{G} \neq \check{0}$ for all dense subsets $\check{D} \subseteq \check{\mathbb{B}}$" is valid in $V^{\mathrm{Coll}(\omega,\kappa)}$.

2. Let $\dot{G}$ be the canonical $\mathbb{B}$-generic name. A straightforward calculation shows that $\|\dot{G}$ is an ultrafilter on $\check{\mathbb{B}}\| = 1$, $\|\check{X} \subseteq \dot{G} \rightarrow \prod \check{X} \in \dot{G}\| = 1$ and $\|\check{b} \in \dot{G}\| = b$ for all $X \subseteq \mathbb{B}$ and $b \in \mathbb{B}^+$.

Both arguments show that for any $b \in \mathbb{B}^+$ it is consistent, relative to ZFC, to have a $\mathbb{B}$-generic ultrafilter with $b \in G$ in some larger universe. Rather than relativizing all of our arguments to the original V inside this larger universe, we shall however still work directly with V (and its generic extensions), assuming that all relevant generic filters exist - in some larger, unspecified universe.

Definition 2.3.1. *We define the forcing relation* $\Vdash_{\mathbb{B}}$ *by*

$$b \Vdash_{\mathbb{B}} \phi(\dot{x}_1, \ldots, \dot{x}_n) :\leftrightarrow b \in \mathbb{B}^+ \text{ and } b \preceq \|\phi(\dot{x}_1, \ldots, \dot{x}_n)\|$$

for all $\mathcal{L}_{\{\in\}}$*-formulae* ϕ *and* $\mathbb{B}$*-names* $\dot{x}_1, \ldots, \dot{x}_n$.
If the Boolean algebra $\mathbb{B}$ *is clear from the context, we sometimes omit the subscript and simply write* $\Vdash$ *instead of* $\Vdash_{\mathbb{B}}$. *We also write* $\Vdash_{\mathbb{B}} \phi$ *iff* $1 \Vdash_{\mathbb{B}} \phi$.

This allows us to restate the previous Theorem in the following way: Let G be $\mathbb{B}$-generic, $\dot{x}_1, \ldots, \dot{x}_k \in V^{\mathbb{B}}$ and let ϕ be a $\mathcal{L}_{\{\in\}}$-formula. Then

$$V[G] \models \phi(\dot{x}_1^G, \ldots, \dot{x}_n^G) \text{ iff there is some } b \in G \text{ with } b \Vdash_{\mathbb{B}} \phi(\dot{x}_1, \ldots, \dot{x}_n).$$

And conversely $b \Vdash_{\mathbb{B}} \phi(\dot{x}_1, \ldots, \dot{x}_n)$ iff $V[G] \models \phi(\dot{x}_1^G, \ldots, \dot{x}_n^G)$ for all $\mathbb{B}$-generic ultrafilters G with $b \in G$.
In the remainder of this chapter, we prove some results about generic extensions that will be needed in later chapters.

Proposition 2.3.1. *Let* $\mathbb{B} \sqsubseteq \mathbb{C}$ *be Boolean algebras such that* $\mathbb{B}$ *is complete and let* $\dot{G}$ *be the canonical* $\mathbb{B}$*-generic name. Then for any* $b \in \mathbb{B}^+$ *and any* $c, d \in \mathbb{C}$

$$b \Vdash_{\mathbb{B}} {}^{\check{c}}\!/_{\dot{G}} \preceq {}^{\check{d}}\!/_{\dot{G}} \text{ if and only if } b \cdot c \preceq d.$$

In particular

$$b \Vdash_{\mathbb{B}} {}^{\check{c}}\!/_{\dot{G}} = 0 \text{ if and only if } b \cdot c = 0.$$

Proof. Observe that

$$\begin{aligned}
\|{}^{\check{c}}\!/_{\dot{G}} \preceq {}^{\check{d}}\!/_{\dot{G}}\| &= \|\exists \check{b} \colon \check{b} \in \dot{G} \wedge \check{b} \preceq -\check{c} + \check{d}\| \\
&= \textstyle\sum_{b \in \mathbb{B}} \underbrace{\|\check{b} \in \dot{G}\|}_{=b} \cdot \|\check{b} \preceq -\check{c} + \check{d}\| \\
&= \textstyle\sum_{b \in \mathbb{B}} b \cdot \underbrace{\|\check{b} \cdot \check{c} \preceq \check{d}\|}_{= \begin{cases} 1 & \text{, if } b \cdot c \preceq d \\ 0 & \text{, otherwise} \end{cases}} \\
&= \textstyle\sum \{b \in \mathbb{B} \mid b \cdot c \preceq d\}
\end{aligned}$$

and hence, for all $b \in \mathbb{B}^+$,

$$b \Vdash_{\mathbb{B}} {}^{\check{c}}\!/_{\dot{G}} \preceq {}^{\check{d}}\!/_{\dot{G}} \leftrightarrow b \preceq \|{}^{\check{c}}\!/_{\dot{G}} \preceq {}^{\check{d}}\!/_{\dot{G}}\| \\ \leftrightarrow b \cdot c \preceq d.$$

□

Proposition 2.3.2. *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$ and let G be $\mathbb{B}$-generic. Then ${}^{\mathbb{C}}\!/_{G}$ is a complete Boolean algebra in $V[G]$.*

A word of warning: While $\mathbb{C}$ still is a Boolean algebra in $V[G]$, it may no longer be complete in this larger universe. Take for example $\mathbb{C} = (\mathcal{P}(\omega); \emptyset, \omega, \cup, \cap, {}^{\complement})$ and properly extend V to $V[G]$ by adding a *Cohen real* $x \subseteq \omega$. In $V[G]$ consider the set $X = \{\{n\} \mid n \in x\}$. Now note that $\preceq = \subseteq$ in $\mathbb{C}$ and if $y \in \mathbb{P}(\omega)^V$ satisfies $\{n\} \subseteq y$ for all $\{n\} \in X$, then $x \subseteq y$. Since $x \notin \mathcal{P}(\omega)^V$, there is some $k < \omega$ such that $x \subseteq y \setminus \{k\} \subsetneq y$ - proving that x doesn't have a least upper bound in $\mathbb{C}$.

Proof. We work inside $V[G]$.
Since we've already seen that ${}^{\mathbb{C}}\!/_{G}$ is a well-defined Boolean algebra, it suffices to prove that it is complete (in $V[G]$). By Proposition 2.0.4 we may reduce this task further to verifying that $\sum_{{}^{\mathbb{C}}\!/_{G}} \dot{X}^G$ exists for all $\dot{X}^G \subseteq {}^{\mathbb{C}}\!/_{G}$.
So let $\dot{X}$ be a $\mathbb{B}$-name such that $\dot{X}^G \subseteq {}^{\mathbb{C}}\!/_{G}$ and fix some $b \in G$ with

$$b \Vdash_{\mathbb{B}} \dot{X} \subseteq {}^{\check{\mathbb{C}}}\!/_{\dot{G}}.$$

Using fullness, we may (in V) choose for each $\dot{x} \in \dot{X}$ some $c_{\dot{x}} \in \mathbb{C}$ such that

$$b \Vdash_{\mathbb{B}} \dot{x} = {}^{\check{c}_{\dot{x}}}\!/_{\dot{G}}.$$

Now $Y := \{c_{\dot{x}} \mid \dot{x} \in \dot{X}\}$ is a subset of $\mathbb{C}$ in V and since $\mathbb{C}$ is complete, we may let $y := \sum_{\mathbb{C}}\{c_{\dot{x}} \mid \dot{x} \in \dot{X}\}$. We finish our proof by verifying the following

Claim.

$${}^{y}\!/_{G} = \sum\nolimits_{{}^{\mathbb{C}}\!/_{G}} \dot{X}^G.$$

Proof. Let $\dot{y}$ be a $\mathbb{B}$-name such that $\dot{y}^G \in \dot{X}^G$. Since $V^{\mathbb{B}}$ is full, there is some $\dot{x} \in \dot{X}$ such that $b \Vdash_{\mathbb{B}} \dot{x} = \dot{y}$. Now $b \Vdash_{\mathbb{B}} \dot{x} = {}^{\check{c}_{\dot{x}}}\!/_{\dot{G}}$ and since $c_{\dot{x}} \preceq y$, we obtain $\dot{y}^G \preceq {}^{y}\!/_{G}$. Hence ${}^{y}\!/_{G}$ is an upper bound for $\dot{X}^G$ in ${}^{\mathbb{C}}\!/_{G}$.
Now let $\dot{u}$ be a $\mathbb{B}$-name such that $\dot{u}^G \in {}^{\mathbb{C}}\!/_{G}$ and $\dot{x}^G \preceq \dot{u}^G$ for all $\dot{x} \in \dot{X}$. Fix $b' \in G$ and some $u \in \mathbb{C}$ such that

$$b' \Vdash_{\mathbb{B}} \dot{u} \in {}^{\check{\mathbb{C}}}\!/_{\dot{G}} \wedge \dot{u} = {}^{\check{u}}\!/_{\dot{G}} \wedge \forall \dot{x} \in \dot{X} \colon \dot{x} \preceq \dot{u}.$$

By replacing b' with $b \cdot b'$, we may assume that $b' \preceq b$. This implies

$$b' \Vdash_{\mathbb{B}} \check{c}_{\dot{x}}/\dot{G} \preceq \check{u}/\dot{G}$$

for all $\dot{x} \in \dot{X}$. By Proposition 2.3.1 this may be equivalently stated as $b' \cdot c_{\dot{x}} \preceq u$ for all $\dot{x} \in X$. But now

$$\textstyle\sum_{\mathbb{C}}\{b' \cdot c_{\dot{x}} \mid \dot{x} \in \dot{X}\} = b' \cdot y \preceq u$$

implies $b' \Vdash_{\mathbb{B}} \check{y}/\dot{G} \preceq \check{u}/\dot{G}$ and therefore $y/G \preceq xu/G = \dot{u}^G$. ■

□

Proposition 2.3.3. *Let $\mathbb{B} \sqsubseteq \mathbb{C}$ be Boolean algebras such that $\mathbb{B}$ is complete and let $h_{\mathbb{B},\mathbb{C}} \colon \mathbb{C} \to \mathbb{B}, c \mapsto \prod\{b \in \mathbb{B} \mid c \preceq b\}$ be the canonical projection. Then for all $c \in \mathbb{C}$*

$$\|\check{c}/\dot{G} \neq 0\| = h_{\mathbb{B},\mathbb{C}}(c).$$

Proof. Note that

$$\begin{aligned}
\|\check{c}/\dot{G} = 0\| &= \textstyle\sum\{b \in \mathbb{B} \mid b \Vdash_{\mathbb{B}} \check{c}/\dot{G} = 0\} \\
&\overset{Proposition\ 2.3.1}{=} \textstyle\sum\{b \in \mathbb{B} \mid b \cdot c = 0\} \\
&= \textstyle\sum\{b \in \mathbb{B} \mid b \preceq -c\} \\
&= -\textstyle\prod\{-b \in \mathbb{B} \mid b \preceq -c\} \\
&= -\textstyle\prod\{b \in \mathbb{B} \mid c \preceq b\} \\
&= -h_{\mathbb{B},\mathbb{C}}(c)
\end{aligned}$$

and thus $\|\check{c}/\dot{G} \neq 0\| = -\|\check{c}/\dot{G} = 0\| = h_{\mathbb{B},\mathbb{C}}(c)$. □

Lemma 2.3.1. *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$, let G be $\mathbb{B}$-generic and let $A \subseteq \mathbb{B}^+$ be an antichain. For each $a \in A$ fix some $b_a \in \mathbb{C}$ and let $b := \sum_{a \in A} a \cdot b_a$. Then for each $\overline{a} \in A$ we have*

$$\overline{a} \Vdash_{\mathbb{B}} \check{b}/\dot{G} = \check{b}_{\overline{a}}/\dot{G}.$$

Proof. Since A is an antichain, we have $\overline{a} \cdot a = \begin{cases} \overline{a} & \text{, if } a = \overline{a} \\ 0 & \text{, otherwise} \end{cases}$ for all $a \in A$. Thus

$$\overline{a} \Vdash_{\mathbb{B}} \check{b}/\dot{G} = (\overline{a} \cdot b)\check{}/\dot{G} = (\textstyle\sum_{a \in A} \overline{a} \cdot a \cdot b_a)\check{}/\dot{G} = (\overline{a} \cdot b_{\overline{a}})\check{}/\dot{G} = \check{b}_{\overline{a}}/\dot{G}.$$

□

Lemma 2.3.2. *Let $\dot{x}, \dot{y} \in V^{\mathbb{B}}$. Then there there a canonical $\mathbb{B}$-names $\underline{\{}\dot{x}, \dot{y}\underline{\}}$ and $\underline{(}\dot{x}, \dot{y}\underline{)} \in V^{\mathbb{B}}$ such that for all $\mathbb{B}$-generic ultrafilters G:*

1. *$\underline{\{}\dot{x}, \dot{y}\underline{\}}^G = \{\dot{x}^G, \dot{y}^G\}$ and*
2. *$\underline{(}\dot{x}, \dot{y}\underline{)}^G = (\dot{x}^G, \dot{y}^G)$.*

Proof. 1. Let $\underline{\{}\dot{x}, \dot{y}\underline{\}} := \{(\dot{x}, 1), (\dot{y}, 1)\}$. Clearly $\underline{\{}\dot{x}, \dot{y}\underline{\}} \in V^{\mathbb{B}}$ and for all z

$$\begin{aligned} z \in \underline{\{}\dot{x}, \dot{y}\underline{\}}^G &\leftrightarrow \exists b \in G \colon b \Vdash_{\mathbb{B}} z = (\dot{x}, 1) \wedge z = (\dot{y}, 1) \\ &\leftrightarrow z = \dot{x}^G \wedge z = \dot{y}^G, \end{aligned}$$

i.e. $\underline{\{}\dot{x}, \dot{y}\underline{\}}^G = (\dot{x}^G, \dot{y}^G)$.

2. Let $\underline{(}\dot{x}, \dot{y}\underline{)} := \underline{\{}\ \underline{\{}\dot{x}\underline{\}}, \underline{\{}\dot{x}, \dot{y}\underline{\}}\ \underline{\}}$. Then $\underline{(}\dot{x}, \dot{y}\underline{)} \in V^{\mathbb{B}}$ and by 1.

$$\begin{aligned} \underline{(}\dot{x}, \dot{y}\underline{)}^G &= \underline{\{}\ \underline{\{}\dot{x}\underline{\}}, \underline{\{}\dot{x}, \dot{y}\underline{\}}\ \underline{\}}^G \\ &= \{\underline{\{}\dot{x}\underline{\}}^G, \underline{\{}\dot{x}, \dot{y}\underline{\}}^G\} \\ &= \{\{\dot{x}^G\}, \{\dot{x}^G, \dot{y}^G\}\} \\ &= (\dot{x}^G, \dot{y}^G). \end{aligned}$$

□

Definition 2.3.2. *Let $\dot{x}_1, \ldots, \dot{x}_k \in V^{\mathbb{B}}$. By induction on $k < \omega$ we may now define the* generic k-tupel *of $\dot{x}_1, \ldots, \dot{x}_k$ by*

$$\underline{(}\dot{x}_1, \ldots, \dot{x}_k\underline{)} := \underline{(}\underline{(}\dot{x}_1, \ldots, \dot{x}_{k-1}\underline{)}, \dot{x}_k\underline{)}.$$

Note that for any $\mathbb{B}$-generic ultrafilter G, we now have

$$\begin{aligned} \underline{(}\dot{x}_1, \ldots, \dot{x}_k\underline{)}^G &= \underline{(}\underline{(}\dot{x}_1, \ldots, \dot{x}_{k-1}\underline{)}, \dot{x}_k\underline{)}^G \\ &= (\underline{(}\dot{x}_1, \ldots, \dot{x}_{k-1}\underline{)}^G, \dot{x}_k^G) \\ &= ((\dot{x}_1^G, \ldots, \dot{x}_{k-1}^G), \dot{x}_k^G) \\ &= (\dot{x}_1^G, \ldots, \dot{x}_{k-1}^G, \dot{x}_k^G). \end{aligned}$$

The following result can be used to prove the fullness of $V[G]$ and generally allows us to "merge" several $\mathbb{B}$-names into a single one.

Lemma 2.3.3. *Let $A \subseteq \mathbb{B}$ be an antichain and for each $a \in A$ fix a $\mathbb{B}$-name $\dot{x}_a$. Then there is a $\mathbb{B}$-name $\dot{x}$ such that*

$$a \Vdash_{\mathbb{B}} \dot{x} = \dot{x}_a$$

for each $a \in A$.

Proof. Let
$$\dot{x}\colon \bigcup_{a\in A} \operatorname{dom}(\dot{x}_a) \to \mathbb{B}, \dot{u} \mapsto \sum\nolimits_{a\in A} a \cdot \|\dot{u} \in \dot{x}_a\|.$$
Fix $a \in A$ and let G be $\mathbb{B}$-generic with $a \in G$. Using that G is $\mathbb{B}$-generic and A is an antichain, we obtain
$$\begin{aligned}\dot{x}^G &= \{\dot{u}^G \colon \dot{x}(\dot{u}) \in G\}\\ &= \{\dot{u}^G \colon \sum\nolimits_{b\in A} b \cdot \|\dot{u} \in \dot{x}_b\| \in G\}\\ &= \{\dot{u}^G \colon a \cdot \|\dot{u} \in \dot{x}_a\| \in G\}\\ &= \{\dot{u}^G \colon \dot{x}_a(\dot{u}) \in G\}\\ &= \dot{x}_a^G.\end{aligned}$$
□

Proposition 2.3.4. *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$, let $\dot{c}$ be a $\mathbb{B}$-name and let $b \in \mathbb{B}^+$ be such that*
$$b \Vdash_{\mathbb{B}} \dot{c} \in {}^{\check{\mathbb{C}}}\!/_{\dot{G}}.$$
Then there is some $c \in \mathbb{C}$ such that
$$b \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}}\!/_{\dot{G}}.$$
If $b = 1$, then there is a unique such c.

Proof. Whenever G is $\mathbb{B}$-generic and $b \in G$, we have $\dot{c}^G \in {}^{\mathbb{B}}\!/_{G}$ and hence some $c \in \mathbb{C}$ with $\dot{c}^G = {}^{c}\!/_{G}$. Thus there is some $0 \prec b' \preceq b$ with
$$b' \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}}\!/_{\dot{G}}.$$
Therefore
$$D := \{b' \in \mathbb{B} \mid b \cdot b' = 0 \vee (0 \prec b' \preceq b \wedge \exists c \in \mathbb{C}\colon b' \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}}\!/_{\dot{G}})\}.$$
is a dense subset in $\mathbb{B}$ and we may fix a maximal antichain $A \subseteq D$ of $\mathbb{B}$. For each $a \in A$ with $a \cdot b \neq 0$ choose some $c_a \in \mathbb{C}$ with
$$a \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}_a}\!/_{\dot{G}}.$$
For $a \in A$ with $a \cdot b = 0$ choose $c_a := 0$ and let $c := \sum_{a\in A} a \cdot c_a$.

Claim.
$$b \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}}\!/_{\dot{G}}.$$

Proof. Let G be $\mathbb{B}$-generic with $b \in G$. Since A is a maximal antichain in $\mathbb{B}$, there is a unique $a' \in A \cap G$. Since $a' \cdot b \in G$, we have $0 \neq a' \cdot b$ and hence $a' \preceq b$ and $a' \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}_{a'}}\!/_{\dot{G}}$. Note that

$$\begin{aligned} a' \cdot \Delta(c_a, c) &= a' \cdot (c_{a'} \cdot (-c) + c \cdot (-c_{a'})) \\ &= \underbrace{a' \cdot c_{a'}}_{\preceq c} \cdot (-c) + (-c_{a'}) \cdot a' \cdot \textstyle\sum_{a \in A} a \cdot c_a \\ &= 0 + (-c_{a'}) \cdot \textstyle\sum_{a \in A} (\underbrace{a' \cdot a}_{= \begin{cases} a' & \text{, if } a = a' \\ 0 & \text{, otherwise} \end{cases}} \cdot c_a) \\ &= -c_{a'} \cdot a' \cdot c_{a'} \\ &= 0, \end{aligned}$$

i.e. $a' \preceq -\Delta(c_{a'}, c)$. Since $a' \in G$, this proves ${}^{c_{a'}}\!/_{G} = {}^{c}\!/_{G}$. ■

If $b = 1$ and $c, d \in \mathbb{C}$ are such that

$$1 \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{c}}\!/_{\dot{G}} \text{ and } 1 \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{d}}\!/_{\dot{G}},$$

then $1 \Vdash_{\mathbb{B}} = {}^{\check{c}}\!/_{\dot{G}} = {}^{\check{d}}\!/_{\dot{G}}$. By Proposition 2.3.1, this yields $c = 1 \cdot c \preceq d = 1 \preceq d \preceq c$ and hence $c = d$. □

Proposition 2.3.5. *Let $\mathbb{A} \sqsubseteq_c \mathbb{B} \sqsubseteq_c \mathbb{C}$ and let G be $\mathbb{A}$-generic over V. Let*

$$h_{\mathbb{B},\mathbb{C}} \colon \mathbb{C} \to \mathbb{B},\ c \mapsto \prod\{b \in \mathbb{B} \mid c \preceq b\}$$

(in V) and let

$$h_{{}^{\mathbb{B}}\!/_{G}, {}^{\mathbb{C}}\!/_{G}} \colon {}^{\mathbb{C}}\!/_{G} \to {}^{\mathbb{B}}\!/_{G},\ {}^{c}\!/_{G} \mapsto \prod\{{}^{b}\!/_{G} \in {}^{\mathbb{B}}\!/_{G} \mid {}^{c}\!/_{G} \preceq {}^{b}\!/_{G}\}$$

(in $V[G]$) be the canonical projections. Let $\tau \colon \mathbb{C} \to {}^{\mathbb{C}}\!/_{G}$, $c \mapsto {}^{c}\!/_{G}$ and $\pi \colon \mathbb{B} \to {}^{\mathbb{B}}\!/_{G}$, $b \mapsto {}^{b}\!/_{G}$ be the associated quotient maps. Then, the following diagram commutes

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{\ h_{\mathbb{B},\mathbb{C}}\ } & \mathbb{B} \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\pi} \\ {}^{\mathbb{C}}\!/_{G} & \xrightarrow[\ h_{{}^{\mathbb{B}}\!/_{G}, {}^{\mathbb{C}}\!/_{G}}\ ]{} & {}^{\mathbb{B}}\!/_{G} \end{array}$$

Proof. Let $c \in \mathbb{C}$. Since $c \preceq b$ implies ${}^{c}\!/\!_{G} \preceq {}^{b}\!/\!_{G}$, for any $b \in \mathbb{B}$, we have

$$\begin{aligned}
\pi(h_{\mathbb{B},\mathbb{C}}(c)) &= \prod\{b \in \mathbb{B} \mid c \preceq b\}\big/_{G} \\
&= \prod\{{}^{b}\!/\!_{G} \in {}^{\mathbb{B}}\!/\!_{G} \mid c \preceq b\} \\
&\succeq \prod\{{}^{b}\!/\!_{G} \in {}^{\mathbb{B}}\!/\!_{G} \mid {}^{c}\!/\!_{G} \preceq {}^{b}\!/\!_{G}\} \\
&= h_{{}^{\mathbb{B}}\!/\!_{G},{}^{\mathbb{C}}\!/\!_{G}}(\tau(c)).
\end{aligned}$$

On the other hand, if $b \in \mathbb{B}$ is such that ${}^{c}\!/\!_{G} \preceq {}^{b}\!/\!_{G}$, then there is some $g \in G$ such that $g \preceq -c + b$. Thus $g \cdot -(-c + b) = g \cdot c \cdot (-b) = 0$ and therefore $c \preceq -(g \cdot (-b)) = b + -g$. But ${}^{b + -g}\!/\!_{G} = {}^{b}\!/\!_{G} + {}^{-g}\!/\!_{G} = {}^{b}\!/\!_{G} + {}^{0}\!/\!_{G} = {}^{b}\!/\!_{G}$. This implies

$$\prod\{{}^{b}\!/\!_{G} \in {}^{\mathbb{B}}\!/\!_{G} \mid c \preceq b\} = \prod\{{}^{b}\!/\!_{G} \in {}^{\mathbb{B}}\!/\!_{G} \mid {}^{c}\!/\!_{G} \preceq {}^{b}\!/\!_{G}\}.$$

□

3 Iterated Forcing

The theory of Iterated Forcings was initially developed by Solovay and Tennenbaum in 1971, in order to construct a model of ZFC without a Suslin tree and quickly became a key tool in a plethora of forcing applications. The "classical" development of this theory works with partially ordered sets (*forcing notions*) and suitable $\mathbb{P}$-names for forcing notions. This approach can be found in [Jec06, ch.16] and is in fact equivalent to our presentation that roughly follows [Jenb] and [Jenc].

3.1 Two Step Iterations

Given a complete Boolean algebra $\mathbb{B}$ and a $\mathbb{B}$-generic filter G, we may form the generic extension $V[G]$. This yields a new class model of ZFC and if $\mathbb{C} \in V[G]$ is such that $V[G] \models \mathbb{C}$ is a complete Boolean algebra and $H \subseteq \mathbb{C}$ is $\mathbb{C}$-generic over $V[G]$, we may again form the generic extension $(V[G])\,[H]$. We call $(V[G])[H]$ a *Two Step Iteration* and we will now develop a method that allows us to obtain $(V[G])[H]$ by forming a single generic extension over V.

