Thilo Volker Weinert

Beschränkte Forcingaxiome

dem Institut für Mathematische Logik und Grundlagenforschung
der Westfälischen Wilhelms–Universität Münster am
Freitag den 11. Januar 2008 als Diplomarbeit eingereicht

Preface

Most of this thesis was written in summer, autumn and winter of the year 2007, sometimes in Münster and sometimes in Wächtersbach. At this point I wish to thank all those people who helped me in doing this. My thanks go to Professor Dr. Ralf-Dieter Schindler who suggested working on Bounded Forcing Axioms which eventually turned out to be a fruitful topic and allowed me to pursue my own lines of thought too. They also go to Dr. Gunter Fuchs who always had time to discuss problems and to give advice when my mathematical considerations temporarily reached a dead end. Furthermore I am grateful for Philipp Doebler for always showing interest and asking inspiring questions, not only personally but often on the telephone as well. He also gave me the valuable hint to look at a certain article.

Further gratitude goes to Philipp Schlicht for inspiring discussions. I am also feeling thankful to my math teachers at school and at university who often encouraged my interest in mathematics and sometimes failed in spoiling it.

When one considers the bibliography one cannot escape the conjecture that at this point a statement of gratitude towards many more women and men is in order. So I now also wish to thank all those who over the decades participated in the intellectual adventure known as set theory.

Finally I wish to thank some people who have got no clue about set theory. These are: my father who encouraged my general curiosity greatly, the men of Knobelsdorff Computer GbR who managed to repair my eleven–year old laptop within twenty four hours, Ian Walsh, who corrected some errors in English grammar, spelling and interpunction and, last but not least, my mother who supported me in many ways but also for some weeks gave me additional time to work on this thesis for not more than some occasional gardening in exchange.

Münster in Westfalia — Thilo V. Weinert

Thursday, January 10th, 2008

Contents

Introduction

One can probably say that the story of Bounded forcing axioms begins in 1970 with the publication of [M–S] by Donald Martin. There he formulates what came to be known as Martin's Axiom. This stood in connection with the publication of [S–T] where the consistency of Souslin's hypothesis was established by iterated forcing. Martin's axiom proved to be a useful tool even for mathematicians whose main focus is not set theory. For this recall that for any regular uncountable κ ZFC $+ 2^{\aleph_0} = \kappa +$ MA is relatively consistent to ZFC. So anybody whose everyday mathematical practice could be formalized within ZFC—we suppose that this is the case for most mathematicians—simply could try to prove a certain statement from Martin's axiom. If she succeeded in this endeavour she would in particular have proved that $\neg p$ is unprovable from ZFC—while not necessarily knowing anything about forcing or constructibility. Although in its formulation Martin's axiom is not a *Bounded* Forcing Axiom it can be considered as such since it is equivalent to its bounded form—see for example [Ku], lemma II.3.1.. In the eighties there were several important developments. Axiom A was defined by James Baumgartner, the theory of proper forcing was developed by Saharon Shelah and the forcing axioms PFA and MM were formulated in [Ba 2], [F–M–S] respectively. Attention shifted to the bounded forms somehow after the publication of [G–S] and [Bag 1]. While the latter article showed that Bounded Forcing Axioms allow an attractive characterization as absoluteness statement between the ground model and generic extensions, the former yielded an equiconsistency result showing that unlike in the case of MA not only is BPFA not equivalent to PFA but in addition it has considerably lower consistency strength.

Mainly this thesis presents the results of [G–S]. The first three chapters provide the basis for this. While in the first chapter the necessary large cardinal notions are introduced—the one of reflecting and regular Σ_n–correct cardinals—the second chapter contains an introduction to proper forcing as well as Axiom A forcing. In the third chapter the main theorem of [Bag 1] is stated and proved and the forcing axiom B$\mathbb{A}$AFA is introduced.

The fourth chapter proves the consistency of BPFA from the existence of a reflecting cardinal κ by an iterated forcing construction of length κ. Here the proof is written down in a new—more semantically–oriented—way.

The goal of the fifth chapter is dual to the one of chapter four. Here it is proved that if BAAFA holds then $\aleph_2$ is reflecting in L.

The sixth chapter then aims at the construction of a model of BAAFA $+\neg$ BPFA.

Finally the seventh and last chapter contains some remarks on questions left open by the thesis as well as historical explanations and semi–philosophical considerations concerning the continuum problem.

In the course of writing this thesis I already decided at an early point not to aim at methodical purity. This shows for example in the fourth chapter where some arguments are carried out for reflecting cardinals and others for regular Σ