Definition 3.0.1. *A* $\mathbb{B}$-name for a (complete) Boolean algebra *is a* $\mathbb{B}$*-name* $\dot{\mathbb{C}}$ *such that*

$$1 \Vdash_{\mathbb{B}} \dot{\mathbb{C}} \textit{ is a (complete) Boolean algebra.}$$

We can further arrange that $\dot{\mathbb{C}} = \underline{(}\dot{C}; \dot{0}, \dot{1}, \dot{+}, \dot{\cdot}, \dot{-}\underline{)}$ *is a generic* 6*-tupel such that*

- $1 \Vdash_{\mathbb{B}} \dot{C}$ *is a nonempty set,*
- $1 \Vdash_{\mathbb{B}} \dot{0}, \dot{1} \in \dot{C}$*,*
- $1 \Vdash_{\mathbb{B}} \dot{0} \neq \dot{1}$*,*
- $1 \Vdash_{\mathbb{B}} \dot{+} \colon \dot{C} \times \dot{C} \to \dot{C}$*,*
- $1 \Vdash_{\mathbb{B}} \dot{\cdot} \colon \dot{C} \times \dot{C} \to \dot{C}$ *and*
- $1 \Vdash_{\mathbb{B}} \dot{-} \colon \dot{C} \to \dot{C}$

and all the identities in Definition 2.0.1 (and the existence of infima and suprema for all $\dot{X} \subseteq \dot{C}$*) are valid, e.g. for all* $\dot{a}, \dot{b} \in \dot{C}$ $1 \Vdash_{\mathbb{B}} \dot{a} \dot{+} \dot{b} = \dot{b} \dot{+} \dot{a}$ *(and for all* $\dot{X} \in V^{\mathbb{B}} \colon 1 \Vdash_{\mathbb{B}} \dot{X} \subseteq \dot{C} \to \sum \dot{X}$ *and* $\prod \dot{X}$ *exist). As before, we often identify* $\dot{\mathbb{C}}$ *with its underlying set* $\dot{C}$*.*

Lemma 3.0.1. *Let G be $\mathbb{B}$-generic and let $\dot{\mathbb{D}}$ be a $\mathbb{B}$-name such that $V[G] \models \dot{\mathbb{D}}^G$ is a complete Boolean algebra. Then there is a $\mathbb{B}$ name $\dot{\mathbb{C}}$ for a compelete Boolean algebra such that $\dot{\mathbb{C}}^G = \dot{\mathbb{D}}^G$. I.e. every complete Boolean algebra in $V[G]$ can be represented by a $\mathbb{B}$-name for a complete Boolean algebra in the sense of Definition 3.0.1.*

Proof. Let $\mathbb{E} \in V$ be the trivial complete Boolean algebra with two elements and fix some $b \in G$ such that

$$b \Vdash_{\mathbb{B}} \dot{\mathbb{D}} \text{ is a complete Boolean algebra.}$$

Now fix a maximal antichain $A \subseteq \mathbb{B}$ with $b \in A$ and for each $a \in A$ let

$$\dot{x}_a = \begin{cases} \dot{\mathbb{D}} & \text{, if } a = b \\ \check{\mathbb{E}} & \text{, otherwise} \end{cases}$$

By Lemma 2.3.3 there is a $\mathbb{B}$-name $\dot{\mathbb{C}}$ such that

$$a \Vdash_{\mathbb{B}} \dot{\mathbb{C}} = \dot{x}_a$$

for all $a \in A$.
Given any $\mathbb{B}$-generic filter G there is now a unique $a \in G \cap A$. If $a = b$, then

$$a \Vdash_{\mathbb{B}} \dot{\mathbb{C}} = \dot{\mathbb{D}} \wedge \dot{\mathbb{D}} \text{ is a complete Boolean algebra.}$$

Otherwise

$$a \Vdash_{\mathbb{B}} \dot{\mathbb{C}} = \check{\mathbb{E}} \wedge \check{\mathbb{E}} \text{ is a complete Boolean algebra.}$$

Hence $\mathbb{C}$ satisfies Definition 3.0.1. □

Proposition 3.0.1. *Let $\mathbb{B} = (B; 0, 1, +, \cdot, -)$ be a complete Boolean algebra and let $\dot{\mathbb{C}} = (\dot{C}; \dot{0}, \dot{1}, \dot{+}, \dot{\cdot}, \dot{-})$ be a $\mathbb{B}$-name for a complete Boolean algebra. Then $\mathbb{B} * \dot{\mathbb{C}} := (B * \dot{C}; 0_{\mathbb{B}*\dot{\mathbb{C}}}, 1_{\mathbb{B}*\dot{\mathbb{C}}}, +_{\mathbb{B}*\dot{\mathbb{C}}}, \cdot_{\mathbb{B}*\dot{\mathbb{C}}}, -_{\mathbb{B}*\dot{\mathbb{C}}})$ with*

a) $B * \dot{C} := \{\dot{c} \mid 1 \Vdash_{\mathbb{B}} \dot{c} \in \dot{C}\}$,

b) $0_{\mathbb{B}*\dot{\mathbb{C}}} := \dot{0}$,

c) $1_{\mathbb{B}*\dot{\mathbb{C}}} := \dot{1}$,

d) $+_{\mathbb{B}*\dot{\mathbb{C}}} : B * \dot{C} \times B * \dot{C} \to B * \dot{C}, (\dot{c}, \dot{d}) \mapsto \dot{e}$, *where* $\dot{e} \in V^{\mathbb{B}}$ *is unique such that* $1 \Vdash_{\mathbb{B}} \dot{c} \dot{+} \dot{d} = \dot{e}$,

e) $\cdot_{\mathbb{B}*\dot{\mathbb{C}}} : B * \dot{C} \times B * \dot{C} \to B * \dot{C}, (\dot{c}, \dot{d}) \mapsto \dot{e}$, *where* $\dot{e} \in V^{\mathbb{B}}$ *is unique such that* $1 \Vdash_{\mathbb{B}} \dot{c} \dot{\cdot} \dot{d} = \dot{e}$ *and* *f)* $-_{\mathbb{B}*\dot{\mathbb{C}}} \colon B * \dot{C} \to B * \dot{C}, \dot{c} \mapsto \dot{e}$, *where* $\dot{e} \in V^{\mathbb{B}}$ *is unique such that* $1 \Vdash_{\mathbb{B}} \dot{-}\dot{c} = \dot{e}$

is a complete Boolean algebra and

$$\sigma \colon \mathbb{B} \to \mathbb{B} * \dot{\mathbb{C}}, b \mapsto \dot{b} := \{(\dot{1}, b), (\dot{0}, -b)\}$$

is a complete embedding.

Proof. Since $V^{\mathbb{B}}$ is a full identity model, the definition of $\mathbb{B}*\dot{\mathbb{C}}$ yields a well-defined Boolean algebra and it suffices to prove its completeness:
If $X \subseteq \mathbb{B} * \dot{\mathbb{C}}$, then

$$1 \Vdash_{\mathbb{B}} \check{X} \subseteq \dot{\mathbb{C}}.$$

By the fullness of $V^{\mathbb{B}}$ there is some $\dot{c} \in \dot{\mathbb{C}}$ such that

$$1 \Vdash_{\mathbb{B}} \textstyle\sum_{\dot{\mathbb{C}}} \check{X} = \dot{c} \in \dot{\mathbb{C}},$$

witnessing $\sum_{\mathbb{B}*\dot{\mathbb{C}}} X = \dot{c} \in \mathbb{B} * \dot{\mathbb{C}}$.
It remains to check that

$$\sigma \colon \mathbb{B} \to \mathbb{B} * \dot{\mathbb{C}}, b \mapsto \dot{b} := \{(\dot{1}, b), (\dot{0}, -b)\}$$

is a complete embedding:
Since $\|\dot{1} = \dot{1}\| = 1$ and $\|\dot{0} = \dot{0}\| = 1$, we have $\sigma(1) = \dot{1}$, $\sigma(0) = \dot{0}$ and $\sigma(b) = \dot{0}$ implies $b = 0$. Hence σ is injective.
Given $b \in \mathbb{B}$ and $\dot{b} \in \mathbb{B} * \dot{\mathbb{C}}$ with $\|\dot{b} = \dot{1}\| = b$ and $\|\dot{b} = \dot{0}\| = -b$, we have $-b = \|\dot{-}\dot{b} = \dot{1}\|$ and $-(-b) = b = \|\dot{-}\dot{b} = \dot{0}\|$ and thus $\sigma(-b) = \dot{-}\dot{b} = \dot{-}\sigma(b)$.
If $a, b \in \mathbb{B}$ and $\sigma(a) = \dot{a}$, $\sigma(b) = \dot{b}$, then

$$\begin{aligned} \|\dot{a} \cdot_{\mathbb{B}*\dot{\mathbb{C}}} \dot{b} = \dot{1}\| &= \|\dot{a} = \dot{1}\| \cdot \|\dot{b} = \dot{1}\| \\ &= a \cdot b \end{aligned}$$

and

$$\begin{aligned} \|\dot{a} \cdot_{\mathbb{B}*\dot{\mathbb{C}}} \dot{b} = \dot{0}\| &= \|\dot{a} = \dot{0}\| + \|\dot{b} = \dot{0}\| \\ &= -a + (-b) \\ &= -(a \cdot b), \end{aligned}$$

i.e. $\sigma(a \cdot b) = \dot{a} \cdot_{\mathbb{B}*\dot{\mathbb{C}}} \dot{b} = \sigma(a) \cdot_{\mathbb{B}*\dot{\mathbb{C}}} \sigma(b)$.
This also yields

$$\begin{aligned} \sigma(a + b) &= \sigma(-(-a \cdot -b)) \\ &= \dot{-}(\dot{-}\sigma(a) \cdot_{\mathbb{B}*\dot{\mathbb{C}}} \dot{-}\sigma(b)) \\ &= \sigma(a) +_{\mathbb{B}*\dot{\mathbb{C}}} \sigma(b). \end{aligned}$$

Using Lemma 2.1.1, it now suffices to show $\prod_{\mathbb{B}*\dot{\mathbb{C}}} \sigma"X \preceq_{\mathbb{B}*\dot{\mathbb{C}}} \sigma(\prod_{\mathbb{B}} X)$ for all $X \subseteq \mathbb{B}$: Suppose that

$$1 \Vdash_{\mathbb{B}} \dot{c} \prec_{\mathbb{B}*\dot{\mathbb{C}}} \prod_{\mathbb{B}*\dot{\mathbb{C}}} \sigma"X.$$

There is then some $x \in X$ such that

$$1 \Vdash_{\mathbb{B}} \dot{c} \prec_{\mathbb{B}*\dot{\mathbb{C}}} \sigma(x).$$

But then

$$x \Vdash_{\mathbb{B}} \dot{c} \prec_{\mathbb{B}*\dot{\mathbb{C}}} \dot{1}$$

and consequently

$$\prod_{\mathbb{B}} X \Vdash_{\mathbb{B}} \dot{c} \prec_{\mathbb{B}*\dot{\mathbb{C}}} \dot{1},$$

i.e. $\dot{c} \prec_{\mathbb{B}*\dot{\mathbb{C}}} \sigma(\prod_{\mathbb{B}} X)$. □

Proposition 3.0.2. *Let $\mathbb{B}$ be a complete Boolean algebra and let $\dot{\mathbb{C}}$ be a $\mathbb{B}$-name for a complete Boolean algebra. Then there is a complete Boolean algebra $\mathbb{C}$ such that $\mathbb{B} \sqsubseteq_c \mathbb{C}$ and $1 \Vdash_{\mathbb{B}} \dot{\mathbb{C}} \cong \check{\mathbb{C}}/\dot{G}$, where $\dot{G}$ is the canonical $\mathbb{B}$-generic name.*

Proof. Consider the complete Boolean algebra $\mathbb{B} * \dot{\mathbb{C}}$ and the complete embedding

$$\sigma\colon \mathbb{B} \to \mathbb{B} * \dot{\mathbb{C}},\ b \mapsto \dot{b} := \{(\dot{1}, b), (\dot{0}, -b)\}.$$

Under the isomorphism $h\colon \mathbb{B} \to \sigma"\mathbb{B}$ we have that G is $\mathbb{B}$-generic iff $h"G$ is $h"\mathbb{B}$-generic and in that case $V[G] = V[h"G]$. In particular, if G is a fixed $\mathbb{B}$-generic ultrafilter and $H := h"G$, then for all $\dot{b}, \dot{c} \in \mathbb{B} * \dot{\mathbb{C}}$

$$\begin{aligned} \dot{b}/H \preceq \dot{c}/H &\leftrightarrow \exists h \in H\colon h \Vdash_{h"\mathbb{B}} \dot{b}/\dot{H} \preceq \dot{c}/\dot{H} \\ &\leftrightarrow \exists h \in H\colon h \cdot \dot{b} \preceq \dot{c} \\ &\leftrightarrow \exists g \in G\colon \sigma(g) \cdot \dot{b} \preceq \dot{c} \\ &\overset{g \Vdash_{\mathbb{B}} \sigma(g) = \dot{1}}{\leftrightarrow} \exists g \in G\colon g \Vdash_{\mathbb{B}} \dot{b} \preceq \dot{c} \\ &\leftrightarrow \dot{b}^G \preceq \dot{c}^G, \end{aligned}$$

where in each instance $\preceq$ is interpreted as the underlying partial order. We therefore have the following isomorphism

$$\pi\colon \mathbb{B} * \dot{\mathbb{C}}/\sigma"G \to \dot{\mathbb{C}}^G,\ \dot{b}/\sigma"G \mapsto \dot{b}^G.$$

By replacing the isomorphic copy of $\mathbb{B}$ in $\mathbb{B} * \dot{\mathbb{C}}$ with $\mathbb{B}$, we now obtain a complete Boolean algebra $\mathbb{C}$ and an isomorphism $f\colon \mathbb{B} * \dot{\mathbb{C}} \to \mathbb{C}$ such that $f \circ \sigma = \mathrm{id} \restriction \mathbb{B}$. Then $\mathbb{B} \sqsubseteq_c \mathbb{C}$, $\mathbb{C}/G \cong \mathbb{B} * \dot{\mathbb{C}}/\sigma"G \cong \dot{\mathbb{C}}^G$ and $\mathbb{C}$ is as desired. □

Proposition 3.0.3. *Suppose that $\mathbb{B} \sqsubseteq_c \mathbb{C}$ are complete Boolean algebras, G is $\mathbb{B}$-generic over V and $\overline{H}$ is ${}^{\mathbb{C}}\!/_{G}$-generic over $V[G]$. Then*

$$G * \overline{H} := \{c \in \mathbb{C} \mid {}^{c}\!/_{G} \in \overline{H}\}$$

is $\mathbb{C}$-generic over V and $V[G * \overline{H}] = (V[G])\,[\overline{H}]$.

Proof. Working in $V[G * \overline{H}]$, we first check that $G * \overline{H}$ is an ultrafilter:
Since ${}^{0}\!/_{G} \notin \overline{H}$, we have $0 \notin G * \overline{H}$. Now for $b, c \in \mathbb{C}$

$$\begin{aligned} b, c \in G * \overline{H} &\leftrightarrow {}^{b}\!/_{G}, {}^{c}\!/_{G} \in \overline{H} \\ &\rightarrow {}^{b}\!/_{G} \cdot {}^{c}\!/_{G} = {}^{b \cdot c}\!/_{G} \in \overline{H} \\ &\rightarrow b \cdot c \in G * \overline{H}. \end{aligned}$$

Morever

$$\begin{aligned} b \in G * \overline{H} \wedge b \preceq c &\rightarrow {}^{b}\!/_{G} \in \overline{H} \wedge {}^{b}\!/_{G} \preceq {}^{c}\!/_{G} \\ &\rightarrow {}^{c}\!/_{G} \in \overline{H} \\ &\rightarrow c \in G * \overline{H}. \end{aligned}$$

Finally, for $c \in \mathbb{C}$, we have ${}^{c}\!/_{G} + {}^{-c}\!/_{G} = {}^{(c + (-c))}\!/_{G} = {}^{1}\!/_{G} \in \overline{H}$. Since $\overline{H}$ is ${}^{\mathbb{C}}\!/_{G}$-generic, we thus have ${}^{c}\!/_{G} \in \overline{H}$ or ${}^{-c}\!/_{G} \in \overline{H}$, i.e. $c \in G * \overline{H}$ or $-c \in G * \overline{H}$. Therefore $G*\overline{H}$ is indeed an ultrafilter on $\mathbb{C}$. We now prove that $G*\overline{H}$ is $\mathbb{C}$-generic over V:
Let $D \subseteq \mathbb{C}^+$ be dense. Since for $c, d \in \mathbb{C}$ with $c \preceq d$ we also have ${}^{c}\!/_{G} \preceq {}^{d}\!/_{G}$, the set

$$\overline{D} := \{{}^{d}\!/_{G} \mid d \in D\}$$

is dense in ${}^{\mathbb{C}}\!/_{G}$. Now $\overline{H}$ is ${}^{\mathbb{C}}\!/_{G}$-generic over $V[G]$ which allows us to fix some $d \in D$ such that ${}^{d}\!/_{G} \in \overline{D} \cap \overline{H}$, i.e. $d \in D \cap (G * \overline{H})$. Finally, let us verify that $V[G * \overline{H}] = (V[G])[\overline{H}]$:
We have $G, \overline{H} \in (V[G])[\overline{H}]$ and thus $G * \overline{H} = \{c \in \mathbb{C} \mid {}^{c}\!/_{G} \in \overline{H}\} \in (V[G])[H]$. Since $V[G*H]$ is the minimal class model of ZFC containing $V \cup \{G*\overline{H}$, it follows that $V[G * \overline{H}] \subseteq (V[G])[\overline{H}]$.
In order to prove equality, it now suffices to check that $G, \overline{H} \in V[G*\overline{H}]$. Working in $V[G * \overline{H}]$, we may define $G^* := \mathbb{B}^+ \cap (G * \overline{H})$. Thus the following yields $G \in V[G * \overline{H}]$:

Claim. $G^* = G$.

Proof. For all $b \in \mathbb{B}$ we have either $b \in G$ or $-b \in G$. In the first case, ${}^{b}\!/\!_{G} = {}^{1}\!/\!_{G} \in \overline{H}$ and in the latter case ${}^{b}\!/\!_{G} = {}^{0}\!/\!_{G} \notin \overline{H}$. Thus for all $b \in \mathbb{B}$:

$$b \in G \leftrightarrow {}^{b}\!/\!_{G} \in \overline{H} \leftrightarrow b \in G * \overline{H},$$

i.e. $G = G^*$. ■

Knowing that $G \in V[G * \overline{H}]$, we may now define $\overline{H}^* := \{{}^{c}\!/\!_{G} \mid c \in G * \overline{H}\}$ in $V[G * H]$. In order to see that $\overline{H} \in V[G * \overline{H}]$, it suffices to prove the following

Claim. $\overline{H} = \overline{H}^*$.

Proof. Let ${}^{c}\!/\!_{G} \in \overline{H}^*$. Then $c \in G * \overline{H}$ and consequently ${}^{c}\!/\!_{G} \in \overline{H}$, i.e. $\overline{H}^* \subseteq \overline{H}$. Suppose that $\overline{H} \nsubseteq \overline{H}^*$. Then there is some $c \in \mathbb{C}$ such that ${}^{c}\!/\!_{G} \in \overline{H} \setminus \overline{H}^*$. By the definition of $\overline{H}^*$, we now have $c \notin G * \overline{H}$ and, since $G * \overline{H}$ is an ultrafilter, also $-c \in G * \overline{H}$. This yields ${}^{-c}\!/\!_{G} = -\left({}^{c}\!/\!_{G}\right) \in \overline{H}$. (Contradiction!) ■

□

Let us outline how the results above can be applied: Suppose that we have some complete Boolean algebra $\mathbb{B} \in V$ and some $\mathbb{B}$-generic filter G (over V). Form the generic extension $V[G]$ and fix some $\dot{\mathbb{C}}^G \in V[G]$ that is a complete Boolean algebra in $V[G]$. By Lemma 3.0.1 we may assume that $1 \Vdash_{\mathbb{B}} \dot{\mathbb{C}}$ is a complete Boolean algebra. For any H that is $\dot{\mathbb{C}}^G$-generic over $V[G]$, we may now form a second generic extension $(V[G])[H]$. The same could have been achieved by first choosing a complete Boolean algebra $\mathbb{C}$ that completely contains $\mathbb{B}$ and such that $1 \Vdash_{\mathbb{B}} \dot{\mathbb{C}} \cong {}^{\check{\mathbb{C}}}\!/\!_{\dot{G}}$:
Let $\overline{H}$ be the pointwise image of H under the aforementioned isomorphism. Then $G * \overline{H}$ is $\mathbb{C}$-generic over V and $V[G * \overline{H}] = (V[G])[H]$.
Conversely suppose that we started with a complete Boolean algebra $\mathbb{C}$ in V and some H that is $\mathbb{C}$-generic over V. It's natural to ask, whether $V[H]$ can be obtained by a two step iteration, first extending V to $V[G]$ for some $\mathbb{B} \sqsubseteq_c \mathbb{C}$ and some G that is $\mathbb{B}$-generic over V and then extending $V[G]$ further by some $\overline{H}$ that is ${}^{\mathbb{C}}\!/\!_{G}$-generic over $V[G]$. Since ${}^{\mathbb{C}}\!/\!_{G}$ is a complete Boolean algebra in $V[G]$ (see Proposition 2.3.2), this approach results in a legitimate generic extension $V[G*\overline{H}]$. We will now show that for any given $\mathbb{B} \sqsubseteq_c \mathbb{C}$ there is a natural way to transform H into some $\mathbb{B}$-generic G and some $\overline{H}$ that is ${}^{\mathbb{C}}\!/\!_{G}$-generic over $V[G]$ such that $V[G * \overline{H}] = (V[G])[\overline{H}] = V[H]$ and in fact $G * \overline{H} = H$.

Proposition 3.0.4. *Suppose that* $\mathbb{B} \sqsubseteq_c \mathbb{C}$ *are complete Boolean algebras and* H *is* $\mathbb{C}$*-generic over* V*. Let*

$$G := \mathbb{B} \cap H$$

and

$$\overline{H} := \{{}^{c}\!/_{G} \mid c \in H\}.$$

Then

1. *G is $\mathbb{B}$-generic over V,*
2. *$\overline{H}$ is ${}^{\mathbb{C}}\!/_{G}$-generic over $V[G]$ and*
3. *$H = G * \overline{H}$.*

Proof. 1. Let $X \subseteq G = \mathbb{B} \cap H$. Since H is $\mathbb{C}$-generic over V, we have $\prod_{\mathbb{C}} H \in H$. Now $\mathbb{B}$ is completely contained in $\mathbb{C}$ and thus $\prod_{\mathbb{B}} X = \prod_{\mathbb{C}} X \in \mathbb{B} \cap H = G$. By Lemma 2.2.1 this implies that G is $\mathbb{B}$-generic over V.

2. Working in $V[G]$, let $G' := \{c \in \mathbb{C} \mid \exists b \in G \colon b \preceq c\}$. As our first step we verify that $\overline{H}$ is a filter over ${}^{\mathbb{C}}\!/_{G'}$:
For any $c \in \mathbb{C}$ we have ${}^{c}\!/_{G'} = {}^{0}\!/_{G'}$ iff $-\Delta(c, 0) = -c \in G'$ iff there is some $b \in G$ such that $b \preceq -c$. Assume, towards a contradiction, that ${}^{0}\!/_{G'} \in \overline{H}$. Then there is some $h \in H$ and some $b \in G$ such that $b \preceq -h$. Since $G \subseteq H$ and since H is a filter, this implies $-h \in H$ and consequently $-h \cdot h = 0 \in H$. (Contradiction!)
Now let $x, y \in \overline{H}$ and fix $g, h \in H$ such that $x = {}^{g}\!/_{G'}$ and $y = {}^{h}\!/_{G'}$. Since $g \cdot h \in H$, we have $x \cdot y = {}^{g \cdot h}\!/_{G'} \in \overline{H}$.
Finally suppose that $x \in \overline{H}$ and $y \in {}^{\mathbb{C}}\!/_{G'}$ are such that $x \preceq y$. Fix $h \in H$ and $c \in \mathbb{C}$ with $x = {}^{h}\!/_{G'}$ and $y = {}^{c}\!/_{G'}$. Observe that

$$\begin{aligned} {}^{h}\!/_{G'} \preceq {}^{c}\!/_{G'} &\leftrightarrow {}^{h \cdot (-c)}\!/_{G'} = {}^{0}\!/_{G'} \\ &\leftrightarrow \exists b \in G \colon b \preceq -(h \cdot (-c)) = -h + c \\ &\leftrightarrow \exists b \in G \colon b + h \preceq c. \end{aligned}$$

Since $b + h \in H$, we have $c \in H$ and thus $y = {}^{c}\!/_{G'} \in \overline{H}$.
Next, let $c \in \mathbb{C}$. Since H is an ultrafilter, we have $c \in H$ or $-c \in H$ and thus ${}^{c}\!/_{G'} \in \overline{H}$ or $-{}^{c}\!/_{G'} = {}^{-c}\!/_{G'} \in \overline{H}$. It follows that $\overline{H}$ is an ultrafilter on ${}^{\mathbb{C}}\!/_{G'}$ and it only remains to check that $\overline{H}$ is ${}^{\mathbb{C}}\!/_{G'}$-generic over $V[G]$: Let $\dot{D}$ be a $\mathbb{B}$-name such that $\dot{D}^G$ is a dense subset of ${}^{\mathbb{C}}\!/_{G}$. Fix $b \in G$ such that $b \Vdash_{\mathbb{B}} \dot{D} \subseteq {}^{\check{\mathbb{C}}}\!/_{\dot{G}}$ is dense. In V, we may now define

$$E := \{e \in \mathbb{C}^+ \mid \{b' \in \mathbb{B}^+ \mid b \cdot b' = 0 \vee [b' \preceq b \wedge (b' \Vdash_{\mathbb{B}} {}^{\check{e}}\!/_{\dot{G}} = {}^{\check{0}}\!/_{\dot{G}} \vee b' \Vdash_{\mathbb{B}} {}^{\check{e}}\!/_{\dot{G}} \in \dot{D})]\} \text{ is dense in } \mathbb{B}\}$$

Claim. *E is dense in $\mathbb{C}$.*

Proof. Fix $c \in \mathbb{C}^+$. We construct some $e \in \mathbb{C}^+$ such that $e \preceq c$ and $e \in E$: Let $F_0 := \{b' \in \mathbb{B}^+ \mid b \cdot b' = 0\}$, $F_1 := \{b' \in \mathbb{B}^+ \mid b' \preceq b \wedge b' \Vdash_{\mathbb{B}} \check{c}/\dot{G} = \check{0}/\dot{G}\}$ and $F_2 := \{b' \in \mathbb{B}^+ \mid b' \preceq b \wedge \exists f \in \mathbb{C}^+ \colon b' \Vdash_{\mathbb{B}} \check{f}/\dot{G} \in \dot{D} \wedge \check{f}/\dot{G} \preceq \check{c}/\dot{G}\}$. Since $b \Vdash_{\mathbb{B}} \dot{D}$ is dense in $\check{\mathbb{C}}/\dot{G}$, we conclude that $F := F_0 \cup F_1 \cup F_2$ is dense in $\mathbb{B}$. Fix a maximal antichain $A \subseteq \mathbb{B}$ and for each $a \in A \cap F_2$ some f_a such that $a \Vdash_{\mathbb{B}} \check{f_a}/\dot{G} \in \dot{D} \wedge \check{f_a}/\dot{G} \preceq \check{c}/\dot{G}$. Let $e := \sum_{a \in A \cap F_1} a \cdot c + \sum_{a \in A \cap F_2} a \cdot f_a$. By Proposition 2.3.1 we have $a \cdot f_a \preceq c$ for all $a \in A \cap F_2$ and thus $e \preceq c$. Using Lemma 2.3.1, we may further conclude the following:
Let $\overline{a} \in A$

- If $\overline{a} \in F_1$, then $\overline{a} \Vdash_{\mathbb{B}} \check{e}/\dot{G} = \check{c}/\dot{G} = \check{0}/\dot{G}$.
- If $\overline{a} \in F_2$, then $\overline{a} \Vdash_{\mathbb{B}} \check{e}/\dot{G} = \check{f_{\overline{a}}}/\dot{G} \in \dot{D} \wedge \check{f_{\overline{a}}}/\dot{G} \preceq \check{c}/\dot{G}$.

Let $A_{\downarrow} := \{b' \in \mathbb{B} \mid \exists a \in A \colon b' \preceq a\}$. Since A is a maximal antichain, Proposition 2.2.2 yields that $A_{\downarrow}$ is dense and by the above, we see that

$$A_{\downarrow} \subseteq \{b' \in \mathbb{B}^+ \mid b \cdot b' = 0 \vee [b' \preceq b \wedge (b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} = \check{0}/\dot{G} \vee b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} \in \dot{D})]\}.$$

Therefore $A_{\downarrow}$ witnesses that $e \in E$, which finishes our proof. ■

Since H is $\mathbb{C}$-generic, we may fix some $e \in H \cap E$ and by the definition of E, we know that

$$E_e :=\subseteq \{b' \in \mathbb{B}^+ \mid b \cdot b' = 0 \vee [b' \preceq b \wedge (b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} = \check{0}/\dot{G} \vee b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} \in \dot{D})]\}$$

is a dense subset of $\mathbb{B}$. Now G ist $\mathbb{B}$-generic and thus there is some $b' \in G \cap E_e$. Since $b \in G$, we cannot have $b \cdot b' = 0$. Furthermore

$$\begin{aligned} b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} = \check{0}/\dot{G} &\leftrightarrow -\Delta(b' \cdot e, 0) \in G' \\ &\leftrightarrow \exists b'' \in G \colon b'' \preceq -\Delta(b' \cdot e, 0) \\ &\leftrightarrow b'' \cdot b' \cdot e = 0, \end{aligned}$$

which is impossible, since $b', b'' \in G \subseteq H$ and $e \in H$. Therefore we must have

$$b' \Vdash_{\mathbb{B}} \check{e}/\dot{G} \in \dot{D},$$

i.e. $e/G \in \dot{D}^G \cap \overline{H}$, verifying that $\overline{H}$ is $\mathbb{C}/G$-generic in $V[G]$.

3.

$$\begin{aligned} G * \overline{H} &= \{c \in \mathbb{C} \mid {}^{c}\!/\!_{G} \in \overline{H}\} \\ &= \{c \in \mathbb{C} \mid c \in H\} \\ &= H. \end{aligned}$$

□

Thus forming iterated generic extensions $V \subseteq V[G] \subseteq (V[G])[\overline{H}]$ can be equivalently described by forming a single generic extension $V[G * \overline{H}]$ via some $\mathbb{C}$ and some $\mathbb{B} \sqsubseteq_c \mathbb{C}$ satisfying $V[G * \overline{H} \cap \mathbb{B}] = V[G]$. More generally, given a finite sequence $V \subseteq V_1 \subseteq \ldots \subseteq V_k$ of generic extensions $V_{i+1} = V_i[G_i]$ [1] for all $i < k$, we can form a sequence $\mathbb{B}_1 \sqsubseteq_c \mathbb{B}_2 \sqsubseteq_c \ldots \sqsubseteq_c \mathbb{B}_k$ of complete Boolean algebras in V and some G that is $\mathbb{B}_k$-generic over V such that $V[G \cap B_i] = V_i[G_i]$ for all $i < k$ and conversely, any finite sequence $\mathbb{B}_1 \sqsubseteq_c \mathbb{B}_2 \sqsubseteq_c \ldots \sqsubseteq_c \mathbb{B}_k$ gives rise to a finite sequence of iterated generic extensions when viewing ${}^{\mathbb{B}_{i+1}}\!/\!_{G_i}$ as a complete Boolean algebra in $V_i[G_i]$ [2] .
Now suppose that we have a countable sequence $\mathbb{B}_1 \sqsubseteq_c \mathbb{B}_2 \sqsubseteq_c \ldots$ of complete Boolean algebras. Picking some complete Boolean algebra $\mathbb{B}$ that completely contains all of the $\mathbb{B}_i$'s allows us to factor any generic extension $V[G]$ via $\mathbb{B}$ into a sequence of iterated generic extensions $V[G \cap \mathbb{B}_1] \subseteq V[G \cap \mathbb{B}_2] \ldots \subseteq V[G]$. In general (e.g. when $\mathbb{B}$ is much "bigger" than all of the $\mathbb{B}_i$'s), there is little hope that $V[G]$ or $\mathbb{B}$ has much in common with any of its "factors". One might hope to be able to choose $\mathbb{B}$ in a way that preserves desired properties of the $\mathbb{B}_i$'s and generic extensions thereof. Before discussing this question any further, we will now introduce *General Iterations* of forcing extensions.

3.2 General Iterations

Since we are working with complete Boolean algebras, there is a 'natural' approach to generalize finite forcing extentions to infinite ones. We will now check that this approach works.

Definition 3.0.2. *A* forcing iteration *of length $\alpha > 0$ is a sequence $(\mathbb{B}_i \mid i < \alpha)$ such that*

1. *$\mathbb{B}_i$ is a complete Boolean algebra for all $i < \alpha$,*
2. *$\mathbb{B}_i \sqsubseteq_c \mathbb{B}_j$ for all $i < j < \alpha$ and*

[1] where each V_{i+1} is a generic extension with respect to V_i
[2] this involves an inessential abuse of notation for $i > 2$

3. $\mathbb{B}_\lambda$ *is generated by* $\bigcup_{i<\lambda} \mathbb{B}_i$ *for all limit ordinals* $\lambda < \alpha$*, i.e. for all* $\mathbb{B} \sqsubseteq_c \mathbb{B}_\lambda$ *with* $\bigcup_{i<\lambda} \mathbb{B}_i \subseteq \mathbb{B}$ *we already have* $\mathbb{B} = \mathbb{B}_\lambda$*.*

In inductive arguments about forcing iterations, it is often useful to require that $\mathbb{B}_0 = \{0,1\}$ is the trivial complete Boolean algebra. Since $V^{\mathbb{B}_0} \cong V$, this is completely harmless.

Notation 3.0.1. *If* $(\mathbb{B}_i \mid i < \alpha)$ *is a forcing iteration, we write* $\dot{G}_i$ *for the canonical* $\mathbb{B}_i$*-generic name and* G_i *denotes a* $\mathbb{B}_i$*-generic ultrafilter. We also write* $\Vdash_i$ *instead of* $\Vdash_{\mathbb{B}_i}$*.*

Before discussing the requirements at limit stages, we would like to justify our definition of forcing iterations at successor levels. In practice, forcing iterations are usually built recursively, first picking some complete Boolean algebra $\mathbb{C}_0 \in V$ and forming the generic extension $V[G_0]$ via some $\mathbb{C}_0$-generic ultrafilter H_0. At stage $i+1$ we would then pick some $\mathbb{C}_{i+1} \in V[H_i]$ such that $V[H_i] \models \mathbb{C}_{i+1}$ is a complete Boolean algebra. This allows us to continue our construction by picking some H_{i+1} that is $\mathbb{C}_{i+1}$-generic over $V[H_i]$ to form $(V[H_i])[H_{i+1}]$. By managing some notational obstacles, we can turn this iterative construction into a forcing iteration $(\mathbb{B}_i \mid i < \alpha)$:
Let $\mathbb{B}_0 := \mathbb{C}_0$ and for $i = 1$ fix a complete Boolean algebra $\mathbb{B}_0 \sqsubseteq_c \mathbb{B}_1$ such that

$$1 \Vdash_{\mathbb{B}_0} {}^{\check{\mathbb{B}}_1}\!/_{\dot{G}_0} \cong \mathbb{C}_1.$$

In the previous section, we've developed the tools to translate any generic extension $(V[H_0])[H_1]$, where H_0 is $\mathbb{C}_0$-generic over V and H_1 is $\mathbb{C}_1$-generic over $V[H_0]$, to a single generic extension $V[G_1]$ for some $\mathbb{B}_1$-generic G_1 and vice versa. If $i > 2$, a minor difficulty arises. For the sake of notational simplicity, we will only discuss the case $i = 2$ in detail, which readily generalizes to all successor steps:
We are in the situation that $(V[H_0])[H_1] = V[G_1] \models \mathbb{C}_2$ is a complete Boolean algebra. Pick a complete Boolean algebra $\mathbb{B}_1 \sqsubseteq_c \mathbb{B}_2 \in V$ such that

$$1 \Vdash_{\mathbb{B}_1} {}^{\check{\mathbb{B}}_2}\!/_{\dot{G}_1} \cong \mathbb{C}_2.$$

We would like to arrange that additionally

$$1 \Vdash_{\mathbb{B}_0} {}^{\check{\mathbb{B}}_1}\!/_{\dot{G}_0} \sqsubseteq_c {}^{\check{\mathbb{B}}_2}\!/_{\dot{G}_0},$$

allowing us to view the sequence of quotients $({}^{\mathbb{B}_i}\!/_{G_0} \mid i < \alpha)$ as a forcing iteration in $V[G_0]$. While this may not be possible in general, the following Lemma shows that we can fix this issue by identifying the respective quotients with an isomorphic copy.

Lemma 3.0.2. *Let $\mathbb{A} \sqsubseteq_c \mathbb{B} \sqsubseteq_c \mathbb{C}$ be complete Boolean algebras and let F be $\mathbb{A}$-generic. Then there is a canonical complete embedding*

$$\pi\colon \mathbb{B}/F \to \mathbb{C}/F$$

(in $V[F]$).

Proof. Working in $V[F]$, let $F' := \{b \in \mathbb{B} \mid \exists a \in F\colon a \preceq b\}$, $F'' := \{b \in \mathbb{C} \mid \exists a \in F\colon a \preceq c\}$ be the upward closure of F in $\mathbb{B}$ and $\mathbb{C}$. Furthermore, let $\sigma\colon \mathbb{B} \to \mathbb{B}/F', b \mapsto a/F$ and $\tau\colon \mathbb{C} \to \mathbb{C}/F'', c \mapsto c/F$ be the associated quotient maps and define

$$\pi\colon \mathbb{B}/F \to \mathbb{C}/F, \sigma(b) \mapsto \tau(b).$$

We have the following commutative diagram

$$\begin{array}{ccc} \mathbb{B} & \xrightarrow{\sigma} & \mathbb{B}/F' \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{C} & \xrightarrow[\tau]{} & \mathbb{C}/F'' \end{array}$$

Since $\mathbb{B}$ is completely contained in $\mathbb{C}$ and the canonical projections are complete homomorphisms, it follows that π is a complete homomorphism. It hence suffices to check that π is injective:
Let $b \in \mathbb{B}$ be such that $\pi(\sigma(b)) = 0/F''$. Then $b/F'' = 0/F''$ and thus $-\Delta(b,0) = -b \in F''$. By definition of F'', there is hence some $a \in F$ with $a \preceq -b$. Thus $\sigma(b) = b/F' = 0/F'$, i.e. $\sigma(b) = 0_{\mathbb{B}/F}$. □

From now on, we will identify $\mathbb{B}/F$ with $\pi``B/F \sqsubseteq_c \mathbb{C}/F$ and hence simply write $\mathbb{B}/F \sqsubseteq_c \mathbb{C}/F$ in these situations. So, by a slight abuse of notation, we may in fact require

$$1 \Vdash_i \check{\mathbb{B}}_j/\dot{G}_i \sqsubseteq_c \check{\mathbb{B}}_k/\dot{G}_i$$

for $i \leq j \leq k < \alpha$ in a given forcing iteration $(\mathbb{B}_i \mid i < \alpha)$.
This convention is particularly useful, when viewing a given forcing iteration inside one of its generic extensions.
For the rest of this section fix a forcing iteration $\mathcal{B} = (\mathbb{B}_i \mid i < \alpha)$ and $\mathbb{B} := \bigcup_{i<\alpha} \mathbb{B}_i$ together with the canonical projections

$$h_i\colon \mathbb{B} \to \mathbb{B}_i, a \mapsto \prod\{b \in \mathbb{B}_i \mid a \preceq b\}.$$

Recall that $\mathbb{B}$ is a well-defined Boolean algebra such that $\mathbb{B}_i \sqsubseteq \mathbb{B}$ and $h_i = h_{\mathbb{B}_i,\mathbb{B}_j} \circ h_j$ for all $i \leq j < \alpha$ (see Proposition 2.1.3).

Proposition 3.0.5. *Let $k < \alpha$ and let G be $\mathbb{B}_k$-generic over V. In $V[G]$ define*

$$\mathcal{B}/G := (\mathbb{B}_i/G \mid k \leq i < \alpha).$$

Then $V[G] \models \mathcal{B}/G$ is a forcing iteration.

Proof. We already know that, for $k \leq i < \alpha$, $\mathbb{B}_i/G$ is a complete Boolean algebra in $V[G]$ and by our convention, we also have $\mathbb{B}_i/G \sqsubseteq_c \mathbb{B}_j/G$ for all $k \leq i \leq j < \alpha$. So let $k \leq \lambda < \alpha$ be a limit ordinal. We have to prove that $\bigcup_{k \leq i < \alpha} \mathbb{B}_i/G$ generates $\mathbb{B}_\lambda/G$. If $k = \lambda$, then $\mathbb{B}_\lambda/G$ is the trivial algebra and thus generated by the empty set. Thus, suppose that $k < \lambda$ and let $\dot{\mathbb{C}}$ be a $\mathbb{B}_k$-name for a complete Boolean algebra such that $\bigcup_{k \leq i < \lambda} \mathbb{B}_i/G \sqsubseteq_c \dot{\mathbb{C}}^G \sqsubseteq_c \mathbb{B}_\lambda/G$. Fix some $g \in G$ such that

$$g \Vdash_k \bigcup_{k \leq i < \lambda} \check{\mathbb{B}}_i/\dot{G} \sqsubseteq_c \dot{\mathbb{C}} \sqsubseteq_c \check{\mathbb{B}}_\lambda/\dot{G}$$

and in V, let $\mathbb{D} := \{d \in \mathbb{B}_\lambda \mid g \Vdash_k \check{d}/\dot{G} \in \dot{\mathbb{C}}\}$. By repeating the proof for Proposition 3.0.1, we see that $\mathbb{D}$ is completely contained in $\mathbb{B}_\lambda$. Moreover, for any $k \leq i < \lambda$ and any $b \in \mathbb{B}_i$ we have $g \Vdash_k \check{b}/\dot{G} \in \dot{\mathbb{C}}$ and hence $\bigcup_{i<\lambda} \mathbb{B}_i = \bigcup_{k \leq i < \lambda} \mathbb{B}_i \subseteq \mathbb{D}$. Since $\mathbb{B}_\lambda$ is generated by $\bigcup_{i<\lambda} \mathbb{B}_i$, we thus have $\mathbb{D} = \mathbb{B}_\lambda$. In particular, for any $d \in \mathbb{B}_\lambda$, this yields $g \Vdash_k \check{d}/\dot{G} \in \dot{\mathbb{C}}$ and consequently $\dot{\mathbb{C}}^G = \mathbb{B}_\lambda/G$. □

We next shift our focus on the limit steps of forcing iterations. Let $\lambda < \alpha$ be a limit ordinal. We require that $\bigcup_{i<\lambda} \mathbb{B}_i \sqsubseteq \mathbb{B}_\lambda$ generates $\mathbb{B}_\lambda$. This property is certainly satisfied, if $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$ and one might be tempted to think that every generating subalgebra of a given complete Boolean algebra is in fact dense. This, however, is not the case. Take for example $\mathbb{C} = (\mathbb{R}; \emptyset, \mathbb{R}, \cup, \cap, ^{\complement})$ and let $\mathbb{B}$ be the subalgebra consisting of all set of the form $[a, b), [a, \infty), (-\infty, b), (-\infty, \infty)$ for $a \leq b$, $a, b \in \mathbb{R}$. Since $\{a\} = \bigcap_{b>a} [a, b)$, $\mathbb{B}$ clearly generates $\mathbb{C}$. On the other hand $\mathbb{B}$ doesn't contain any singletons and therefore isn't dense in $\mathbb{C}$. The reader may also notice that $\text{card}(\mathbb{B}) = 2^{\aleph_0} < 2^{2^{\aleph_0}} = \text{card}(\mathbb{C})$, demonstrating that the size of a complete Boolean algebra, in general, isn't bound by the size of a generating set.
In fact, our definition of forcing iterations gives us a lot of freedom as how to choose $\mathbb{B}_\lambda$ at limit steps. In these notes, however, all limit steps will be obtained in one of the following ways:
We begin with the smallest limit construction.

Definition 3.0.3. *Let $\lambda < \alpha$ be a limit ordinal. $\mathbb{B}_\lambda$ is a* direct limit *iff $\bigcup_{i<\lambda} \mathbb{B}_i$ is dense in $\mathbb{B}_\lambda$ and $\mathcal{B} = (\mathbb{B}_i \mid i < \lambda)$ is a* direct limit iteration *iff $\mathbb{B}_\lambda$ is a direct limit for all limit $\lambda < \alpha$.*

So, up to isomorphism, the direct limit at stage λ is just the Boolean completion of $\bigcup_{i<\lambda} \mathbb{B}_i$ and any forcing iteration $(\mathbb{B}_i \mid i < \lambda)$ for some limit ordinal λ can be extended to a forcing iteration $(\mathbb{B}_i \mid i < \lambda + 1)$, by taking $\mathbb{B}_\lambda$ as the direct limit. Abstracting direct limit iterations provides a useful hint as how to obtain "bigger" limit stages.

Definition 3.0.4. *Let $\lambda \leq \alpha$ be a limit ordinal. A* thread *through $\mathcal{B}$ of* length λ *is a sequence $\mathfrak{b} = (b_i \mid i < \lambda) \in \Pi_{i<\lambda}\mathbb{B}_i$ such that*

- *$b_0 \neq 0$ and*
- *$h_i(b_j) = b_i$ for all $i \leq j < \alpha$.*

If $\mathfrak{b} = (b_i \mid i < \lambda)$ is a thread through $\mathcal{B}$ and $\lambda < \alpha$, we let $\mathfrak{b}^* := \prod_{\mathbb{B}_\lambda}\{b_i \mid i < \lambda\}$.

Note that a $\mathfrak{b} = (b_i \mid i < \lambda)$ is uniquely determinded by any of its tail ends. In fact:

Proposition 3.0.6. *Let $\lambda \leq \alpha$ be a limit ordinal and let $\mathfrak{b} = (b_i \mid i < \lambda), \mathfrak{c} = (c_i \mid i < \lambda)$ be two threads through $(\mathbb{B}_i \mid i < \lambda)$ such that $\{i < \lambda \mid b_i = c_i\}$ is cofinal. Then $\mathfrak{b} = \mathfrak{c}$.*

Proof. For $j < \lambda$ fix $j < k \in \{i < \lambda \mid b_i = c_i\}$. Then

$$b_j = h_j(b_k) = h_j(c_k) = c_j.$$

□

Proposition 3.0.7. *Let $\lambda \leq \alpha$ be a limit ordinal and let $(b_i \mid i < \lambda)$ be a thread through $\mathcal{B}$. Then*

1. *$h_i(b_j) \neq 0$ for all $i \leq j < \lambda$ and*
2. *$b_i \succeq b_j$ for all $i \leq j < \lambda$.*

Proof. 1.
$$0 \neq b_0 = h_0(b_j) = h_{\mathbb{B}_0,\mathbb{B}_i} \circ h_i(b_j)$$
and thus $h_i(b_j) \neq 0$.

2.
$$b_i = h_i(b_j) = \prod\{b \in \mathbb{B}_i \mid b_j \preceq b\} \succeq b_j.$$

□

Lemma 3.0.3. *Let $\lambda < \alpha$ be a limit ordinal, let $h < \lambda$ and let G be $\mathbb{B}_h$-generic over V. Let $\mathcal{T}$ be the set of all threads through $(\mathbb{B}_i \mid i < \lambda)$ and let ${}^{\mathcal{T}}\!/_{G}$ be the set of all threads through $({}^{\mathbb{B}_i}\!/_{G} \mid h \leq i < \lambda)$ (in $V[G]$). The map*

$$\mathcal{T} \to {}^{\mathcal{T}}\!/_{G}, (b_i \mid i < \lambda) \mapsto ({}^{b_i}\!/_{G} \mid h \leq i < \lambda)$$

is surjective.

Proof. Work in $V[G]$. Let $\mathfrak{c} = ({}^{c_i}\!/_{G} \mid h \leq i < \lambda)$ be a thread through $({}^{\mathbb{B}_i}\!/_{G} \mid h \leq i < \lambda)$ and let $\dot{c}$ be a $\mathbb{B}_h$-name such that $\prod\{{}^{c_i}\!/_{G} \mid h \leq i < \lambda\} = \dot{c}^G$. By Proposition 2.3.4, we may fix some $b \in G$ and some $c \in \mathbb{B}_\lambda$ such that

$$b \Vdash_{\mathbb{B}_h} \dot{c} = {}^{\check{c}}\!/_{\dot{G}}.$$

In V, let $\mathfrak{b} = (b_i \mid i < \lambda)$ be the unique thread through $(\mathbb{B}_i \mid i < \lambda)$ such that $b_i = h_i(c)$. Back in $V[G]$, we use Proposition 2.3.5 to conclude that, for $h \leq i < \lambda$, we have

$$\begin{aligned} {}^{c_i}\!/_{G} &= h_{{}^{\mathbb{B}_i}\!/_{G}, {}^{\mathbb{B}_\lambda}\!/_{G}}({}^{c}\!/_{G}) \\ &= {}^{h_{\mathbb{B}_i,\mathbb{B}_\lambda}(c)}\!/_{G} \\ &= {}^{b_i}\!/_{G} \end{aligned}$$

and thus $\mathfrak{c} = ({}^{b_i}\!/_{G} \mid h \leq i < \lambda)$. □

Let us stress that, in general, the sequence $(c_i \mid h \leq i < \lambda)$ is not an element of V and $\prod\{c_i \mid h \leq i < \lambda\}$ may not exist in $\mathbb{B}_\lambda$. However, since ${}^{\mathbb{B}_\lambda}\!/_{G}$ is complete in $V[G]$(!), we are still able to form its infinum - modulo G - and thereby obtain a suitable thread that maps to $\mathfrak{c}$.

Definition 3.0.5. *Let $\lambda \leq \alpha$ be a limit ordinal an let $\mathfrak{b} = (b_i \mid i < \lambda)$ be a thread through $\mathcal{B}$. Then*

$$\operatorname{supp}(\mathfrak{b}) := \{j < \lambda \mid \forall i < j \colon b_i \neq b_j\}$$

is the support *of $\mathfrak{b}$.*
$\mathfrak{b}$ is eventually constant *iff* $\operatorname{supp}(\mathfrak{b})$ *is bounded, i.e. there is some $j < \lambda$ such that $b_i = b_j$ for all $j \leq i < \lambda$.*
Finally, for each $b \in \bigcup_{i<\lambda} \mathbb{B}_i$ we define the eventually constant thread [3] $c(b) = (c_i \mid i < \lambda)$ by

$$c_i = \begin{cases} h_i(b) & , \text{ if } b \notin \mathbb{B}_i \\ b & , \text{ otherwise.} \end{cases}$$

[3] note that this yields a well-defined thread, since $h_i = h_{\mathbb{B}_i,\mathbb{B}_j} \circ h_j$ for all $i \leq j < \lambda$

As an immediate Corollary of Proposition 3.0.6 we get the following: Let $\mathfrak{b} = (b_i \mid i < \lambda)$ be an eventually constant thread and let $j < \lambda$ be such that $b_i = b_j$ for all $j \leq i < \lambda$. Then $c(b_j) = \mathfrak{b}$.

Proposition 3.0.8. *Let $\lambda < \alpha$ be a limit ordinal. The following are equivalent:*

1. $\mathbb{B}_\lambda$ *is a direct limit and*

2. $\mathfrak{b}^* \neq 0$ *for every eventually constant thread* $\mathfrak{b}$ *and*
   $\{\mathfrak{b}^* \mid \mathfrak{b}$ *is an eventually constant thread* $\}$ *is dense in* $\mathbb{B}_\lambda$.

Proof. If $\mathbb{B}_\lambda$ is the a direct limit and $\mathfrak{b} = (b_i \mid i < \lambda)$ is an eventually constant thread through $(\mathbb{B}_i \mid i < \lambda)$, let $j < \lambda$ be such that $b_i = b_j$ for all $j \leq i < \lambda$. Then $\mathfrak{b}^* = b_j$. Since $h_0(b_j) = b_0 \neq 0$, we have $b_j \neq 0$ and the set $\{\mathfrak{b}^* \mid \mathfrak{b}$ is eventually constant$\}$ in dense in $\mathbb{B}_\lambda$.
On the other hand, if $\{\mathfrak{b}^* \mid \mathfrak{b}$ is eventually constant $\}$ is dense in $\mathbb{B}_\lambda$, then $\mathbb{B}_\lambda$ is a direct limit. To see this, just note that $\{\mathfrak{b}^* \mid \mathfrak{b}$ is eventually constant $\} = \bigcup_{i<\lambda} \mathbb{B}_i^+$. □

Dropping the restriction on the support of our threads entirely, yields the biggest limit construction, we shall introduce.

Definition 3.0.6. *Let $\lambda < \alpha$ be a limit ordinal. We say that* $\mathbb{B}_\lambda$ *is an* inverse limit *iff*

1. $\mathfrak{b}^* \neq 0$ *for all threads* $\mathfrak{b}$ *of length* λ *and*

2. $\{\mathfrak{b}^* \mid \mathfrak{b}$ *is a thread of length* $\lambda\}$ *is dense in* $\mathbb{B}_\lambda$.

$\mathcal{B}$ *is an* inverse limit iteration *iff* $\mathbb{B}_\lambda$ *is an inverse limit for all limit* $\lambda < \alpha$.

Every limit in our notes will be a subalgebra of the associated inverse limit and this is in fact true of most (but not all) iterated forcings, that have been constructed to this date.

Definition 3.0.7. *Let $\lambda < \alpha$ be a limit ordinal and let $\mathfrak{b} = (b_i \mid i < \lambda)$ be a thread through $\mathcal{B}$.* $\mathfrak{b}$ *has* countable support *iff* $\mathrm{card}(\mathrm{supp}(\mathfrak{b})) \leq \aleph_0$. *In this case, we also say that* $\mathfrak{b}$ *is a* countable support thread *(or just* CS-thread*).*
$\mathbb{B}_\lambda$ *is a* countable support limit *(*CS-limit*) iff*

1. $\mathfrak{b}^* \neq 0$ *for every* CS-*thread* $\mathfrak{b}$ *of length* λ *and*

2. $\{\mathfrak{b}^* \mid \mathfrak{b}$ *is a* CS-*thread*$\}$ *is dense in* $\mathbb{B}_\lambda$.

Finally, $\mathcal{B}$ *is a* countable support iteration *(*CS-*iteration) iff* $\mathbb{B}_\lambda$ *is a* CS-*limit for all limit* $\lambda < \alpha$.

While CS-iterations behave well in many applications, there is one particular issue that arises, when the cofinality of some cardinal $\kappa < \alpha$ with uncountable cofinality is changed to ω by some $\mathbb{B}_i$ with index $i < \kappa$: Let G_i be $\mathbb{B}_i$-generic such that $V[G_i] \models \mathrm{cf}(\kappa) = \omega$. It may no longer be the case that the associated forcing iteration ${}^{\mathcal{B}}\!/_{G}$ is a CS-iteration in $V[G_i]$, i.e. there may very well be some thread $\mathfrak{b}$ of length κ in $V[G]$ whose support has order-type $\mathrm{cf}(\kappa) = \omega$ such that ${}^{\mathfrak{b}^*}\!/_{G} = 0$. To overcome this issue, Shelah introduced a very technical notion of *revised countable support* iterations (see [She98, X.1]) that has later been simplified by Donder (see [Fuc08]) to its current form.

Definition 3.0.8. *Let $\lambda < \alpha$ be a limit ordinal and let $\mathfrak{b} = (b_i \mid i < \lambda)$ be a thread through $\mathcal{B}$. $\mathfrak{b}$ is a* revised countable support thread *(*RCS-thread*) iff it is eventually constant or there is some $i < \lambda$ such that $b_i \Vdash_i \mathrm{cf}(\check{\lambda}) = \check{\omega}$.*
$\mathbb{B}_\lambda$ is a revised countable support limit *(*RCS-limit*) iff*

1. *$\mathfrak{b}^* \neq 0$ for all* RCS-*threads of length λ and*

2. *$\{\mathfrak{b}^* \mid \mathfrak{b}$ is a* RCS *-thread of length $\lambda\}$ is dense in $\mathbb{B}_\lambda$.*

Finally, $\mathcal{B}$ is a revised countable support iteration *(*RCS-iteration*) iff $\mathbb{B}_\lambda$ is a* RCS-*limit for all limit $\lambda < \alpha$.*

The following theorem is a key tool for RCS-iterations.

Theorem 3.1 (The RCS Factor Property)**.** *Let $h < \alpha$ and let G_h be $\mathbb{B}_h$-generic over V. If $\mathcal{B} = (\mathbb{B}_i \mid i < \alpha)$ is an* RCS-*iteration, then*

$$ {}^{\mathcal{B}}\!/_{G_h} = ({}^{\mathbb{B}_i}\!/_{G_h} \mid h \leq i < \alpha) $$

is an RCS-*iteration in $V[G_h]$.*

Proof. Work in $V[G_h]$. We already know that ${}^{\mathcal{B}}\!/_{G_h}$ is a forcing iteration (see Proposition 3.0.5). So, let $\lambda < \alpha$ be a limit ordinal and let $\mathfrak{c} = ({}^{b_i}\!/_{G_h} \mid h \leq i < \lambda)$ be an RCS-thread through $({}^{\mathbb{B}_i}\!/_{G_h} \mid h \leq i < \lambda)$. By Lemma 3.0.3, we may assume that $(b_i \mid h \leq i < \lambda)$ is in V and uniquely extends to a thread $\mathfrak{b} = (b_i \mid i < \lambda)$ through $(\mathbb{B}_i \mid i < \lambda)$. We have to show that $\mathfrak{c}^* = \prod\{{}^{b_i}\!/_{G_h} \mid h \leq i < \lambda\} = {}^{\mathfrak{b}^*}\!/_{G_h} \neq 0$. There are two cases:
If $\mathfrak{c}$ is eventually constant, there is some $h \leq j < \lambda$ and some $b \in \mathbb{B}_j$ such that, for $j \leq i < \lambda$: ${}^{b_i}\!/_{G_h} = {}^{b}\!/_{G_h}$. Then $\mathfrak{c}^* = {}^{b}\!/_{G_h} \neq 0$.
Otherwise, there is some $h \leq j < \lambda$ such that

$$ {}^{b_j}\!/_{G_h} \Vdash_{{}^{\mathbb{B}_j}\!/_{G_h}} \mathrm{cf}(\check{\lambda}) = \check{\omega}. $$

Fix some $b \in G_h$ such that

$$b \Vdash_{\mathbb{B}_h} \ulcorner {}^{\check{b}_j}\!/\!_{\dot{G}_h} \Vdash_{\check{\mathbb{B}}_j/\dot{G}_h} \mathrm{cf}(\check{\lambda}) = \check{\omega} \urcorner.$$

Let $h \leq k < l < \lambda$. Then $h_k(b \cdot b_l) = b \cdot h_k(b_l) = b \cdot b_k$. Since $b \Vdash_{\mathbb{B}_h} {}^{\check{b}_j}\!/\!_{\dot{G}_h} \neq 0$, we also have $b \cdot b_j \neq 0$ and thus $b \cdot b_i \neq 0$ for all $h \leq i < \lambda$. Hence there is (in V) a unique thread $\mathfrak{d} = (d_i \mid i < \lambda)$ through $(\mathbb{B}_i \mid i < \lambda)$ such that $d_i = b \cdot b_i$ for $h \leq i < \lambda$. Since $b \in G_h$, we have ${}^{b_i}\!/\!_{G_h} = {}^{d_i}\!/\!_{G_h}$ for all $h \leq i < \lambda$ and thus $\mathfrak{c} = ({}^{d_i}\!/\!_{G_h} \mid h \leq i < \lambda)$. Working in V, we now verify that $\mathfrak{d}$ is an RCS-thread:
Let G_j be $\mathbb{B}_j$-generic over V such that $d_j = b \cdot b_j \in G_j$. Let $G := G_j \cap B_h$ and $H := \{{}^{d}\!/\!_{G} \mid d \in G_j\}$. Recall (Proposition 3.0.4) that G is $\mathbb{B}_h$-generic over V and H is ${}^{\mathbb{B}_j}\!/\!_{G}$-generic over $V[G]$. Now $b \in G$ and ${}^{b_j}\!/\!_{G} \in H$ imply that

$$(V[G])\,[H] \models \mathrm{cf}(\lambda) = \omega.$$

Since $(V[G])\,[H] = V[G_j]$, this yields

$$d_j = b \cdot b_j \Vdash_{\mathbb{B}_j} \mathrm{cf}(\check{\lambda}) = \check{\omega}$$

and $\mathfrak{d}$ is indeed an RCS-thread. In particular, we have $\mathfrak{d}^* = \prod\{d_i \mid i < \lambda\} = \prod\{b \cdot b_i \mid h \leq i < \lambda\} \neq 0$.
In $V[G_h]$, we now have $\mathfrak{c}^* = {}^{\mathfrak{d}^*}\!/\!_{G_h}$. Assume, towards a contradiction, that ${}^{\mathfrak{d}^*}\!/\!_{G_h} = 0$. By Proposition 2.3.1, there is some $\bar{b} \in G_h$ such that $\bar{b} \cdot \mathfrak{d}^* = 0$. By the same argument as before, there is a unique thread $\mathfrak{e} = (e_i \mid i < \lambda)$ through $(\mathbb{B}_i \mid i < \lambda)$ such that $e_i = \bar{b} \cdot b \cdot b_i$ for all $h \leq i < \lambda$. Since $e_j \preceq d_j$, $\mathfrak{e}$ is an RCS-thread through $(\mathbb{B}_i \mid i < \lambda)$. However

$$\begin{aligned}
\mathfrak{e}^* &= \prod\{e_i \mid i < \lambda\} \\
&= \prod\{\bar{b} \cdot b \cdot b_i \mid h \leq i < \lambda\} \\
&= \bar{b} \cdot \mathfrak{d}^* \\
&= 0,
\end{aligned}$$

contradicting the fact that $\mathbb{B}_\lambda$ is an RCS-limit.
The only thing remaining is to check that

$$\{\mathfrak{c}^* \mid \mathfrak{c} \text{ is an RCS-thread through } ({}^{\mathbb{B}_i}\!/\!_{G_h} \mid h \leq i < \lambda)\}$$

is dense in ${}^{\mathbb{B}_\lambda}\!/\!_{G_h}$. Since

$$\{\mathfrak{c} \mid \mathfrak{c} \text{ is an RCS-thread through } (\mathbb{B}_i \mid i < \lambda)\}$$

is dense in $\mathbb{B}_\lambda$, it suffices to prove the following

Claim. *Let $\mathfrak{b} = (b_i \mid i < \lambda)$ be an* RCS-*thread through* $(\mathbb{B}_i \mid i < \lambda)$ *(in V). Then*

$$\mathfrak{b}/G_h = (b_i/G_h \mid h \leq i < \lambda)$$

is an RCS-*thread through* $(\mathbb{B}_i/G_h \mid h \leq i < \lambda)$ *in* $V[G_h]$.

Proof. Work in $V[G_h]$. If $\mathfrak{b}$ is eventually constant, then $\mathfrak{b}/G_h$ is eventually constant as well. Otherwise there is some $j < \lambda$ such that

$$b_j \Vdash_{\mathbb{B}_j} \operatorname{cf}(\check{\lambda}) = \check{\omega}.$$

By Proposition 2.2.1 and our convention to identify $V^{\mathbb{B}_j}$ with its isomorphic copy in $V^{\mathbb{B}_i}$ for $j \leq i < \lambda$, this implies

$$b_j \Vdash_{\mathbb{B}_i} \operatorname{cf}(\check{\lambda}) = \check{\omega}$$

for $j \leq i < \lambda$ and, because $b_i \preceq b_j$, also

$$b_i \Vdash_{\mathbb{B}_i} \operatorname{cf}(\check{\lambda}) = \check{\omega}.$$

We may thus pick j such that $h \leq j$. We finish our proof by verifying that

$$b_j/G_h \Vdash_{\mathbb{B}_j/G_h} \operatorname{cf}(\check{\lambda}) = \check{\omega}.$$

Let H be $\mathbb{B}_h/G_h$-generic over $V[G_h]$ such that $b_j/G_h \in H$. By Proposition 3.0.3 $G_h * H = \{b \in \mathbb{B}_j \mid b/G_h \in H\}$ is $\mathbb{B}_j$-generic over V and clearly $b_j \in G_h * H$. Thus $V[G_h * H] = (V[G_h])[H] \models \operatorname{cf}(\lambda) = \omega$, as desired. ■

□

If one wants to construct a forcing iteration, it is very useful to know that the existence of suitable complete Boolean algebras at limit stages is automatic in all of the above constructions.

Theorem 3.2. *Let λ be a limit ordinal and let $\mathcal{B} = (\mathbb{B}_i \mid i < \lambda)$ be a forcing iteration. Let $\mathcal{F}$ be set of threads of length λ through $(\mathbb{B}_i \mid i < \lambda)$ such that*

1. *every eventually constant thread is in $\mathcal{F}$ and*
2. *for all $j < \lambda$, for all $\mathfrak{b} = (b_i \mid i < \lambda) \in \mathcal{F}$ and for all $b \in \mathbb{B}_j$:*

$$b \preceq b_j \rightarrow \exists \mathfrak{c} = (c_i \mid i < \lambda) \in \mathcal{F}\colon \forall i < \lambda \; c_i \preceq_{\mathbb{B}_i} b_i \cdot c(b)_i.$$

Then there is a complete Boolean algebra $\mathbb{B}_\lambda$ such that *i)* $\mathbb{B}_i \sqsubseteq_c \mathbb{B}_\lambda$ *for all* $i < \lambda$,

ii) $\mathfrak{b}^* := \prod_{\mathbb{B}_\lambda}\{b_i \mid i < \lambda\} \neq 0_{\mathbb{B}_\lambda}$ *for all* $\mathfrak{b} = (b_i \mid i < \lambda) \in \mathcal{F}$ *and*

iii) $\{\mathfrak{b}^* \mid \mathfrak{b} \in \mathcal{F}\}$ *is dense in* $\mathbb{B}_\lambda$.

Proof.

Claim. *Define a relation* $\preceq^* \subseteq \mathcal{F} \times \mathcal{F}$ *by*

$$(b_i \mid i < \lambda) \preceq^* (c_i \mid i < \lambda) :\leftrightarrow \forall i < \lambda\colon b_i \preceq_{\mathbb{B}_i} c_i.$$

Then $(\mathcal{F}, \preceq^*)$ *is a separative partially ordered set.*

Proof. Since $(\mathcal{F}, \preceq^*)$ is the product of the partial orders $(\mathbb{B}_i, \preceq_{\mathbb{B}_i})$, it certainly is a partial order and hence it suffices to verify that it is separative:
Let $\mathfrak{b} = (b_i \mid i < \lambda), \mathfrak{c} = (c_i \mid i < \lambda) \in \mathcal{F}$ such that $\mathfrak{b} \not\preceq^* \mathfrak{c}$. Let $j < \lambda$ be minimal such that $b_j \not\preceq_{\mathbb{B}_j} c_j$ and let $b := b_j \cdot (-c_j)$. Then $0 \neq b \preceq_{\mathbb{B}_j} b_j$ and $b \perp_{\mathbb{B}_j} c_j$. By our assumption, we may now fix some $\mathfrak{d} = (d_i \mid i < \lambda) \in \mathcal{F}$ such that $d_i \preceq_{\mathbb{B}_i} c(b)_i \cdot b_i$ for all $i < \lambda$. Now $\mathfrak{d} \preceq^* \mathfrak{b}$ and since $d_j \preceq b \cdot b_j = b \perp_{\mathbb{B}_j} c_j$, we also have $\mathfrak{d} \perp \mathfrak{c}$. ∎

By Theorem 2.1, we may now fix a complete Boolean algebra $\mathbb{B}(\mathcal{F})$ such that

- $\preceq^* \subseteq \preceq_{\mathbb{B}(\mathcal{F})}$ and
- $\mathcal{F}$ is dense in $(\mathbb{B}(\mathcal{F}), \preceq_{\mathbb{B}(\mathcal{F})})$.

Claim. *For each* $j < \lambda$

$$\pi_j \colon \mathbb{B}_j \to \mathbb{B}(\mathcal{F}), b \mapsto c(b)$$

is a complete embedding such that $\pi_i = \pi_j \restriction \mathbb{B}_i$, *where* $c(b)$ *is defined as in Definition 3.0.5.*

Proof. Clearly $\pi_i = \pi_j \restriction \mathbb{B}_i$ for all $i \leq j < \lambda$. Since π_i is injective and $b \preceq_{\mathbb{B}_i} c$ iff $\pi_i(b) \preceq^* \pi_i(c)$, Proposition 2.0.5 yields that π_i is an embedding. It hence suffices to verify that it is complete:
By Lemma 2.1.1 it suffices to show that $\pi_j(\sum_{\mathbb{B}_j} X) \preceq \sum_{\mathbb{B}(\mathcal{F})} \pi_j"X$ for every $X \subseteq \mathbb{B}_j$. Suppose this fails. Then $0 \prec \pi_j(\sum_{\mathbb{B}_j} X) \cdot \left(- \sum_{\mathbb{B}(\mathcal{F})} \pi_j"X\right)$ and by the density of $\mathcal{F}$, there is some $\mathfrak{f} = (f_i \mid i < \lambda) \in \mathcal{F}$ such that

$$0 \prec \mathfrak{f} \preceq \pi_j(\sum_{\mathbb{B}_j} X) \cdot \left(- \sum_{\mathbb{B}(\mathcal{F})} \pi_j"X\right).$$

Then $0 \prec f_j \preceq \sum_{\mathbb{B}_j} X$ and we may thus fix some $x \in X$ such that $b := x \cdot f_j \neq 0$. By our assumption on $\mathcal{F}$ there is now some $\mathfrak{c} = (c_i \mid i < \lambda) \in \mathcal{F}$ such that $\mathfrak{c} \preceq \mathfrak{f} \cdot c(b)$. Since $c(b) \preceq \sum_{\mathbb{B}(\mathcal{F})} \pi_j"X$, this implies $\mathfrak{f} \cdot \sum_{\mathbb{B}(\mathcal{F})} \pi_j"X \neq 0$. (Contradiction!) ∎

Next, we replace the isomorphic copy of $\bigcup_{i<\lambda} \mathbb{B}_i$ in $\mathbb{B}(\mathcal{F})$ with $\bigcup_{i<\lambda} \mathbb{B}_i$ to obtain a complete Boolean algebra $\mathbb{B}_\lambda$ and an

$$\pi \colon \mathbb{B}(\mathcal{F}) \to \mathbb{B}_\lambda$$

such that $\pi \circ \pi_i = \mathrm{id}$, i.e. $\mathbb{B}_i \sqsubseteq_c \mathbb{B}_\lambda$, for all $i < \lambda$.

Claim. *Let* $\mathfrak{b} = (b_i \mid i < \lambda) \in \mathcal{F}$. *Then* $\mathfrak{b}^* := \prod_{\mathbb{B}_\lambda}\{b_i \mid i < \lambda\} \neq 0$ *and* $\{\mathfrak{b}^* \mid \mathfrak{b} \in \mathcal{F}\}$ *is dense in* $\mathbb{B}_\lambda$.

Proof. In $\mathbb{B}(\mathcal{F})$, we have $0 \neq \mathfrak{b} = \prod_{\mathbb{B}(\mathcal{F})}\{c(b_i) \mid i < \lambda\}$ and thus

$$\begin{aligned} 0 &\neq \pi(\mathfrak{b}) \\ &= \pi\Big(\prod\nolimits_{\mathbb{B}(\mathcal{F})}\{c(b_i) \mid i < \lambda\}\Big) \\ &= \prod\nolimits_{\mathbb{B}_\lambda}\{\pi(c(b_i)) \mid i < \lambda\} \\ &= \prod\nolimits_{\mathbb{B}_\lambda}\{b_i \mid i < \lambda\} \\ &= \mathfrak{b}^*. \end{aligned}$$

Now $\mathcal{F} \subseteq \mathbb{B}(\mathcal{F})$ is dense and hence $\{\pi(\mathfrak{b}) \mid \mathfrak{b} \in \mathcal{F}\} = \{\mathfrak{b}^* \mid \mathfrak{b} \in \mathcal{F}\}$ is dense in $\mathbb{B}_\lambda$. ■

Finally, since $\{\mathfrak{b}^* \mid \mathfrak{b} \in \mathcal{F}\}$ is dense in $\mathbb{B}_\lambda$, it follows that $\bigcup_{i<\lambda} \mathbb{B}_i$ generates $\mathbb{B}_\lambda$. □

Proposition 3.2.1. *Let* $(\mathbb{B}_i \mid i < \lambda)$ *be a forcing iteration of limit length. Then there exist complete Boolean algebras* $\mathbb{D}, \mathbb{C}, \mathbb{R}, \mathbb{I}$ *such that*

1. $\mathbb{D}$ *is the direct limit of* $(\mathbb{B}_i \mid i < \lambda)$ *and* $(\mathbb{B}_i \mid i < \lambda)^\frown\mathbb{D}$ *is a forcing iteration,*
2. $\mathbb{C}$ *is the* CS-*limit of* $(\mathbb{B}_i \mid i < \lambda)$ *and* $(\mathbb{B}_i \mid i < \lambda)^\frown\mathbb{C}$ *is a forcing iteration,*
3. $\mathbb{R}$ *is the* RCS-*limit of* $(\mathbb{B}_i \mid i < \lambda)$ *and* $(\mathbb{B}_i \mid i < \lambda)^\frown\mathbb{R}$ *is a forcing iteration and*
4. $\mathbb{I}$ *is the inverse limit of* $(\mathbb{B}_i \mid i < \lambda)$ *and* $(\mathbb{B}_i \mid i < \lambda)^\frown\mathbb{I}$ *is a forcing iteration.*

Proof.
1. Take $\mathbb{D}$ as the Boolean completion of $\bigcup_{i<\lambda} \mathbb{B}_i$.
2. If $cf(\lambda) > \omega$, then every CS-thread through $(\mathbb{B}_i \mid i < \lambda)$ is eventually constant and we may thus take $\mathbb{C}$ to be the direct limit of $(\mathbb{B}_i \mid i < \omega)$. On the other hand, if $\mathrm{cf}(\lambda) = \omega$ then the set $\{\mathfrak{b}^* \mid \mathfrak{b} \text{ is an RCS-thread of length } \lambda\}$ is dense in the inverse limit of $(\mathbb{B}_i \mid i < \lambda)$ and hence, we can take $\mathbb{C}$ as the inverse limit of $(\mathbb{B}_i \mid i < \lambda)$. This is justified, because inverse limits always exist (see below).

3. It suffices to show that the set $\mathcal{F}$ of all RCS-threads through $(\mathbb{B}_i \mid i < \lambda)$ satisfies the premises of Theorem 3.2:
   The first premise is trivially satisfied. To verify the second one, fix an RCS-thread $\mathfrak{b} = (b_i \mid i < \lambda)$, $j < \lambda$ and some $b \in \mathbb{B}_j$ such that $b \preceq b_j$. We have to show that there is some RCS-thread $\mathfrak{c} = (c_i \mid i < \lambda)$ such that $c_i \preceq c(b)_i$ and $c_i \preceq b_i$ for all $i < \lambda$. First suppose that $\mathfrak{b}$ is eventually constant. Then $\mathfrak{c} := c(b \cdot b_j)$ is as desired.
   Otherwise there is some $k < \omega$ such that $b_k \Vdash_k \mathrm{cf}(\check{\lambda}) = \check{\omega}$. Let $\mathfrak{c}$ be the unique thread through $(\mathbb{B}_i \mid i < \lambda)$ such that for all $j \leq i < \lambda$: $c_i = b \cdot b_i$. Such a thread exists, because for $j \leq i < k < \lambda$

   $$h_i(c_k) = h_i(b \cdot b_k) = b \cdot h_i(b_k) = b \cdot b_i = c_i.$$

   Now $c_k \preceq b_k$ and thus $c_k \Vdash_k \mathrm{cf}(\check{\lambda}) = \check{\omega}$ witnesses that $\mathfrak{c}$ is a RCS-thread.

4. This is an immediate consequence of Theorem 3.2, since the set $\mathcal{F}$ of all threads through $(\mathbb{B}_i \mid i < \lambda)$ satisfies the required conditions.

$\square$

So, every forcing iteration $\mathcal{B} = (\mathbb{B}_i \mid i < \lambda)$ of limit length can be extended by its induced direct/CS/RCS/inverse limit, i.e. each of these limit constructions yields a complete Boolean algebra $\mathbb{B}_\lambda$ such that $(\mathbb{B}_i \mid i < \lambda + 1)$ is a forcing iteration.

4 Subcomplete Boolean Algebras

In [Jenb], [Jenc] and [Jen12] Jensen developed the notions of *subcomplete*, *suproper* and *semi-subproper* Boolean algebras and proved a strong iteration theorem for each of them and we now prove his main iteration theorem for subcomplete Boolean algebras.

4.1 Examples

Definition 4.0.1. *Let $\mu > 0$ be an ordinal. A transitive model $\overline{N}$ of* ZFC$^-$ *is* regular *in $L_\mu(\overline{N})$ iff for all $f\colon x \to \overline{N}$ with $x \in \overline{N}$ and $f \in L_\mu(\overline{N})$, we already have $f \in \overline{N}$.*
$\overline{N}$ is full *iff there is some $\mu > 0$ such that $L_\mu(\overline{N}) \models$* ZFC$^-$ *and $\overline{N}$ is regular in $L_\mu(\overline{N})$.*

Definition 4.0.2. *Let $\mathcal{L}$ be a language, let $\mathcal{M}, \mathcal{N}$ be $\mathcal{L}$-structures and let M, N be their respective universes. An* elementary embedding *$\sigma\colon \mathcal{M} \prec \mathcal{N}$* [1] *is a function $\sigma\colon M \to N$ such that for every $\mathcal{L}$-formula ϕ and all $x_1, \dots, x_l \in M$:*

$$\mathcal{M} \models \phi(x_1, \dots, x_l) \text{ iff } \mathcal{N} \models \phi(\sigma(x_1), \dots, \sigma(x_l)).$$

We say that $\mathcal{M}$ is an elementary substructure *of $\mathcal{N}$ iff* id*$\colon \mathcal{M} \prec N$ is an elementary embedding and in this case we simply write $\mathcal{M} \prec \mathcal{N}$.*

Notation 4.0.1. *Let $\sigma\colon \mathcal{M} \prec \mathcal{N}$ be an elementary embedding of $\mathcal{L}$-structures $\mathcal{M}, \mathcal{N}$ with respective universes M, N. We write*

$$\sigma(x_1, \dots, x_l) = y_1, \dots, y_l$$

as a shorthand for the following statement:
There exist $x_1, \dots, x_l \in M$ and $y_1, \dots, y_l \in N$ such that for all $i \in \{1, \dots, l\}\colon \sigma(x_i) = y_i$.
When the language $\mathcal{L}$ and the $\mathcal{M}$-interpretations of all the $\mathcal{L}$-symbols is clear from the context, we also identify the $\mathcal{L}$-structure $\mathcal{M}$ with its universe M.

[1] It will always be clear from the context, whether $\prec$ denotes an elementary embedding or the strict partial order associated to a given Boolean algebra.

Definition 4.0.3. *A complete Boolean algebra* $\mathbb{B}$ *is* z-subcomplete*, for some set* z*, iff for all sufficiently large* θ*, all regular* $\tau > \theta$ *and sets* A *s.t.* $\mathbb{B} \in H_\theta$ *and* $A, H_\theta \subseteq L_\tau[A]$ *the following hold:*
Let $\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$ *be an elementary embedding such that* $\overline{N}$ *is transitive, countable and full with* $\overline{A} \subseteq \overline{N}$ *and* $\theta, \mathbb{B}, z \in \sigma"\overline{N}$*. Let* $\lambda_1, \ldots, \lambda_n \in \sigma"\overline{N}$ *be regular cardinals with* $\max\{\mathrm{card}(\mathbb{B}), \omega_1\} < \lambda_i < \theta$*,* $i = 1, \ldots, n$*, and let* $s \in \sigma"\overline{N}$ *be an additional parameter s.t.* $\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{s}, \overline{z}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, s, z, \lambda_1, \ldots, \lambda_n$*. Finally, let* $\overline{G}$ *be* $\overline{\mathbb{B}}$*-generic over* $\overline{N}$ *and set* $\overline{\lambda_0} := \overline{N} \cap \mathrm{Ord}$*.*
Then there is an $b \in \mathbb{B}^+$ *s.t. for all* $\mathbb{B}$*-generic* G *with* $b \in G$*, there is some* $\sigma_0 \in V[G]$ *s.t.*

a) $\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$*,*

b) $\sigma_0(\overline{\theta}, \overline{\mathbb{B}}, \overline{s}, \overline{z}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, s, z, \lambda_1, \ldots, \lambda_n$*,*

c) $\sup \sigma_0"\overline{\lambda_i} = \sup \sigma"\overline{\lambda_i}$*, for* $i = 0, 1, \ldots, n$ *and*

d) $\sigma_0"\overline{G} \subseteq G$*.*

We call $\mathbb{B}$ subcomplete *iff it is* $\emptyset$*-subcomplete.*

Since we plan to extend elementary embeddings $\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$ to elementary embeddings of their generic extensions

$$\sigma'\colon (\overline{N}[\overline{G}]; \in, \overline{A}) \prec ((L_\tau[A])\,[G]; \in, A),$$

we have to require that $\overline{A} \subseteq \overline{N}$ and $A \subseteq L_\tau[A]$ and from now on, this is part of our convention.

Definition 4.0.4. *We say that* μ verifies the z-subcompleteness of $\mathbb{B}$ *iff the above holds for all cardinals* $\theta \geq \mu$ *and we also say that* $\sigma\colon (\overline{N}; \in \overline{A}) \prec (L_\tau[A]; \in, A)$ witnesses the z-subcompleteness of $\mathbb{B}$*.*

Lemma 4.0.1. *If* $\mathbb{B}$ *is* z*-subcomplete for some* z*, then it is subcomplete.*

Proof. Let θ_∞ be minimal verifying the z-subcompleteness of $\mathbb{B}$ for some $z \in H_{\theta_\infty}$ and for ordinals $\theta_\infty < \mu$ with $\mathrm{card}(V_\mu) = \mu$ let θ_μ be s.t.

$$V_\mu \models \text{"}\theta_\mu \text{ is minimal verifying the } z\text{-subcompleteness of } \mathbb{B} \text{ for some } z \in H_{\theta_\mu}\text{"}.$$

Then $\mu \leq \mu'$ implies $\theta_\mu \leq \theta_{\mu'} \leq \theta_\infty$ and we may let μ_0 be minimal s.t. $\theta_{\mu_0} = \theta_\infty$.
Next, for $\mu \geq \mu_0$, let A_μ be the set of all $z \in H_{\theta_0}$ such that for all cardinals θ, $\theta_0 \leq \theta < \mu$, all regular $\tau > \theta$ and all A with $H_\theta \subseteq L_\tau[A]$, the following holds:
If $\sigma\colon \overline{X} \prec L_\tau[A]$ is an elementary embedding, $\overline{X}$ is countable, transitive and full with $z, \theta, \mathbb{B} \in \sigma"\overline{X}$, then σ witnesses the z-subcompleteness of $\mathbb{B}$. Let A_∞ be defined as above, but without the restriction "$\theta < \mu$".
Now $\theta_0 < \mu \leq \mu'$ implies $A_\mu \supseteq A_{\mu'} \supseteq A_\infty$ and we may fix μ_1 minimal s.t. $A_{\mu_1} = A_\mu$ for all $\mu_1 \leq \mu$, i.e. $A_{\mu_1} = A_\infty$. The following claim completes our proof.

Claim. *μ_1^+ verifies the subcompleteness of $\mathbb{B}$.*

Proof. Let $\theta \geq \mu_1^+$ be a cardinal, let $\tau > \theta$ be regular and let A be a set s.t. $H_\theta \subseteq L_\tau[A]$. The sequence $(\theta_\mu \mid \mu < \theta)$ is definable in $L_\tau[A]$ from parameters $\theta, \mathbb{B}$ and since $\theta > \mu_0$, μ_0 is definable in $L_\tau[A]$ from the same parameters.
Therefore $(A_\mu \mid \mu = \text{card}(V_\mu), \mu_0 \leq \mu < \theta)$ is definable in $L_\tau[A]$ from $\theta, \mathbb{B}$ as well as μ_1 and $A_{\mu_1} = A_\infty$. If $\sigma\colon \overline{X} \prec L_\tau[A]$ is an elementarty embedding, $\overline{X}$ is countable, transitive and full with $\theta, \mathbb{B} \in \sigma"\overline{X}$, then elementarity and the above calculation yield $A_{\mu_1} \in \sigma"\overline{X}$ and furthermore $\sigma"\overline{X} \cap A_{\mu_1} \neq \emptyset$. By the definition of $A_{\mu_1} = A_\infty$, σ now witnesses the subcompleteness of $\mathbb{B}$. ■

□

Proposition 4.0.1. *Let $\mathbb{B}$ be subcomplete and let $\mathbb{C}$ be another complete Boolean algebra such that $\mathbb{B} \cong \mathbb{C}$. Then $\mathbb{C}$ is subcomplete.*

Proof. Fix an isomorphism $f\colon \mathbb{B} \to \mathbb{C}$. By Lemma 4.0.1, it suffices to prove that $\mathbb{C}$ is $\{f, \mathbb{B}\}$-generic. Let θ be large enough such that $\{\mathbb{B}, \mathbb{C}, f\} \subseteq H_\theta$ and such that θ verifies the subcompleteness of $\mathbb{B}$. Let $\tau > \theta$ be regular and let A be a set such that $H_\theta \subseteq L_\tau[A]$. Furthermore, let $\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$ be an elementary embedding such that $\overline{N}$ is transitive, countable and full with $\theta, \mathbb{B}, \mathbb{C} \in \sigma"\overline{N}$. Let $\lambda_1, \ldots, \lambda_n \in \sigma"\overline{N}$ be regular cardinals with $\max\{\text{card}(\mathbb{C}), \omega_1\} < \lambda_i$ for $i = 1, \ldots, n$ and let $s \in \sigma"\overline{N}$ be an additional parameter such that

$$\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{f}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, f, s, \lambda_1, \ldots, \lambda_n.$$

Let $\overline{G}$ be $\overline{\mathbb{C}}$-generic over $\overline{N}$ and let $\overline{\lambda}_0 := \overline{N} \cap \text{Ord}$. Then $\overline{H} := \overline{f}"G$ is $\overline{\mathbb{B}}$-generic over $\overline{N}$ and by the subcompleteness of $\mathbb{B}$ there is some $b \in \mathbb{B}^+$ such that for all $\mathbb{B}$-generic ultrafilters H over V with $b \in H$ there is some $\sigma_0 \in V[H]$ such that

1. $\sigma_0\colon (\overline{N}; \in \overline{A}) \prec (L_\tau[A]; \in, A)$,
2. $\sigma_0(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{f}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, f, s, \lambda_1, \ldots, \lambda_n$,
3. $\sup \sigma_0"\overline{\lambda}_i = \sup \sigma"\overline{\lambda}_i$ for $i = 0, 1, \ldots, n$ and
4. $\sigma_0"\overline{H} \subseteq H$.

Now let $c := f^{-1}(b)$ and let G be $\mathbb{C}$-generic over V such that $c \in V$. Then $H := f"G$ is $\mathbb{B}$-generic over V and $b \in G$. So, there is some $\sigma_0 \in V[G] = V[H]$ that satisfies properties a)-d). However, σ_0 also witnesses the subcompleteness of $\mathbb{C}$:

Properties a)-c) are literally the same as for $\mathbb{B}$ and the only thing to check is that σ_0"$\overline{G} \subseteq G$. Towards this end, fix $\overline{g} \in \overline{G}$. Then $\overline{f}^{-1}(g) \in \overline{H}$ and thus

$$\begin{aligned}\sigma_0(\overline{f}^{-1}(\overline{g})) &= \sigma_0(\overline{f}^{-1})(\sigma_0(\overline{g})) \\ &= f^{-1}(\sigma_0(\overline{g})) \in H.\end{aligned}$$

This yields $\sigma_0(\overline{g}) \in G$ and hence σ_0"$\overline{G} \subseteq G$, as desired. □

Proposition 4.0.2. *Forcing with subcomplete Boolean algebras doesn't add reals, i.e. if $\mathbb{B}$ is subcomplete and G is $\mathbb{B}$-generic. Then $V[G] \cap^\omega \omega \subseteq V$.*

Proof. Assume, towards a contradiction, that there is some $\dot{f}^G \in (V[G] \cap^\omega \omega) \setminus V$. Fix $b \in G$ such that

$$b \Vdash_{\mathbb{B}} \dot{f} \colon \check{\omega} \to \check{\omega} \wedge \dot{f} \notin \check{V}.$$

Let θ be large enough, $\mathbb{B} \in H_\theta$, $\tau > \theta$ regular and let A be a set such that $H_\theta \subseteq L_\tau[A]$. Let $\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$ be an elementary embedding such that $\overline{N}$ is transitive and full with $b, \mathbb{B}, \preceq, \dot{f} \in \sigma$"$\overline{N}$. Fix $\overline{b}, \overline{\mathbb{B}}, \overline{\preceq}, \overline{f} \in \overline{N}$ with $\sigma(\overline{b}, \overline{\mathbb{B}}, \overline{\preceq}, \overline{f}) = b, \mathbb{B}, \preceq, \dot{f}$ and fix a $\overline{\mathbb{B}}$-generic ultrafilter $\overline{G} \in V$ such that $\overline{b} \in \overline{G}$.
$\mathbb{B}$ is subcomplete and we may thus fix some $c \in \mathbb{B}^+$ such that for all $\mathbb{B}$-generic H with $c \in H$ there is some $\sigma_0 \in V[H]$ such that $\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$, $\sigma_0(\overline{b}, \overline{\mathbb{B}}, \overline{f}) = b, \mathbb{B}, \dot{f}$ and σ_0"$\overline{G} \subseteq H$.
For each $n < \omega$ fix some $\overline{b_n} \overline{\preceq} \overline{b}$ and m_n such that

$$(\overline{N}; \in \overline{A}) \models \overline{b_n} \Vdash_{\overline{\mathbb{B}}} \overline{f}(\check{n}) = \check{m}_n.$$

Let $g\colon \omega \to \omega, n \mapsto m_n$. Then $g \in V$ and for all $n < \omega$

$$\sigma_0(\overline{b_n}) \Vdash_{\sigma_0(\overline{\mathbb{B}})} \sigma_0(\overline{f})(\sigma_0(\check{n})) = \sigma_0(\check{m}_n).$$

Since $\sigma_0 \upharpoonright \omega = \mathrm{id}$ and $\sigma_0(\overline{b}, \overline{\mathbb{B}}, \overline{f}) = b, \mathbb{B}, \dot{f}$, this yields

$$\sigma_0(\overline{b_n}) \Vdash_{\mathbb{B}} \dot{f}(\check{n}) = \check{m}_n,$$

i.e.

$$\sigma_0(\overline{b_n}) \Vdash_{\mathbb{B}} \dot{f}(\check{n}) = g(n).$$

But σ_o"$\overline{G} \subseteq H$ and thus $\dot{f}^H = g \in V$. (Contradiction!) □

In a brief discussing with Jensen during the conference on "Inner Model Theory, Core Model Induction, and HOD Mice" last July in Münster, we had the opportunity to learn a little bit about the history of subcomplete forcings. Initially meant to generically add reals with specific properties to the ground model, Jensen developed the theory of $\mathcal{L}$-forcings in the early 1990's. He later discovered that a certain class of these forcings, the so called “revisable” $\mathcal{L}$-forcings, provably don't add reals and began to look for a natural iteration theorem for this class. This development lead to the introduction of subcomplete forcings and the iteration theorems, that we shall discuss next. Before doing so, we would like to highlight some of the results, Jensen was able to derive.

Theorem 4.1. *Let $A \subseteq \omega_2$ be stationary such that $\mathrm{cf}(\alpha) = \omega$ for all $\alpha \in A$ and let P_A be the set of all striclty increasing and continuous function $p\colon \alpha + 1 \to A$, where $\alpha < \omega_1$. Define a separative parial order $\mathbb{P}_A = (P_A; \leq)$ by letting*

$$p \leq q :\leftrightarrow q \subseteq p$$

for all $p, q \in P_A$. Then the Boolean $\mathbb{B}_A$ completion of $\mathbb{P}_A$ is subcomplete.

In fact, for any $\sigma\colon (\overline{N}; \in \overline{A}) \preceq (L_\tau[A]; \in, A)$ as in the definition of subcompleteness and any $\mathbb{B}_{\overline{A}}$-generic ultrafilter $\overline{G}$ over $\overline{N}$, there is some suitable

$$\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$$

with σ_0"$\overline{G} \subseteq G$ *in V.*

Proof. [Jenc, §3.4 Lemma 1]. □

Theorem 4.2. *Let κ be a measurable cardinal, let U be a normal measure on κ and let $\mathbb{P}_U = (P_U; \leq_U)$ be the* Prikry forcing *given by U. I.e. let P_U be the set of all pairs (s, X) such that $X \in U$, $s\colon n \to \kappa$ is a strictly increasing map for some $n < \omega$ and define $\leq$ by*

$$(s, X) \leq (t, Y) :\leftrightarrow t \subseteq s \wedge X \subseteq Y \wedge \mathrm{ran}(s) \setminus \mathrm{ran}(t) \subseteq Y$$

for all $(s, X), (t, Y) \in P_U$. Let $\mathbb{B}_U$ be the Boolean completion of $\mathbb{P}_U$. Then $\mathbb{B}_U$ is subcomplete.

Proof. [Jenc, §3.5 Lemma 2]. □

Theorem 4.3. *Assume that $2^{\aleph_0} = \aleph_1$, $2^{\aleph_1} = \aleph_2$ and let $\mathbb{B}$ be the Boolean completion of Namba forcing. Then $\mathbb{B}$ is subcomplete.*

Proof. [Jen12, 6.1]. □

Theorem 4.4. *Assume $2^{\aleph_0} = \aleph_1$ and let κ be strongly inaccessible. Then there is a subcomplete Boolean algebra $\mathbb{B}$ such that for all $\mathbb{B}$-generic G we have*

a) $V[G] \models \kappa = \omega_2$,

b) $V[G] \models \mathrm{cf}(\theta) = \omega$ for all regular $\omega_1 < \theta < \kappa$ and

c) every stationary subset $S \subseteq \kappa$ remains stationary in $V[G]$.

Since $\mathbb{B}$ doesn't add reals, this provides an unexpected, positive answer to the “Extended Namba Problem” that is discussed in the introduction of [Jena]

Proof. [Jena, Theorem 1]. □

4.2 Two Step Iterations

Lemma 4.4.1. *Let $\overline{M}, M$ be a transitive models of* ZFC$^-$*, $\mathbb{B} \in M$ be such that*

$$(M; \in) \models \mathbb{B} \text{ is a complete Boolean algebra}$$

and let $\sigma\colon (\overline{M}; \in) \prec (M; \in)$ be an elementary embedding such that $\sigma(\overline{\mathbb{B}}) = \mathbb{B}$. Furthermore, let $\overline{G}$ be $\overline{\mathbb{B}}$-generic over $\overline{M}$ and let G be $\mathbb{B}$-generic over M such that $\sigma"\overline{G} \subseteq G$. Then there is a unique elementary embedding

$$\sigma^*\colon \overline{M}[\overline{G}] \prec M[G],$$

such that $\sigma^* \restriction \overline{M} = \sigma$ and $\sigma(\overline{G}) = G$.

Proof. If $\pi\colon \overline{M}[\overline{G}] \prec M[G]$ satifies $\pi \restriction \overline{M} = \sigma$, $\pi(\overline{\mathbb{B}}) = \mathbb{B}$ and $\pi(\overline{G}) = G$, then for every $\overline{\mathbb{B}}$-name $\dot{x}$ in $\overline{M}$ we have $\pi(\dot{x}^{\overline{G}}) = \pi(\dot{x})^{\pi(\overline{G})} = \sigma(\dot{x})^G$. This yields the uniqueness of σ^* and it now suffices to prove that

$$\sigma^*\colon \overline{M}[\overline{G}] \to M[G], \dot{x}^{\overline{G}} \mapsto \sigma(\dot{x})^G$$

is an elementary embedding such that $\sigma^* \restriction \overline{M} = \sigma$ and $\sigma^*(\overline{G}) = G$.
First of all, for any $x \in \overline{M}$ we have that $\sigma^*(x) = \sigma^*(\check{x}^{\overline{G}}) = \sigma(\check{x})^G = \check{\sigma(x)}^G = \sigma(x)$. Now let $\dot{\overline{G}}$ be the canonical $\overline{\mathbb{B}}$-generic name as defined in $\overline{M}$ and let $\dot{G}$ be the canonical $\mathbb{B}$-generic name as defined in M. Since this definition only depends on the underlying Boolean algebra, we have $\sigma(\dot{\overline{G}}) = \dot{G}$ and thus $\sigma^*(\overline{G}) = \sigma^*(\dot{\overline{G}}^{\overline{G}}) = \sigma(\dot{\overline{G}})^G = \dot{G}^G = G$.
Now let $\phi(v_1, \ldots, v_n)$ be a formula with exactly $v_1, \ldots, v_n$ free and let $\dot{x}_1, \ldots, \dot{x}_n$ be $\overline{\mathbb{B}}$-names in $\overline{M}$. Then

$$\begin{aligned}
\overline{M}[\overline{G}] \models \phi(\dot{x}_1^{\overline{G}}, \ldots, \dot{x}_n^{\overline{G}}) &\leftrightarrow \exists \bar{b} \in \overline{G}\colon \overline{M} \models \bar{b} \Vdash_{\overline{\mathbb{B}}} \phi(\dot{x}_1, \ldots, \dot{x}_n) \\
&\to \exists \bar{b} \in \overline{G}\colon M \models \sigma(\bar{b}) \Vdash_{\mathbb{B}} \phi(\sigma(\dot{x}_1), \ldots, \sigma(\dot{x}_n)) \\
&\overset{\sigma(\bar{b}) \in G}{\to} M[G] \models \phi(\sigma(\dot{x}_1)^G, \ldots, \sigma(\dot{x}_n^G)) \\
&\to M[G] \models \phi(\sigma^*(\dot{x}_1^{\overline{G}}), \ldots, \sigma^*(\dot{x}_n^{\overline{G}})).
\end{aligned}$$

□

Lemma 4.4.2. *Let θ be uncountable, regular and $\mathbb{B} \in H_\theta$. Then for any $\mathbb{B}$-generic ultrafilter G we have*

$$H_\theta^V[G] = H_\theta^{V[G]}.$$

Proof. Note that $H_\theta^V \subseteq H_\theta^{V[G]}$, $G \in H_\theta^{V[G]}$ and therefore $H_\theta^V \cup \{G\} \subseteq H_\theta^{V[G]}$. Since $H_\theta^V[G]$ is the $\subseteq$-minimal transitive ZFC$^-$-model containing $H_\theta \cup \{G\}$, it follows that $H_\theta^V[G] \subseteq H_\theta^{V[G]}$.
To show $H_\theta^{V[G]} \subseteq H_\theta^V[G]$, it suffices to prove that for all $x \in H_\theta^{V[G]}$ there is some $\mathbb{B}$-name $\dot{z} \in H_\theta^V$ with $x \subseteq \dot{z}^G$. By an induction on $\mathrm{rk}(x)$ we may fix a $\mathbb{B}$-name $\dot{x} \subseteq H_\theta^V$ with $\dot{x}^G = x$. Since $x \in H_\theta^{V[G]}$ there is some $b \in G$, a $\mathbb{B}$-name $\dot{f}$ and some $\kappa < \theta$ such that

$$b \Vdash \dot{f} \colon \dot{x} \to \check{\kappa} \text{ is an injective function.}$$

For all $\xi < \kappa$ and $b' \preceq b$ let

$$X_\xi^{b'} := \{\dot{y} \in H_\theta^V \mid b' \Vdash \dot{y} \in \dot{x} \text{ and } \dot{f}(\dot{y}) = \check{\xi}\}.$$

If $\dot{y}, \dot{z} \in X_\xi^{b'}$ then $b' \Vdash \dot{y} = \dot{z}$ because $b' \preceq b$ and $b \Vdash \dot{f}$ is injective. For each $\xi < \kappa$ and $b' \preceq b$ for which $X_\xi^{b'} \neq \emptyset$ choose some $\dot{y}_\xi^{b'} \in X_\xi^{b'}$ and let X be the set of all these $\dot{y}_\xi^{b'}$. Because $X \subseteq H_\theta^V$ and $\mathrm{card}(X) \leq \kappa \cdot \mathrm{card}(\mathbb{B}) < \theta$, the $\mathbb{B}$-name

$$\dot{z} \colon X \to \mathbb{B},\ y_\xi^{b'} \mapsto 1$$

is an element of H_θ^V. If $\dot{y}^G \in \dot{x}^G$, then there is some $c \in G$ with

$$c \Vdash \dot{y} \in \dot{x}.$$

Now $0 \neq b \cdot c \in G$ and there is some $\xi < \kappa$ and $b' \preceq b \cdot c \preceq b$, $b' \in G$ such that

$$b' \Vdash \dot{f}(\dot{y}) = \check{\xi}.$$

It follows that $b' \Vdash \dot{y} = \dot{y}_\xi^{b'}$ and thus $\dot{y}^G = (\dot{y}_\xi^{b'})^G \in \dot{z}^G$. This proves $\dot{x}^G \subseteq \dot{z}^G$. □

We are now able to prove the main theorem of this section.

Theorem 4.5 (Two Step Iteration Theorem)**.** *Let $\mathbb{B} \sqsubseteq_c \mathbb{C}$ be such that $\mathbb{B}$ is subcomplete and $\Vdash_\mathbb{B} {}^{\check{\mathbb{C}}}\!/\!_{\dot{G}}$ is subcomplete. Then $\mathbb{C}$ is subcomplete.*

Proof. By Lemma 4.0.1, it suffices to show that $\mathbb{C}$ is $\mathbb{B}$-subcomplete: Fix θ large enough such that $\mathbb{C} \in H_\theta$, θ verifies the subcompleteness of $\mathbb{B}$ and

$$1 \Vdash_\mathbb{B} \check{\theta} \text{ verifies the subcompleteness of } {}^{\check{\mathbb{C}}}\!/\!_{\dot{G}}.$$

Let $\tau > \theta$ be regular and A be a set such that $H_\theta \subseteq L_\tau[A]$. Furthermore, let $\sigma \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$ be an elementary embedding, where $\overline{N}$ is countable, transitive and full with $\theta, \mathbb{B}, \mathbb{C} \in \sigma"\overline{N}$, and let $\lambda_1, \ldots, \lambda_n \in \sigma"\overline{N}$ be regular cardinals with $\max\{\mathrm{card}(\mathbb{C}), \omega_1\} < \lambda_i < \theta$, $i = 1, \ldots, n$. Let $s \in \sigma"\overline{N}$ be an

additional parameter such that $\sigma(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, s, \lambda_1, \ldots, \lambda_n$. Set $\overline{\lambda}_0 := \overline{N} \cap \mathrm{Ord}$ and fix an ultrafilter $\overline{E}$ that is $\overline{\mathbb{C}}$-generic over $\overline{N}$.
Now $\overline{G} := \overline{E} \cap \overline{\mathbb{B}}$ is $\overline{\mathbb{B}}$-generic over $\overline{N}$ and by the subcompleteness of $\mathbb{B}$ we may fix an $b \in \mathbb{B}^+$ such that for all $\mathbb{B}$-generic ultrafilter G with $b \in G$ there is some $\sigma_0 \in V[G]$ satisfying

a) $\sigma_0 \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\sigma_0(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \sigma_0 "\overline{\lambda_i} = \sup \sigma "\overline{\lambda_i}$, for $i = 0, 1, \ldots, n$ and

d) $\sigma_0 "\overline{G} \subseteq G$.

From now on, work in $V[G]$. Let $\sigma_0' \colon (\overline{N}[\overline{G}]; \in) \prec ((L_\tau[A])\,[G]; \in)$ be the unique elementary embedding such that $\sigma_0' \upharpoonright \overline{N} = \sigma_0$ and $\sigma_0'(\overline{G}) = G$, given by Lemma 4.4.1. Since $\overline{A} \subseteq \overline{N}$ and $A \subseteq L_\tau[A]$, this yields $\sigma_0' \colon (\overline{N}[\overline{G}]; \in, \overline{A}) \prec ((L_\tau[A])\,[G]; \in, A)$.

Claim. *$\overline{N}[\overline{G}]$ is full.*

Proof. Since $\overline{N}$ is full, we may fix some μ such that $L_\mu(\overline{N}) \models \mathrm{ZFC}^-$ and $\overline{N}$ is regular in $L_\mu(\overline{N})$. In order to show that $L_\mu(\overline{N}[\overline{G}]) \models \mathrm{ZFC}^-$ we prove the following

Subclaim. $L_\mu(\overline{N}[\overline{G}]) = L_\mu(\overline{N})[\overline{G}]$.

Proof. $\subseteq$: $L_0(\overline{N}[\overline{G}]) = \mathrm{tc}(\{\overline{N}[\overline{G}]\}) = \{\overline{N}[\overline{G}]\} \cup \overline{N}[\overline{G}] \in L_\mu(\overline{N})[\overline{G}]$. Since $L_\mu(\overline{N})[\overline{G}]$ is closed under Gödel-Operations and $\mu \subseteq L_\mu(\overline{N})[\overline{G}]$, we have $L_\alpha(\overline{N}[\overline{G}]) \in L_\mu(\overline{N})[\overline{G}]$ for all $\alpha < \mu$ and thus $L_\mu(\overline{N}[\overline{G}]) = \bigcup_{\alpha<\mu} L_\alpha(\overline{N}[\overline{G}]) \subseteq L_\mu(\overline{N})[\overline{G}]$.

$\supseteq$: $L_\mu(\overline{N})[\overline{G}]$ is the $\subseteq$-least transitive model of ZFC^- containing $L_\mu(\overline{N}) \cup \{\overline{G}\}$. As $L_\mu(\overline{N}) \cup \{\overline{G}\} \subseteq L_\mu(\overline{N}[\overline{G}])$, the claim follows.

■

Subclaim. *$\overline{N}[\overline{G}]$ is regular in $L_\mu(\overline{N}[\overline{G}])$.*

Proof. Let $\dot{x}^{\overline{G}} \in \overline{N}[\overline{G}]$ and let $f \in L_\mu(\overline{N}[\overline{G}])$ be a function

$$f \colon \dot{x}^{\overline{G}} \to \overline{N}[\overline{G}].$$

For each $y \in \dot{x}^{\overline{G}}$ fix $\dot{y}, \dot{\overline{y}} \in \overline{N}^{\overline{\mathbb{B}}}$ with $y = \dot{y}^{\overline{G}}$ and $f(y) = \dot{\overline{y}}^{\overline{G}}$. By the fullness of $\overline{N}$

$$g \colon \dot{x}^{\overline{G}} \to \overline{N}, y \mapsto (\ \underline{(\dot{y}, \dot{\overline{y}})}\ , 1)$$

is an element of $\overline{N}$. Thus the $\overline{\mathbb{B}}$-name $\dot{f} := g\text{"}\dot{x}^{\overline{G}}$ is an element of $\overline{N}$ and we have

$$\begin{aligned}\dot{f}^{\overline{G}} &= \{\ \underline{(\dot{y}, \dot{\overline{y}})}^{\overline{G}} \mid y \in \dot{x}^{\overline{G}}\} \\ &= \{(\dot{y}^{\overline{G}}, \dot{\overline{y}}^{\overline{G}}) \mid y \in \dot{x}^{\overline{G}}\} \\ &= f,\end{aligned}$$

i.e. $f = \dot{f}^{\overline{G}} \in \overline{N}[\overline{G}]$. ■

■

Claim. $(L_\tau[A])\,[G] = L_\tau[A \times \{0\} \cup G \times \{1\}] =: L_\tau[A, G]$.

Proof. $\subseteq$: We have $L_\tau[A] \cup \{G\} \subseteq L_\tau[A, G]$ and since $G \in L_\tau[A, G]$ it follows that $\dot{x}^G \in L_\tau[A, G]$ for all $\dot{x} \in L_\tau[A]^{\mathbb{B}}$.

$\supseteq$: Construct $(L_\alpha[A, G]; \in, A, G)$ for all $\alpha < \tau$ inside $((L_\tau[A])\,[G]; \in, A)$. Since this construction is correct, we may conclude that $L_\tau[A, G] = \bigcup_{\alpha<\tau} L_\alpha[A, G] \subseteq (L_\tau[A])\,[G]$.

□

Claim. $H_\theta^{V[G]} \subseteq (L_\tau[A])\,[G]$.

Proof. If θ is regular, then Lemma 4.4.2 yields $H_\theta^{V[G]} = H_\theta^V[G]$. Since $H_\theta \subseteq L_\tau[A]$, we get $H_\theta^{V[G]} \subseteq (L_\tau[A])\,[G]$.
If θ is singular, let $(\theta_i \mid i < \mathrm{cf}(\theta))$ be a strictly increasing sequence of regular cardinals such that $\mathbb{B} \in H_{\theta_0}$. Then for each $i < \mathrm{cf}(\theta)$ we have $H_{\theta_i}^{V[G]} = H_{\theta_i}^V[G] \subseteq (L_\tau[A])\,[G]$ and thus $H_\theta^{V[G]} = \bigcup_{i<\mathrm{cf}(\theta)} H_{\theta_i}^{V[G]} \subseteq (L_\tau[A])\,[G]$. □

By repeating the proof of Proposition 3.0.4 we see that

$$\overline{H} := \{\overline{b}/\overline{G} \mid \overline{b} \in \overline{E}\}$$

is $\overline{\mathbb{C}}/\overline{G}$-generic over $N[\overline{G}]$. Since $\mathbb{C}/G$ is subcomplete (inside $V[G]$), $\sigma\colon \overline{N}[\overline{G}] \prec L_\tau[A \times \{0\} \cup G \times \{1\}]$ is an elementary embedding, $\overline{N}[\overline{G}]$ is countable, transitive and full and $H_\theta^{V[G]} \subseteq L_\tau[A \times \{0\} \cup G \times \{1\}]$, we may now fix some $c \in {\mathbb{C}/G}^+$ such that for all $\mathbb{C}/G$-generic ultrafilter H (over $V[G]$) with $c \in H$ there is some $\sigma_1 \in (V[G])\,[H]$ satisfying

a) $\sigma_1\colon \overline{N}[\overline{G}] \prec (L_\tau[A])\,[G]$,

b) $\sigma_1(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{\mathbb{C}}/\overline{G}, \overline{G}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, \mathbb{C}/G, G, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \sigma_1``\overline{\lambda_i} = \sup \sigma``\overline{\lambda_i}$, for $i = 0, 1, \ldots, n$ and

d) $\sigma_1``\overline{H} \subseteq H$.

Let $c = \dot{c}^G$. By extending b, if neccessary, and Proposition 2.3.4 we may fix some $b_1 \in \mathbb{B}$ such that

$$b \Vdash_{\mathbb{B}} \dot{c} = {}^{\check{b}_1}\!/_{G}$$

and b forces all the above to hold, i.e.

$$b \Vdash_{\mathbb{B}} {}^{\check{b}_1}\!/_{\dot{G}} \Vdash_{\check{\mathbb{C}}/\dot{G}} \exists \dot{\sigma}_1 \colon \check{\overline{N}}[\dot{G}] \prec \left(L_\tau[\check{A}]\right)[\dot{G}] \ldots$$

We have

$$b \Vdash_{\mathbb{B}} {}^{\check{b}_1}\!/_{\dot{G}} \neq {}^{\check{0}}\!/_{\dot{G}}$$

and by Proposition 2.3.1 also $b \cdot b_1 \neq 0$.
Let E be a $\mathbb{C}$-generic ultrafilter over V with $b \cdot b_1 \in E$. Then $G := E \cap \mathbb{B}$ is $\mathbb{B}$-generic over V. Since $b \in G$ there is some $\sigma_0 \in V[G]$ satisfying

a) $\sigma_0 \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\sigma_0(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \sigma_0``\overline{\lambda_i} = \sup \sigma``\overline{\lambda_i}$, for $i = 0, 1, \ldots, n$ and

d) $\sigma_0``\overline{G} \subseteq G$,

where $\overline{G} = \overline{E} \cap \overline{\mathbb{B}}$ is $\mathbb{B}$-generic over $\overline{N}$.
Again, there is a unique extension $\sigma_0' \colon (\overline{N}[\overline{G}]; \in, \overline{A}) \prec ((L_\tau[A])[G]; \in, A)$ with $\sigma_0' \upharpoonright \overline{N} = \sigma_0$ and $\sigma_0'(\overline{G}) = G$. Now $H := {}^{E}\!/_{G}$ is ${}^{\mathbb{C}}\!/_{G}$-generic over $V[G]$ and since ${}^{b_1}\!/_{G} \in H$ there is some $\sigma_1 \in (V[G])[H]$ satisfying

a) $\sigma_1 \colon (\overline{N}[\overline{G}]; \in, \overline{A}) \prec ((L_\tau[A])[G]; \in, A)$,

b) $\sigma_1(\overline{\theta}, \overline{\mathbb{B}}, \overline{\mathbb{C}}, {}^{\overline{\mathbb{C}}}\!/_{\overline{G}}, \overline{G}, \overline{s}, \overline{\lambda_1}, \ldots, \overline{\lambda_n}) = \theta, \mathbb{B}, \mathbb{C}, {}^{\mathbb{C}}\!/_{G}, G, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \sigma_1``\overline{\lambda_i} = \sup \sigma``\overline{\lambda_i}$, for $i = 0, 1, \ldots, n$ and

d) $\sigma_1``\overline{H} \subseteq H$,

where $\overline{H} = {}^{\overline{E}}\!/_{\overline{G}}$ is ${}^{\overline{\mathbb{C}}}\!/_{\overline{G}}$-generic over $\overline{N}[\overline{G}]$. Since $(V[G])[H] = V[E]$ and $\sigma_2 := \sigma_1 \upharpoonright \overline{N} \in V[E]$ satisfies both $\sigma_2 \colon (\overline{N}; \in, \overline{N}) \prec (L_\tau[A]; \in, A)$ and $\sigma_2``\overline{E} \subseteq E$, this finishes our proof.

□

4.3 RCS-Iterations

Theorem 4.6. *Let $\mathcal{B} = (\mathbb{B}_i \mid i < \alpha)$ be an* RCS-*iteration such that $\mathbb{B}_0 = \{0, 1\}$ is trivial and for all $i + 1 < \alpha$*

1. $\mathbb{B}_i \neq \mathbb{B}_{i+1}$,
2. $\Vdash_{\mathbb{B}_i} \check{\mathbb{B}}_{i+1}/\dot{G}_i$ *is subcomplete and*
3. $\Vdash_{\mathbb{B}_{i+1}} \operatorname{card}(\check{\mathbb{B}}_i) \leq \omega_1$.

Then every $\mathbb{B}_i$ is subcomplete.

Proof. By induction on i, we will prove the following stronger result.

Claim (1)**.** *Let $h \leq i < \alpha$ and let G_h be $\mathbb{B}_h$-generic over V. Then $\mathbb{B}_i/G_h$ is subcomplete in $V[G_h]$.*

Since $\mathbb{B}_0$ is trivial, letting $h = 0$ then yields the theorem.

Proof. If $h = i$, then $\mathbb{B}_i/G_h \cong \{0, 1\}$ is trivial and hence subcomplete. In particular, the claim holds for $i = 0$ and from now on, we may assume that $h < i$.
Now suppose that $i = j+1$ for some $j < \alpha$. Let $h < i$ and let G_h be $\mathbb{B}_h$-generic over V. Recall that $\mathbb{B}_j/G_h \sqsubseteq_c \mathbb{B}_i/G_h$ and let H_j be $\mathbb{B}_j/G_h$-generic over $V[G_h]$. Then (see Proposition 3.0.3) $G_j := G_h * H_j$ is $\mathbb{B}_j$-generic over V and $\mathbb{B}_i/G_j \cong \left(\mathbb{B}_i/G_h\right)/H_j$ is subcomplete in $V[G_j] = (V[G_h])\,[H_j]$. This proves

$$\Vdash_{\mathbb{B}_j/G_h} \left(\check{\mathbb{B}}_i/\check{G}_h\right)/\dot{H}_j \text{ is subcomplete,}$$

where $\dot{H}_j$ is the canonical $\mathbb{B}_j/G_h$-generic name. By our induction hypothesis, $\mathbb{B}_j/G_h$ is subcomplete in $V[G_h]$ and we may now apply the Two Step Iteration Theorem in $V[G_h]$ (see Theorem 4.5) to conclude that $\mathbb{B}_i/G_h$ is subcomplete in $V[G_h]$.
The case for limit ordinals $\lambda < \alpha$ is substantially more difficult. As a first step, let us prove the following

Claim (2)**.** *If $\operatorname{cf}(\lambda) \leq \omega_1$ and $\mathbb{B}_i$ is subcomplete for all $i < \lambda$, then $\mathbb{B}_\lambda$ is subcomplete.*

Proof. Fix a strictly increasing and cofinal function $f\colon \omega_1 \to \lambda$ such that $f(0) = 0$. By Lemma 4.0.1 it suffices to prove that $\mathbb{B}_\lambda$ is $\{f, \mathcal{B}, \lambda, \mathbb{B}_\lambda\}$-subcomplete. By the fullness of $V^{\mathbb{B}_i}$, for $i < \lambda$, we may fix a cardinal θ such that $\lambda < \theta$, $\mathbb{B}_\lambda, \mathbb{B} \in H_\theta$ and such that for all $i \leq j < \lambda$

$$\Vdash_{\mathbb{B}_i} \check{\theta} \text{ verifies the subcompleteness of } \check{\mathbb{B}}_j.$$

Let $\tau > \theta$ be regular and let A be a set such that $H_\theta \subseteq L_\tau[A]$ and let

$$\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$$

be an elementary embedding, where $\overline{N}$ is countable, transitive and full. Furthermore, let $\lambda_1, \ldots, \lambda_n$ be regular cardinals such that $\text{card}(\mathbb{B}_\lambda) < \lambda_k$ for $k = 1, \ldots, n$ and let s be an additional parameter such that

$$\sigma(\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_1) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_n.$$

Finally, let $\overline{G}$ be $\overline{\mathbb{B}}_{\overline{\lambda}}$-generic over $\overline{M}$. We have to prove that there is some $b \in \mathbb{B}_\lambda^+$ such that for all $\mathbb{B}_\lambda$-generic ultrafilters G over V with $b \in G$ there is some $\sigma_0 \in V[G]$ that satisfies

a) $\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\sigma_0(\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_n) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \sigma_0\text{"}\overline{\lambda_i} = \sup \sigma\text{"}\overline{\lambda_i}$, for $i = 0, 1, \ldots, n$, where $\overline{\lambda_0} := \overline{N} \cap \text{Ord}$, and

d) $\sigma_0\text{"}\overline{G} \subseteq G$.

Let g be the $<_{L_\tau[A]}$-minimal [2] bijection $g\colon \omega \to \overline{\lambda}$ and recursively construct a strictly increasing sequence $(\nu_n \mid n < \omega)$ of ordinals $\nu_n \in \omega_1^{\overline{N}}$ by letting $\nu_0 := 0$ and ν_{n+1} being minimal such that $\overline{f}(\nu_{n+1}) > \max\{\overline{f}(\nu_n), g(n)\}$. For $n < \omega$, let $\overline{\xi}_n := \overline{f}(\nu_n)$. By construction $(\overline{\xi}_n \mid n < \omega)$ is stricly increasing and cofinal in $\overline{\lambda}$ and $\overline{\xi}_0 = 0$. Furthermore, if $\pi\colon \overline{N} \prec L_\tau[A]$ is an elementary embedding such that $\pi(\overline{f}) = f$, then, by the construction of $(\nu_n \mid n < \omega)$,

$$\begin{aligned} \pi(\overline{\xi}_n) &= \pi(\overline{f}(\nu_n)) \\ &= \pi(\overline{f})(\pi(\nu_n)) \\ &= f(\pi(\nu_n)) \\ &= f(\sigma(\nu_n)) \\ &= \sigma(\overline{\xi}_n). \end{aligned}$$

[2] $<_{L_\tau[A]}$ denotes the canonical well-order of $L_\tau[A]$

Letting $\xi_n := \sigma(\overline{\xi}_n)$, for $n < \omega$, thus yields a striclty increasing and cofinal sequence $(\xi_n \mid n < \omega)$ in $\sup \sigma_0``\lambda$ such that $\xi_n = \pi(\overline{\xi}_n)$ for all elementary embeddings $\pi\colon \overline{N} \prec L_\tau[A]$ with $\pi(\overline{f}) = f$.
For $i = 0, \dots, n$ fix a strictly increasing and cofinal sequence $(\overline{\xi}^i_n \mid n < \omega)$ in $\overline{\lambda}_i$ and let $(\xi^i_n \mid n < \omega) = (\sigma_0(\overline{\xi}^i_n) \mid n < \omega)$. Then $(\xi^i_n \mid n < \omega)$ is strictly increasing and cofinal in $\sup \sigma_0``\overline{\lambda}_i$. Finally fix an enumeration $(x_l \mid l < \omega)$ of $\overline{N}$.
We will now recursively construct a sequence $((\dot{\tau}_k, b_k) \mid k < \omega)$ such that each $\dot{\tau}_k$ is a $\mathbb{B}_{\xi_k}$-name and $(b_{\xi_k} \mid k < \omega)$ is a thread through $(\mathbb{B}_{\xi_k} \mid k < \omega)$ satisfying the following properties:
We have $\dot{\tau}_0 = \check{\sigma}_0$ and $b_0 = 1$. If $k > 0$ and G_k is $\mathbb{B}_{\xi_k}$-generic over V such that $b_k \in G_k$, let $G_j := G_k \cap \mathbb{B}_{\xi_j}$ and $\tau_j := \dot{\tau}_j^{G_k} = \dot{\tau}_j^{G_j}$ for $j \le k$. Then

a) $\tau_k\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\tau_k(\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \dots, \overline{\lambda}_1) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \dots, \lambda_n$,

c) $\sup \tau_l``\overline{\lambda}_l = \sup \sigma``\overline{\lambda}_l$ for $l = 0, \dots, n$,

d) $\tau_k``\overline{G}_k \subseteq G_k$, where $\overline{G}_k := \overline{G} \cap \overline{\mathbb{B}}_{\xi_k}$,

e) $\tau_k(x_l) = \tau_j(x_l)$ for $l \le j \le k$ and

f) let $i \in \{0, \dots, n\}$, $k = j + 1$ and m be such that

$$\tau_j(\overline{\xi}^i_m) \le \xi^i_k < \tau_j(\overline{\xi}^i_{m+1}).$$

Then $\tau_k(\overline{\xi}^i_l) = \tau_j(\overline{\xi}^i_l)$ for $l \le m + 1$.

Given $((\dot{\tau}_j, b_j) \mid j < k)$, we now construct $(\dot{\tau}_k, b_k)$.
Let G_{k-1} be $\mathbb{B}_{\xi_{k-1}}$-generic over V such that $b_{k-1} \in G_{k-1}$. By d) and Lemma 4.4.1, there is a unique elementary embedding

$$\tau^*_{k-1}\colon (\overline{N}[\overline{G}_{k-1}]; \in, \overline{A}) \prec ((L_\tau[A])\,[G_{k-1}]; \in, A)$$

such that $\tau^*_{k-1} \upharpoonright \overline{N} = \tau_{k-1}$ and $\tau^*_{k-1}(\overline{G}_{k-1}) = G_{k-1}$. Since θ verifies the subcompleteness of ${\mathbb{B}_{\xi_k}}/{G_{k-1}}$ in $V[G_{k-1}]$, there is some $c \in {\mathbb{B}_{\xi_k}}/{G_{k-1}}^+$ such that for all H_k that are ${\mathbb{B}_{\xi_k}}/{G_{k-1}}$-generic over $V[G_{k-1}]$ with $c \in H_k$, there is some $\tau \in (V[G_{k-1}])\,[H_k]$ satisfying

a) $\tau\colon (\overline{N}[\overline{G}_{k-1}]; \in, \overline{A}) \prec ((L_\tau[A])\,[G_{k-1}]; \in, A)$,

b) $\tau(\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \dots, \overline{\lambda}_1) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \dots, \lambda_n$,

c) $\sup \tau``\overline{\lambda}_l = \sup \sigma``\overline{\lambda}_l$ for $l = 0, \dots, n$ and

d) $\tau``\overline{H}_k \subseteq H_k$, where $\overline{H}_k := {}^{\overline{G}_k}\!\!/_{\overline{G}_{k-1}}$.

By replacing the parameter s with

$$\{s\} \cup \{\tau_{k-1}(x_l) \mid l \leq k-1\} \cup \{\tau_{k-1}(\overline{\xi}^i_l) \mid i, l \text{ are as in } f)\},$$

we may further require that

e) $\tau(x_l) = \tau_{k-1}(x_l)$ for $l \leq k-1$ and

f) $\tau(\overline{\xi}^i_l) = \tau_{k-1}(\overline{\xi}^i_l)$ for all $l \leq m+1$ where $i \in \{0, \ldots, n\}$ and m are such that

$$\tau_{k-1}(\overline{\xi}^i_m) \leq \xi^i_k < \tau_{k-1}(\overline{\xi}^i_{m+1}).$$

By Proposition 3.0.3 $G_k := G_{k-1} * H_k$ is $\mathbb{B}_{\xi_k}$-generic over V, $\overline{G}_k := \overline{G}_{k-1} * \overline{H}_k$ is $\overline{\mathbb{B}}_k$-generic over $\overline{N}$. Let $b \in \overline{G}_k$. Then ${}^{b}\!/_{\overline{G}_{k-1}} \in \overline{H}_k = {}^{\overline{G}_k}\!/_{\overline{G}_{k-1}}$ and, since $\tau``\overline{H}_k \subseteq H_k$, $\tau(b) = {}^{\tau(b)}\!/_{G_{k-1}} \in {}^{G_k}\!/_{G_{k-1}}$. Recall that by ${}^{G_k}\!/_{G_{k-1}}$ we actually mean ${}^{G_k}\!/_{G_{k-1\uparrow}}$, where $G_{k-1\uparrow}$ is the upward closure of G_{k-1} in $\mathbb{B}_{k-1}$ (see Definition 2.1.4). Hence, there is some $b' \in G_k$ and some $c \in G_{k-1}$ such that $c \preceq -\Delta(\tau(b), b') = (-\tau(b) + b') \cdot (-b' + \tau(b))$. Since $b' \in G_k$, this yields $G_k \ni (-b' + \tau(b)) \cdot b' \preceq \tau(b)$ and thus $\tau(b) \in G_k$. In particular, this proves $\tau``\overline{G}_k \subseteq G$.
Now $\tau_k := \tau \upharpoonright \overline{N}$ fulfills properties a) - f) and it suffices to find a suitable $\mathbb{B}_{\xi_k}$-name $\dot{\tau}_k$ thereof and some $b_k \in \mathbb{B}_{\xi_k}$ that forces all the required conditions while also extending $(b_j \mid j < k)$ to a partial thread $(b_j \mid j < k)^\frown b_k$, i.e. such that $h_{\xi_{k-1}}(b_k) = b_{k-1}$.
Let $\dot{c}$ be a $\mathbb{B}_{\xi_{k-1}}$ name such that $c = \dot{c}^{G_{k-1}}$ and $\Vdash_{\mathbb{B}_{\xi_{k-1}}} \check{b}_{k-1} \notin \dot{G}_{k-1} \rightarrow \dot{c} = 0$. By Proposition 2.3.4, we may fix some $b_k \in \mathbb{B}_{\xi_k}$ such that

$$b_{k-1} \Vdash_{\mathbb{B}_{\xi_{k-1}}} \dot{c} = {}^{\check{b}_k}\!/_{\dot{G}_{k-1}} \wedge c \Vdash_{{}^{\check{\mathbb{B}}_k}\!/_{\dot{G}_{k-1}}} \exists \dot{\tau}_k \text{ such that } a) - f) \text{ hold}.$$

We have

$$\begin{aligned} h_{\xi_{k-1}}(b_k) &= \| {}^{\check{b}_k}\!/_{\dot{G}_{k-1}} \neq 0 \| \\ &= \| \dot{c} \neq 0 \| \\ &= b_{k-1} \end{aligned}$$

and whenever G_k is $\mathbb{B}_{\xi_k}$-generic over V such that $b_k \in G_k$, we have ${}^{b_k}\!/_{G_k \cap \mathbb{B}_{\xi_{k-1}}} = \dot{c}^{G_k \cap \mathbb{B}_{\xi_{k-1}}} \in {}^{\mathbb{B}_{\xi_k}}\!/_{G_k \cap \mathbb{B}_{\xi_{k-1}}}$ and since $b_{k-1} \in G_k \cap \mathbb{B}_{\xi_{k-1}}$, there is some $\tau_k \in V[G_k] =$

$\left(V[G_k \cap \mathbb{B}_{\xi_{k-1}}]\right)[{}^{G_k}\!/\!{}_{G_k \cap \mathbb{B}_{\xi_{k-1}}}]$ satisfying a)-f). We may thus fix some $\mathbb{B}_{\xi_k}$-name $\dot\tau_k$ such that

$$b_k \Vdash_{\mathbb{B}_{\xi_k}} \dot\tau_k \text{ satisfies } a) - f).$$

Because $(\dot\tau_k, b_k)$ now has all the desired properties, this finishes our construction and we may fix a sequence $((\dot\tau_k, b_k) \mid k < \omega)$ such that:
$\dot\tau_0 = \check\sigma_0$ and $b_0 = 1$. If $k > 0$ and G_k is $\mathbb{B}_{\xi_k}$-generic over V such that $b_k \in G_k$, let $G_j := G_k \cap \mathbb{B}_{\xi_j}$ and $\tau_j := \dot\tau_j^{G_k} = \dot\tau_j^{G_j}$ for $j \leq k$. Then

a) $\tau_k\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\tau_k(\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_1) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_n$,

c) $\sup \tau_l``\overline{\lambda}_l = \sup \sigma``\overline{\lambda}_l$ for $l = 0, \ldots, n$,

d) $\tau_k``\overline{G}_k \subseteq G_k$, where $\overline{G}_k := \overline{G} \cap \overline{\mathbb{B}}_{\xi_k}$,

e) $\tau_k(x_l) = \tau_j(x_l)$ for $l \leq j \leq k$ and

f) let $i \in \{0, \ldots, n\}$, $k = j + 1$ and m be such that
$$\tau_j(\overline{\xi}^i_m) \leq \xi^i_k < \tau_j(\overline{\xi}^i_{m+1}).$$
Then $\tau_k(\overline{\xi}^i_l) = \tau_j(\overline{\xi}^i_l)$ for $l \leq m + 1$.

Let $b := \prod_{\mathbb{B}_\lambda}\{b_k \mid k < \omega\}$ and let $\mathfrak{c} = (c_i \mid i < \lambda)$ be the unique thread through $(\mathbb{B}_i \mid i < \lambda)$ such that $\operatorname{supp}(\mathfrak{c}) \subseteq \sup\{\xi_k \mid k < \omega\}$ and $c_{\xi_k} = b_k$ for all $k < \omega$. If $\operatorname{cf}(\lambda) = \omega$, then $c_0 = b_0 \Vdash_{\mathbb{B}_0} \operatorname{cf}(\check\lambda) = \check\omega$. Otherwise $(\xi_k \mid k < \lambda)$ is bounded below λ and $\mathfrak{c}$ is eventually constant. In both cases $\mathfrak{c}$ is an RCS-thread trough $(\mathbb{B}_i \mid i < \lambda)$ and $b = \mathfrak{c}^* \neq 0$.
Let G be $\mathbb{B}_\lambda$-generic over V such that $b \in G$. For each $k < \omega$ let $G_k := G \cap \mathbb{B}_{\xi_k}$ be the associated $\mathbb{B}_{\xi_k}$-generic filter and let $\tau_k := \dot\tau_k^{G_k} = \dot\tau_k^G$. Define

$$\sigma_0\colon \overline{N} \to L_\tau[A]$$

by letting $\sigma_0(x_l) := \tau_l(x_l)$. By e), we have $\sigma_0(x_l) = \tau_j(x_l)$ for all $l \leq j < \omega$. Let us verify that $\sigma_0 \in V[G]$ now has all the desired properties.
We use the Tarski-Vaught test to verify that $\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$:
Let ϕ be a $\mathcal{L}_{\{\in, A\}}$-formula and let $y_1, \ldots, y_m \in \mathcal{N}$ such that there is some $y \in L_\tau[A]$ with

$$(L_\tau[A]; \in, A) \models \phi(\sigma_0(y_1), \ldots, \sigma_0(y_m), y).$$

Fix $l_1, \ldots, l_m < \omega$ such that $y_1 = x_{l_1}, \ldots, y_m = x_{l_m}$ and let $l = \max\{l_1, \ldots, l_m\}$. Then $\sigma_0(x_{l_1}) = \tau_l(x_{l_1}), \ldots, \sigma_0(x_{l_m}) = \tau_l(x_{l_m})$ and by the elementarity of τ_l there is some $y_{m+1} \in \overline{N}$ such that

$$(\overline{N}; \in, \overline{A}) \models \phi(x_{l_1}, \ldots, x_{l_m}, y_{m+1})$$

Fix $l_{m+1} < \omega$ such that $y_{m+1} = x_{l_{m+1}}$ and let $l' = \max\{l, m+1\}$. By the elementarity of $\tau_{l'}$, we have

$$(L_\tau[A]; \in, A) \models \phi(\tau_l(x_{l_1}), \dots, \tau_l(x_{l_m}), \tau_l(x_{l_{m+1}}))$$

and thus

$$(L_\tau[A]; \in, A) \models \phi(\sigma_0(x_{l_1}), \dots, \sigma_0(x_{l_m}), \sigma_0(x_{l_{m+1}})).$$

Hence $\sigma_0 \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$.
Now let $k < \omega$ be such that $\{\overline{\theta}, \overline{f}, \overline{\mathcal{B}}, \overline{\lambda}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \dots, \overline{\lambda}_1\} \subseteq \{x_l \mid l < k\}$. Then

$$\begin{aligned}\sigma_0(\overline{\theta}, \overline{\mathcal{B}}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \dots, \overline{\lambda}_{n+1}) &= \tau_k(\overline{\theta}, \overline{\mathcal{B}}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \dots, \overline{\lambda}_{n+1}) \\ &= \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \dots, \lambda_{n+1}.\end{aligned}$$

Now, for $\overline{\xi}^i_k$, fix $l < \omega$ be large enough such that $\sigma_0(\overline{\xi}^i_k) = \tau_l(\overline{\xi}^i_k)$. Since $\tau_k(\overline{\xi}^i_k) < \sup \tau_k{}''\overline{\lambda}_i = \sup \sigma \overline{\lambda}_i$, this implies $\sup \sigma_0'' \overline{\lambda}_i \leq \sup \sigma'' \overline{\lambda}_i$. On the other hand, fix ξ^i_k and let m be such that $\tau_k(\overline{\xi}^i_m) \leq \xi^i_k < \tau_k(\overline{\xi}^i_{m+1})$. Then, by property f), $\sigma_0(\overline{\xi}^i_{m+1}) = \tau_k(\overline{\xi}^i_{m+1}) > \xi^i_k$, witnessing that $\sigma_0{}''\overline{\lambda}_i$ is in fact cofinal in $\sup \sigma''\overline{\lambda}_i$.
Finally, we have to see that $\sigma_0{}''\overline{G} \subseteq G$. First, let $\overline{b} \in G_k$ for some $k < \omega$. Fix $l < \omega$ large enough such that $\sigma_0(\overline{b}) = \tau_l(\overline{b})$. By d, this yields $\sigma_0(\overline{b}) = \tau_l(\overline{b}) \in G_k \subseteq G$ and thus $\sigma_0{}'' \bigcup\{\overline{G}_k \mid k < \omega\} \subseteq G$.
If $\mathrm{cf}(\lambda) = \omega_1$, then, by elementarity,

$$(\overline{N}; \in, \overline{A}) \models \mathrm{cf}(\overline{\lambda}) = \omega_1^{(\overline{N}; \in, A)}$$

and hence $\bigcup\{\overline{G}_k \mid k < \omega\}$ is dense in $\overline{G}$. Thus, for $\overline{g} \in \overline{G}$, we may fix some $k < \omega$ and some $\overline{b} \in G_k$ such that $\overline{b} \preceq \overline{g}$. Then $\sigma_0(\overline{b}) \preceq \sigma_0(\overline{g})$ and since $\sigma_0(\overline{b}) \in G$, this yields $\sigma_0(\overline{g}) \in G$.
Otherwise $\mathrm{cf}(\lambda) = \omega$ and, again by elementarity, there is some $(\nu_n \mid n < \omega) \in \overline{N}$ that is strictly increasing and cofinal in $\overline{\lambda}$. Now let $\overline{g} \in$. Since $(\overline{N}; \in, \overline{A}) \models$ $\mathbb{B}_{\overline{\lambda}}$ is an RCS-limit , there is some thread $\overline{\mathfrak{b}} = (\overline{b}_n \mid n < \omega) \in \overline{N}$ through $(\mathbb{B}_{\nu_n} \mid n < \omega)$ such that $\overline{\mathfrak{b}}^* := \prod_{\mathbb{B}_{overline\lambda}}\{\overline{b}_{\nu_n} \mid n < \omega\} \in \overline{G}$ and $\overline{\mathfrak{b}}^* \preceq \overline{g}$. Since, for $n < \omega$, $\overline{\mathfrak{b}}^* \preceq \overline{b}_n$, we have $\overline{b}_n \in \overline{G}$ and furthermore, by elementarity,

$$\begin{aligned}\sigma_0(\overline{\mathfrak{b}}^*) &= \sigma_0(\textstyle\prod_{\mathbb{B}_{\overline{\lambda}}}\{\overline{b}_n \mid n < \omega\}) \\ &= \textstyle\prod_{\mathbb{B}_\lambda}\{\sigma_0(\overline{b}_n) \mid n < \omega\} \\ &\preceq \sigma_0(\overline{g}).\end{aligned}$$

Since G is $\mathbb{B}$-generic and $\{\sigma_0(\overline{b}_n) \mid n < \omega\} \subseteq G$, Lemma 2.2.1 implies that $\prod_{\mathbb{B}_\lambda}\{\sigma_0(\overline{b}_n) \mid n < \omega\} \in G$ and thus $\sigma_0(\overline{g}) \in G$, as desired. ■

Combined with the following claim, this allows to prove our initial claim at limit ordinals λ for which there is some $i < \lambda$ such that $\mathrm{cf}(\lambda) \leq \mathrm{card}(\mathbb{B}_i)$.

Claim (3)**.** *Let $h \leq k \leq i < \lambda$ and let H_k be ${}^{\mathbb{B}_k}\!/_{G_h}$-generic over $V[G_h]$. Then $\left({}^{\mathbb{B}_i}\!/_{G_h}\right)/_{H_k}$ is subcomplete in $(V[G_h])[H_k]$.*

Proof. By Proposition 3.0.3 $G_k := G_h * H_k$ is $\mathbb{B}_k$-generic over V and now, by what we've already proved, $\left({}^{\mathbb{B}_i}\!/_{G_h}\right)/_{H_k} \cong {}^{\mathbb{B}_i}\!/_{G_k}$ is subcomplete in $V[G_k] = (V[G_h])[H_k]$. ■

In other words: Our induction hypothesis is also satisfied for $({}^{\mathbb{B}_i}\!/_{G_h} \mid h \leq i < \lambda)$ in $V[G_h]$.

So, let $\lambda < \alpha$ be a limit ordinal such that there is some $j < \lambda$ with $\mathrm{cf}(\lambda) \leq \mathrm{card}(\mathbb{B}_j)$ and such that for all $h \leq i < \lambda$ and all $\mathbb{B}_h$-generic G_h the complete Boolean algebra ${}^{\mathbb{B}_i}\!/_{G_h}$ is subcomplete in $V[G_h]$. We have to show that ${}^{\mathbb{B}_\lambda}\!/_{G_h}$ is subcomplete in $V[G_h]$.
There are two cases:
If $j < h$, then $cf(\lambda) \leq \mathrm{card}(\mathbb{B}_j) \leq \omega_1$ in $V[G_h]$ and since $({}^{\mathbb{B}_i}\!/_{G_h} \mid h \leq i < \lambda)$ is an RCS-iteration such that ${}^{\mathbb{B}_i}\!/_{G_h}$ is subcomplete in $V[G_h]$ for all $h \leq i < \lambda$, our previous claim yields that ${}^{\mathbb{B}_\lambda}\!/_{G_h}$ is subcomplete in $V[G_h]$.
Otherwise $h \leq j$. Let H_{j+1} be any ${}^{\mathbb{B}_{j+1}}\!/_{G_h}$ generic ultrafilter over $V[G_h]$. Then $G_{j+1} := G_h * H_{j+1}$ is $\mathbb{B}_{j+1}$-generic over V and in $V[G_{j+1}] = (V[G_h])[H_j]$, we have $\mathrm{cf}(\lambda) \leq \mathrm{card}(\mathbb{B}_j) \leq \omega_1$. Since $({}^{\mathbb{B}_i}\!/_{G_{j+1}} \mid j+1 \leq i < \lambda)$ is an RCS-iteration in $V[G_{j+1}]$ and ${}^{\mathbb{B}_i}\!/_{G_{j+1}}$ is subcomplete in $V[G_{j+1}]$ for all $j+1 \leq i < \lambda$, we may apply our previous claim to conclude that ${}^{\mathbb{B}_\lambda}\!/_{G_{j+1}} \cong \left({}^{\mathbb{B}_\lambda}\!/_{G_h}\right)/_{H_j}$ is subcomplete in $V[G_{j+1}] = (V[G_h])[H_{j+1}]$. This proves

$$\Vdash_{{}^{\mathbb{B}_{j+1}}\!/_{G_h}} \left({}^{\check{\mathbb{B}}_\lambda}\!/_{\check{G}_h}\right)/_{\dot{H}_{j+1}} \text{ is subcomplete,}$$

where $\dot{H}_{j+1}$ is the canonical ${}^{\mathbb{B}_{j+1}}\!/_{G_h}$-generic name. Since ${}^{\mathbb{B}_j}\!/_{G_h}$ is subcomplete in $V[G_h]$, the Two Step Iteration Theorem (Theorem 4.5) yields that ${}^{\mathbb{B}_\lambda}\!/_{G_h}$ is subcomplete in $V[G_h]$.
The only remaining case is that $\lambda < \alpha$ is an uncountable limit ordinal such that $\mathrm{card}(\mathbb{B}_i) < \mathrm{cf}(\lambda)$ for all $i < \lambda$. Since $\mathbb{B}_i \subsetneq \mathbb{B}_{i+1}$ for all $i < \lambda$, this implies that λ is regular. First, let us handle the case that $h = 0$:

Claim (4). *Let $\lambda < \alpha$ be a regular cardinal such that* $\text{card}(\mathbb{B}_i) < \lambda$ *and such that* $\mathbb{B}_i$ *is subcomplete for all* $i < \lambda$*. Then* $\mathbb{B}_\lambda$ *is subcomplete.*

Proof. By Lemma 4.0.1 it suffices to prove that $\mathbb{B}_\lambda$ is $\{\mathcal{B}, \lambda, \mathbb{B}_\lambda\}$-subcomplete. By the fullness of $V^{\mathbb{B}_i}$, for $i < \lambda$, we may fix a cardinal θ such that $\lambda < \theta$, $\mathbb{B}_\lambda, \mathbb{B} \in H_\theta$ and such that for all $i \leq j < \lambda$

$$\Vdash_{\mathbb{B}_i} \check{\theta} \text{ verifies the subcompleteness of } \check{\mathbb{B}}_j.$$

Let $\tau > \theta$ be regular and let A be a set such that $H_\theta \subseteq L_\tau[A]$ and let

$$\sigma\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$$

be an elementary embedding, where $\overline{N}$ is countable, transitive and full. Furthermore, let $\lambda_1, \ldots, \lambda_{n+1}$ be regular cardinals such that $\text{card}(\mathbb{B}_\lambda) < \lambda_k$ for $k = 1, \ldots, n$, $\lambda_{n+1} = \lambda$ and let s be an additional parameter such that

$$\sigma(\overline{\theta}, \overline{\mathcal{B}}, \overline{\lambda}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, f, \mathcal{B}, \lambda, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}.$$

Finally, let $\overline{G}$ be $\overline{\mathbb{B}}_{\overline{\lambda}}$-generic over $\overline{M}$ and set $\overline{\lambda}_0 := \overline{N} \cap \text{Ord}$. We prove that there is some $b \in \mathbb{B}_\lambda^+$ such that for all $\mathbb{B}_\lambda$-generic ultrafilters G over V with $b \in G$ there is some $\sigma_0 \in V[G]$ that satisfies

a) $\sigma_0\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\sigma_0(\overline{\theta}, \overline{\mathcal{B}}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}$,

c) $\sup \sigma_0 "\overline{\lambda_i} = \sup \sigma "\overline{\lambda_i}$, for $i = 0, 1, \ldots, n+1$, where $\overline{\lambda_0} := \overline{N} \cap \text{Ord}$, and

d) $\sigma_0 "\overline{G} \subseteq G$.

Our strategy is the same as in the case that $\text{cf}(\lambda) \leq \omega_1$. However, the absence of $\overline{f}$ complicates our argument and forces us to add some additional induction hypotheses.
As before fix, for $i = 0, 1, \ldots, n+1$, a striclty increasing and cofinal sequence $(\overline{\xi}^i_k \mid k < \omega)$ in $\overline{\lambda}_i$ and let $\xi^i_k := \sigma(\overline{\xi}^i_k)$. Then $(\xi^i_k \mid k, \omega)$ is striclty increasing and cofinal in $\sup \sigma "\lambda_i$. Let $\xi_k := \xi^{n+1}_k$ and $\overline{\xi}_k := \overline{\xi}^{n+1}_k$ for $k < \omega$ and fix an enumeration $(x_l \mid l < \omega)$ of $\overline{N}$.
We aim to recursively construct a sequence $((\dot{\tau}_k, b_k) \mid k < \omega)$ of $\mathbb{B}_{\xi_k}$-names $\dot{\tau}_k$ and elements $b_k \in \mathbb{B}^+_{\xi_k}$ such that

I) a) $b_0 = 1$, $\dot{\tau}_0 = \check{\sigma}$,

b) $h_{\xi_{k-1}}(b_k) = b_{k-1}$ for $0 < k$,

II) If G is $\mathbb{B}_{\xi_k}$-generic over V with $b_k \in G$, let, for $\eta \leq \xi_k$, $G_\eta := G \cap \mathbb{B}_\eta$ and, for $\eta \leq \overline{\xi}_k$, $\overline{G}_\eta := \overline{G} \cap \overline{\mathbb{B}}_\eta$ be the associated generic ultrafilters. For $j \leq k$ let $\tau_j := \dot{\tau}_j^G = \dot{\tau}_j^{G_{\xi_j}}$. Then

   a) $\tau_k \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

   b) $\tau_k(\overline{\theta}, \overline{\mathcal{B}}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}$,

   c) $\sup \tau_k'' \overline{\lambda}_i = \sup \sigma" \overline{\lambda}_i$ for $i = 0, 1, \ldots, n+1$,

   d) $\tau_k" \overline{G}_{\overline{\xi}_m} \subseteq G$, whenever $\tau_k(\overline{\xi}_m) \leq \xi_k < \tau_k(\overline{\xi}_{m+1})$,

   e) $\tau_{k-1}(x_l) = \tau_k(x_l)$ for all $l < k$,

   f) Let $i \in \{0, 1, \ldots, n+1\}$ and m be such that $\tau_{k-1}(\overline{\xi}^i_m) \leq \xi^i_k < \tau_{k-1}(\overline{\xi}^i_{m+1})$. Then $\tau_k(\overline{\xi}^i_l) = \tau_{k-1}(\overline{\xi}^i_l)$ for $l \leq m+1$.

Note that II) e) implies $\tau_k(x_l) = \tau_j(x_l)$ for $l \leq j \leq k$.
Before showing how to construct $((\dot{\tau}_k, b_k) \mid k < \omega)$, let us show that I) and II) imply the existence of a suitable $\sigma_0 \in V[G]$:
Let $b := \prod\{b_k \mid k < \omega\}$. Since λ is regular and uncountable, $\{\xi_k \mid k < \omega\}$ is bounded below λ and there is hence some eventually constant thread $\mathfrak{c} = (c_i \mid i < \lambda)$ through $(\mathbb{B}_i \mid i < \lambda)$ such that $c_{\xi_k} = b_k$ for all $k < \omega$ and consequently $\mathfrak{c}^* = b$. In particular, this yields $b \neq 0$. Let G be $\mathbb{B}_\lambda$-generic over V such that $b \in G$. For $k < \omega$ let $G_{\xi_k} := G \cap \mathbb{B}_{\xi_k}$ be the associated $\mathbb{B}_{\xi_k}$-generic filter and let $\tau_k := \dot{\tau}_k^G = \dot{\tau}_k^{G_{\xi_k}}$. As before define a map

$$\sigma_0 \colon \overline{N} \to L_\tau[A]$$

by letting $\sigma_0(x_l) = \tau_l(x_l)$ for $l < \omega$. The same proof as above yields that

$$\sigma_0(\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$$

is an elementary embedding that satisfies

$$\sigma_0(\overline{\theta}, \overline{\mathcal{B}}, \overline{\mathbb{B}}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}$$

and $\sup \sigma_0" \overline{\lambda}_i = \sup \sigma" \overline{\lambda}_i$ for $i = 0, 1, \ldots, n+1$. Furthermore, by the same argument as before, we have $\sigma_0" \overline{G}_{\overline{\xi}_k} \subseteq G$ for all $k < \omega$ and since $\bigcup\{\overline{\mathbb{B}}_i \mid i < \overline{\lambda}\} = \bigcup\{\overline{\mathbb{B}}_{\overline{\xi}_k} \mid k < \omega\}$ is dense in $\overline{\mathbb{B}}_{\overline{\lambda}}$, this already implies $\sigma_0" \overline{G} \subseteq G$.
We now finish the proof of our claim by constructing a sequence $((\dot{\tau}_k, b_k \mid k < \omega)$ that satisfies I) and II). To do so, we actually construct a sequence $((\dot{\tau}_k, b_k, c_k) \mid k < \omega)$ such that

III) a) $c_0 = b_0 = 1$,

   b) $h_{\xi_{k-1}}(c_k) = b_{k-1}$ for $0 < k$,

c) IIa) - IIf) hold whenever $c_k \in G$ and

d) $b_k \preceq c_k$.

This will be done recursively, by first defining $(\dot{\tau}_k, c_k)$ and then shrinking c_k to a suitable b_k.
Let $\nu \leq \xi_k < \mu < \sigma"\overline{\lambda}$ be such that $\xi_j < \nu$ for all $j < k$. Then let

$$a_k^{(i,\nu,\mu)} := c_k \cdot \|\dot{\tau}_k(\check{\overline{\xi}}_i) = \check{\nu} \wedge \dot{\tau}_k(\check{\overline{\xi}}_{i+1}) = \check{\mu}\|_{\mathbb{B}_{\xi_k}}.$$

Subclaim (1)**.** *Let $(i,\nu,\mu),(i',\nu',\mu')$ be such that $a_k^{(i,\nu,\mu)}$ and $a_k^{(i',\nu',\mu')}$ are defined and $(i,\nu,\mu) \neq (i',\nu',\mu')$. Then $a_k^{(i,\nu,\mu)} \cdot a_k^{(i',\nu',\mu')} = 0$.*

Proof. Suppose not. Let $(i,\nu,\mu) \neq (i',\nu',\mu')$ be such that $a_k^{(i,\nu,\mu)} \cdot a_k^{(i',\nu',\mu')} \neq 0$ and let G be $\mathbb{B}_{\xi_k}$-generic such that $a_k^{(i,\nu,\mu)} \cdot a_k^{(i',\nu',\mu')} \in G$.
If $i \neq i'$, say $i < i'$, then $\mu = \tau_k(\overline{\xi}_{i+1}) \leq \tau_k(\overline{\xi}_{i'}) = \nu' \leq \xi_k$. (Contradiction!)
Thus $i = i'$ and hence, we have both $\nu = \tau_k(\overline{\xi}_i) = \nu'$ and as $\mu = \tau_k(\overline{\xi}_{i+1}) = \mu'$. (Contradiction!) ■

Additionally, we shall inductively verify that

IV) $a_k^{(i,\nu,\mu)} \cdot \|\dot{\tau}_k(\check{x}) = \check{y}\|_{\mathbb{B}_{\xi_k}} \in \mathbb{B}_\nu$ for all $x, y \in V$ and all ν, μ with $\sup\{\xi_j \mid j < k\} < \nu \leq \xi_k < \mu$.

Now let A_k be the antichain of all $a_k^{(i,\nu,\mu)} \neq 0$ such that $\sup\{\xi_j \mid j < k\} < \nu \leq \xi_k < \mu$. By IV), there is, for each $a_k^{(i,\nu,\mu)} \in A_k$, some $\mathbb{B}_\nu$-name $\dot{\tau}(a_k^{(i,\nu,\mu)})$ such that $\dot{\tau}(a_k^{(i,\nu,\mu)})^{G_\nu} = \tau_k^G$ whenever G is $\mathbb{B}_{\xi_k}$-generic over V with $a_k^{(i,\nu,\mu)} \in G$.
On the other hand, if H is $\mathbb{B}_\nu$-generic over V such that $a_k^{(i,\nu,\mu)} \in H$, then there is some $H \subseteq G$ such that G is $\mathbb{B}_{\xi_k}$-generic over V and $H = G_\nu$ (see Proposition 3.0.3). In particular, this yields

$$\dot{\tau}(a_k^{(i,\nu,\mu)})^H = \dot{\tau}(a_k^{(i,\nu,\mu)})^G = \dot{\tau}_k^G.$$

We therefore have the following statement:
(†) If G is $\mathbb{B}_\nu$-generic over V such that $a = a_k^{(i,\nu,\mu)} \in G$, then II) also holds if we replace τ_k with $\tau(a_k^{(i,\nu,\mu)}) := \dot{\tau}(a_k^{(i,\nu,\mu)})^G$ and we $\tau_j := \dot{\tau}_j^G = \dot{\tau}_j^{G_{\xi_j}}$ for $j < k$, where $G_\eta := G \cap \mathbb{B}_\eta$, for $\eta \leq \nu$, and $\overline{G}_\eta := \overline{G} \cap \overline{\mathbb{B}}_\eta$ for $\eta \leq \overline{\xi}_k$.

Let $\nu < \mu < \lambda$ and let G_ν be $\mathbb{B}_\nu$-generic over V. By our induction hypothesis we know that ${}^{\mathbb{B}_\mu}\!/_{G_\nu}$ is subcomplete in $V[G_\nu]$ and by (†), we may now repeat the previous construction of $(\dot{\tau}_{k+1}, b_{k+1})$ from $(\dot{\tau}_k, b_k)$ in our first limit case. This yields:

($\star$) Let $a = a_k^{(i,\nu,\mu)} \in A_k$. Then there are $\tilde{a} \in \mathbb{B}_\mu^+$ and $\dot{\tau}_0(a) \in V^{\mathbb{B}_\mu}$ such that $h_\nu(\tilde{a}) = a$ and whenever G_μ is $\mathbb{B}_\mu$-generic over V with $\tilde{a} \in G$, we have for $\tau(a) := \dot{\tau}(a)^G$ and $\tau_0(a) := \dot{\tau}_0(a)^G$

a) $\tau_0(a)\colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\tau_0(a)(\overline{\theta}, \overline{\mathcal{B}}, \overline{\mathbb{B}}_{\overline{\lambda}}), \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}$,

c) $\sup \tau_0(a)\text{"}\overline{\lambda}_i = \sup \sigma\text{"}\overline{\lambda}_i$ for $i = 0, 1, \ldots, n+1$,

d) $\tau_0(a)\text{"}\overline{G}_{\overline{\xi}_{i+1}} \subseteq G$ (and by construction $\tau_0(a)(\overline{\xi}_{i+1}) = \mu$),

e) if $r < \omega$ is minimal such that $\mu \leq \xi_r$, then $\tau_0(a)(x_l) = \tau(a)(x_l)$ for $l < r$,

f) if $r < \omega$ is minimal such that $\mu \leq \xi_r$, $i \in \{0, 1, \ldots, n+1\}$ and m is such that $\tau(a)(\overline{\xi}^i_m) \leq \xi_r < \tau(a)(\overline{\xi}^i_{m+1})$, then $\tau_0(a)(\overline{\xi}^i_l) = \tau(a)(\overline{\xi}^i_l)$ for $l \leq m+1$.

For each $a \in A_k$ fix such a pair $(\tilde{a}, \dot{\tau}_0(a))$ and let $r < \omega$ be minimal such that $\mu \leq \xi_r$. We aim to construct a $\mathbb{B}_{\xi_r}$-name $\dot{\tau}_r$ and some $c_r \in \mathbb{B}^+_{\xi_r}$ such that for all $\mathbb{B}_{\xi_r}$-generic G_{ξ_r} with $a \cdot c_r \in G_{\xi_r}$ we have

$$\tilde{a} \in G \text{ and } \dot{\tau}_r^{G_{\xi_r}} = \dot{\tau}_0(a)^{G_{\xi_r}}.$$

We have to arrange that $\tilde{a} = a \cdot c_r$ and $h_{\xi_k}(c_r) = b_k$. This means that $h_{\xi_k}(\tilde{a}) = h_{\xi_k}(a \cdot c_r) = a \cdot h_{\xi_k}(c_r) = a \cdot b_k$. Given c_k, we thus let $\overline{c} := c_k \cdot -\sum A_k$ and $b_k := \overline{c} + \sum\{h_{\xi_k}(\tilde{a}) \mid a \in A_k\}$.
We continue our proof by the following induction on k.
Assume that I)-IV) hold for all $j < k$ and IIIa)-IIIc) and IV) hold at k. We verify that I), II) and IIId) hold at k:
IIId) follows immediatly from the defintion of b_k and since IIIc) holds at k, this implies II). To verify Ib), first note that for any $a = a_k^{i,\nu,\mu} \in A_k$

$$\begin{aligned} h_{\xi_{k-1}} \circ h_{\xi_k}(\tilde{a}) &= h_{\xi_{k-1}}(\tilde{a}) \\ &= h_{\xi_{k-1}} \circ h_\nu(\tilde{a}) \\ &= h_{\xi_{k-1}}(a) \end{aligned}$$

and thus

$$\begin{aligned}h_{\xi_{k-1}}(b_k) &= h_{\xi_{k-1}}(\bar{c} + \sum\{h_{\xi_k}(\tilde{a}) \mid a \in A_k\}) \\ &= h_{\xi_{k-1}}(\bar{c}) + \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A_k\} \\ &= h_{\xi_{k-1}}(c_k \cdot -\sum A_k) + \sum\{h_{\xi_{k-1}}(a) \mid a \in A_k\} \\ &= h_{\xi_{k-1}}(c_k \cdot -\sum A_k) + h_{\xi_{k-1}}(\sum A_k) \\ &= h_{\xi_{k-1}}(c_k \cdot -\sum A_k + \sum A_k) \\ &= h_{\xi_{k-1}}(b_k) \\ &= b_{k-1}.\end{aligned}$$

Furthermore, using IIc) and $\sigma_0 = \sigma$, we have $A_0 = \{a^{0,0,\xi_1}\} = \{1\}$ and thus $b_0 = h_0(\tilde{1}) = 1$, i.e. Ia) holds true.
Finally, assuming that I)-IV) hold for all $j < k$, we define c_k and $\dot{\tau}_k$ and verify IIIa)-IIIc) and IV) at k.
For $k = 0$ we let $c_k := 1$ and $\dot{\tau}_k := \check{\sigma}$. In this case, IIIa)-IIIc) and IV hold trivially.
If $k = j + 1$, use A_l and $\{\tilde{a} \mid A_l\}$ for $l < k$ to define A_j^* as the set of all $a_l^{i,\nu,\mu} \in \bigcup\{A_l \mid l < k\}$ such that $\xi_j < \mu$.

Subclaim (2). *Let $l < k$ and $a = a_l^{i,\nu,\mu} \in A_l$. If G is $\mathbb{B}_{\xi_k}$-generic over V such that $a \in G$, then $\tau_{k-1}(\bar{\xi}_h) = \tau_l(\bar{\xi}_h)$ for all $h \leq i + 1$.*

Proof. Let $k = j + 1$. By induction on j, we prove that $\tau_j(\bar{\xi}_h) = \tau_l(\bar{\xi}_h)$ for all $h \leq i + 1$ and for all $l \leq j$ that satisfy I)-IV).
This is trivial for $j = l$. If $j = n + 1$ and $l \leq n$, then

$$\tau_n(\bar{\xi}_i) = \nu \leq \xi_j < \mu = \tau_n(\bar{\xi}_{i+1})$$

and hence IIf) yields the claim. ∎

Repeating the proof of Subclaim(1), we again have

Subclaim (3). *Let $(i, \nu, \mu), (i', \nu', \mu')$ be such that $a_j^{(i,\nu,\mu)}, a_j^{(i',\nu',\mu')} \in A_{k-1}^*$ are defined and $(i, \nu, \mu) \neq (i', \nu', \mu')$. Then $a_j^{(i,\nu,\mu)} \cdot a_j^{(i',\nu',\mu')} = 0$.*

Let

$$c_k := \sum\{h_{\xi_k}(\tilde{a}) \mid a \in A_{k-1}^*\}$$

and $\tilde{A}_k$ be the set of all $a_{k-1}^{(i,\nu,\mu)} \in A_{k-1}^*$ such that $\mu \leq \xi_k$. By Lemma 2.3.3, we may now fix a $\mathbb{B}_{\xi_k}$-name $\dot{\tau}_k$ such that $\|\dot{\tau}_k = \dot{\tau}_0(a)\| = \tilde{a}$ for all $a \in \tilde{A}_k$ and $\|\dot{\tau}_k = \dot{\tau}_{k-1}\| \cdot c_k = c_k \cdot -\sum \tilde{A}_k$.
IIIa) holds trivially at $k > 0$ and we continue by verifying IIIc), i.e.

Subclaim (4). *Let G be $\mathbb{B}_{\xi_k}$-generic over V such that $c_k \in G$. For $\eta \le \xi_k$ let $G_\eta := G \cap \mathbb{B}_\eta$ and for $\eta \le \overline{\xi}_k$ let $\overline{G}_\eta := \overline{G} \cap \overline{\mathbb{B}}_\eta$. Let $\tau_j := \dot{\tau}_j^G = \dot{\tau}_j^{G_{\xi_j}}$, for $j \le k$. Then*

a) $\tau_k \colon (\overline{N}; \in, \overline{A}) \prec (L_\tau[A]; \in, A)$,

b) $\tau_k(\overline{\theta}, \overline{\mathcal{B}}, \mathbb{B}_{\overline{\lambda}}, \overline{s}, \overline{\lambda}_1, \ldots, \overline{\lambda}_{n+1}) = \theta, \mathcal{B}, \mathbb{B}_\lambda, s, \lambda_1, \ldots, \lambda_{n+1}$,

c) $\sup \tau_k'' \overline{\lambda}_i = \sup \sigma" \overline{\lambda}_i$ *for* $i = 0, 1, \ldots, n+1$,

d) $\tau_k" \overline{G}_{\overline{\xi}_m} \subseteq$, *whenever* $\tau_k(\overline{\xi}_m) \le \xi_k < \tau_k(\overline{\xi}_{m+1})$,

e) $\tau_{k-1}(x_l) = \tau_k(x_l)$ *for all* $l < k$,

f) Let $i \in \{0, 1, \ldots, n+1\}$ *and* m *be such that* $\tau_{k-1}(\overline{\xi}^i_m) \le \xi^i_k < \tau_{k-1}(\overline{\xi}^i_{m+1})$. *Then* $\tau_k(\overline{\xi}^i_l) = \tau_{k-1}(\overline{\xi}^i_l)$ *for* $l \le m+1$.

Proof. There are two cases. First suppose that there is some $a \in \tilde{A}_k$ such that $\tilde{a} \in G$. Fix $l < k$ such that $a = a_l^{(i,\nu,\mu)} \in A_l$. Since $a \in \tilde{A}_k$ we have $\mu \le \xi_k$ and thus $\xi_{k-1} < \mu \le \xi_k$. By the definition of τ_k, we have $\tau_k = \tau_0(a)$ and hence a)-d) follow from properties a)-d) in ($\star$).
k is the minimal r such that $\mu \le \xi_r$, and thus ($\star$) yields that $\tau_0(a)(x_l) = \tau(a)(x_l)$ for all $l < k$. Since $\tau(a) = \tau_l$, we have

$$\tau_l(\overline{\xi}_i) = \nu \le \xi_l \le \xi_{l'} < \tau_l(\overline{\xi}_{i+1}) = \mu$$

for $l \le l' < k$ and thus $\tau(a) = \tau_{l'}$ for $l \le l' < k$. In particular, for $l < k$

$$\begin{aligned} \tau_k(x_l) &= \tau(a)_0(x_l) \\ &= \tau(a)(x_l) \\ &= \tau_{k-1}(x_l) \end{aligned}$$

and thus e) holds true.
Finally, let $i \in \{0, 1, \ldots, n+1\}$ and let m be such that $\tau_{k-1}(\overline{\xi}^i_m) \le \xi^i_k < \tau_{k-1}(\overline{\xi}^i_{m+1})$ and let $l \le m+1$. Then

$$\begin{aligned} \tau_k(\overline{\xi}^i_l) &= \tau_0(a)(\overline{\xi}^i_l) \\ &= \tau(a)(\overline{\xi}^i_l) \\ &= \tau_{k-1}(\overline{\xi}^i_l), \end{aligned}$$

i.e. f) holds true.
The second case is easier. If $\tilde{a} \notin G$ for all $a \in \tilde{A}_k$, then $c_k \cdot - \sum \tilde{A}_k \in G$ and thus, by the definition of $\dot{\tau}_k$, $\tau_k = \tau_{k-1}$. This trivially implies a)-f). ■

Next, we verify IIIb), i.e.

Subclaim (5). $h_{\xi_{k-1}}(c_k) = b_{k-1}$

Proof. Suppose not and let $k > 0$ be minimal such that $h_{\xi_{k-1}}(c_k) \neq b_{k-1}$. Recall that

$$\begin{aligned} h_{\xi_{k-1}}(c_k) &= h_{\xi_{k-1}}(\sum\{h_{\xi_k}(\tilde{a}) \mid a \in A^*_{k-1}\}) \\ &= \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A^*_{k-1}\} \end{aligned}$$

and $b_{k-1} = \overline{c} + \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A_{k-1}\}$, where $\overline{c} = c_{k-1} \cdot -\sum A_{k-1}$. Let $A' := A^*_{k-1} \setminus A_{k-1}$. We derive a contradiction by proving that

$$\overline{c} = \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A'\}. \tag{$\Diamond$}$$

Indeed, if $\overline{c} = \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A'\}$, then

$$\begin{aligned} b_{k-1} &= \overline{c} + \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A_{k-1}\} \\ &= \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in \underbrace{A' \cup A_{k-1}}_{=A^*_{k-1}}\} \\ &= h_{\xi_{k-1}}(c_k). \end{aligned}$$

(Contradiction!). So, let us prove ($\Diamond$):
First, let $a \in A' = A^*_{k-1} \setminus A_{k-1}$. Then $h_{\xi_{k-1}}(\tilde{a}) \preceq c_{k-1} = \sum\{h_{\xi_k}(\tilde{a}) \mid a \in A^*_{k-1}\}$. On the other hand, for any $b \in A_{k-1}$, Subclaim(3) yields $a \cdot b = 0$ and thus $h_{\xi_{k-1}}(\tilde{a}) \cdot h_{\xi_{k-1}}(\tilde{b}) = 0$. Hence

$$\begin{aligned} h_0(h_{\xi_{k-1}}(\tilde{a}) \cdot \sum A_{k-1}) &= \sum h_0(\{h_{\xi_{k-1}}(\tilde{a}) \cdot h_\nu(\tilde{b})) \mid b = a^{(i,\nu,\mu)}_{k-1} \in A_{k-1}\}) \\ &= \sum\{h_0(\tilde{a} \cdot \tilde{b}) \mid b = a^{(i,\nu,\mu)}_{k-1} \in A_{k-1}\}) \\ &= 0. \end{aligned}$$

Therefore $h_{\xi_{k-1}}(\tilde{a}) \preceq -\sum A_{k-1}$ and consequently $\sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A'\} \preceq \overline{c}$.
Conversely, suppose that $\overline{c} \not\preceq \sum\{h_{\xi_{k-1}}(\tilde{a}) \mid a \in A'\}$. Then there is some l and some $a = a^{i,\nu,\mu}_l \in A^*_l \setminus A'$ such that $h_{\xi_l}(\tilde{a}) \cdot \overline{c} \neq 0$ and hence $a \cdot \overline{c} \neq 0$. This leads to a contradiction:
Let G be $\mathbb{B}_{\xi_{k-1}}$-generic over V such that $a \cdot \overline{c} \in G$. Since $a \notin A'$, we have $\nu \leq \xi_l < \mu \leq \xi_{k-1}$ and thus $\tau_l(\overline{\xi}_i) = \nu \leq \xi_l < \tau_l(\overline{\xi}_{i+1}) = \mu$. By Subclaim(4)f), this implies $\tau_{k-1}(\overline{\xi}_r) = \tau_l(\overline{\xi}_r)$ for all $r \leq i{+}1$. In particular, we have $\tau_{k-1}(\overline{\xi}_{i+1}) \leq \overline{\xi}_{k-1}$ and there is thus some $i < n$ such that $\tau_{k-1}(\overline{\xi}_n) \leq \xi_{k-1} < \tau_{k-1}(\overline{\xi}_{n+1})$. Let $\nu^* := \tau_{k-1}(\overline{\xi}_n)$ and $\mu^* := \tau_{k-1}(\overline{\xi}_{n+1})$. Then $a^* := a^{(m,\nu^*,\mu^*)}_{k-1} \in G \cap A_{k-1}$a therefore $a^* \cdot \overline{c} \neq 0$, contradicting the definition of $\overline{c}$. ∎

Before verifying IV) at k, let us prove the following

Subclaim (6)**.** *Let $a = a_k^{(i,\nu,\mu)} \in A_k$ and let $\overline{a} = a^{(\overline{i},\overline{\nu},\overline{\mu})} \in A^*_{k-1}$ such that $\nu < \overline{\mu}$. Then $a \cdot \overline{a} = 0$.*

Proof. Suppose not. Let G be $\mathbb{B}_{\xi_k}$-generic over V such that $a \cdot \overline{a} \in G$. Then $\tau_{k-1}(\overline{\xi}_{\overline{i}}) = \overline{\nu} \leq \xi_{k-1} < \tau_{k-1}(\overline{\xi}_{\overline{i}+1})$ and therefore, by Subclaim(2), $\tau_k(\overline{\xi}_l) = \tau_{k-1}(\overline{\xi}_l)$ for all $l \leq \overline{i}+1$.
In particular, we now have

$$\tau_k(\overline{\xi}_i) = \nu < \overline{\mu} = \tau_{k-1}(\overline{\xi}_{\overline{i}+1}) = \tau_k(\overline{\xi}_{\overline{i}+1}) \leq \xi_{k-1}$$

and thus $i \leq \overline{i}$. This implies

$$\nu = \tau_k(\overline{\xi}_i) \leq \tau_k(\overline{\xi}_{\overline{i}}) = \overline{\nu} \leq \xi_{k-1} < \xi_k,$$

contrary to $a_k^{(i,\nu,\mu)} \in A_k$. (Contradiction!) ∎

We finish our proof of Claim(4) by verifying IV), i.e.

Subclaim (7)**.** *$a_k^{(i,\nu,\mu)} \cdot \|\dot{\tau}_k(\check{x}) = \check{y}\|_{\mathbb{B}_{\xi_k}} \in \mathbb{B}_\nu$ for all $x, y \in V$ and all ν, μ with $\sup\{\xi_j \mid j < k\} < \nu \leq \xi_k < \mu$.*

Proof. Let $a = a_k^{(i,\nu,\mu)} \in A_k$ and let A' be the set of all $b = a_l^{(\overline{i},\overline{\nu},\overline{\mu})} \in A^*_{k-1}$ such that $\overline{\mu} \leq \nu$. Using Subclaim(6), we now obtain

$$\begin{aligned} a \cdot c_k &= a \cdot \sum\{\underbrace{h_{\xi_k}(\tilde{b})}_{=\tilde{b}} \mid b \in A^*_{k-1}\} \\ &= \sum\{a \cdot \tilde{b} \mid b \in A^*_{k-1}\}. \end{aligned}$$

Now $\tilde{b} = \|\dot{\tau}_k = \dot{\tau}_0(b)\| \in \mathbb{B}_{\overline{\mu}} \subseteq \mathbb{B}_\nu$ for all $b = a_l^{(\overline{i},\overline{\nu},\overline{\mu})} \in A'$. Furthermore, since $\dot{\tau}_0(b), \check{x}, \check{y} \in V^{\mathbb{B}_{\overline{\mu}}}$ and since $\dot{\tau}_0(b)(\check{x}) = \check{y}$ is a Σ_0 statement, we have, by Proposition 2.2.1, $\|\dot{\tau}_0(b)(\check{x}) = \check{y}\| \in \mathbb{B}_{\overline{\mu}} \subseteq \mathbb{B}_\nu$ and therefore

$$a \cdot \|\dot{\tau}_k(\check{x}) = \check{y}\| \cdot \tilde{b} = \tilde{b} \cdot \|\dot{\tau}_0(b)(\check{x}) = \check{y}\| \cdot \|\dot{\tau}_0(b)(\check{\overline{\xi}}_i) = \check{\nu}\| \cdot \|\dot{\tau}_0(b)(\check{\overline{\xi}}_{i+1}) = \check{\mu}\| \in \mathbb{B}_\nu.$$

Hence $a \cdot \|\dot{\tau}_k(\check{x}) = \check{y}\| = \sum\{a \cdot \|\dot{\tau}_k(\check{x}) = \check{y}\| \cdot \tilde{b} \mid b \in A'\} \in \mathbb{B}_\nu$. ∎

∎

Given the length of this proof, it may be a good idea to summarize our progress. By induction on i, we aimed to prove

Claim (1)**.** *Let $h \leq i < \alpha$ and let G_h be $\mathbb{B}_h$-generic over V. Then ${}^{\mathbb{B}_i}\!/_{G_h}$ is subcomplete in $V[G_h]$.*

This was trivial for $i = h$ and and a straightforward application of the Two Step Iteration Theorem solved the case $i = j + 1$. If $\lambda < \alpha$ is a limit stage of cofinality $\leq \omega_1$ and the claim holds below λ, we were able to prove that $\mathbb{B}_\lambda$ is subcomplete in V. Building on this, we verified the induction step for limit ordinals λ if there is some $i < \lambda$ such that $\mathrm{cf}(\lambda) \leq \mathrm{card}(\mathbb{B}_i)$.
The only remaining case is that λ is regular. So far, we proved that $\mathbb{B}_\lambda$ is in fact subcomplete in V. This was the most difficult part and we now have to check ${}^{\mathbb{B}_\lambda}\!/_{G_h}$ is also subcomplete in $V[G_h]$. Fortunately, there are no further complications:
Let $h < \lambda$ and let G_h be $\mathbb{B}_h$-generic over $V[G_h]$. Applying our induction hypothesis and working in $V[G_h]$, we now have that $({}^{\mathbb{B}_i}\!/_{G_h} \mid h \leq i < \lambda)$ is an RCS-iteration such that ${}^{\mathbb{B}_i}\!/_{G_h}$ is subcomplete for all $i < \lambda$. Since ${}^{\mathbb{B}_h}\!/_{G_h} \cong \{0,1\}$, ${}^{\mathbb{B}_i}\!/_{G_h} \neq {}^{\mathbb{B}_{i+1}}\!/_{G_h}$ and

$$\Vdash_{\mathbb{B}_{i+1}/G_h} \mathrm{card}(\check{{}^{\mathbb{B}_i}\!/_{G_h}}) \leq \omega_1$$

for all $h \leq i < \lambda$, we may simply repeat the proofs of Claim(2) and Claim(4) inside $V[G_h]$ to obtain that ${}^{\mathbb{B}_\lambda}\!/_{G_h}$ is subcomplete in $V[G_h]$. ■

□

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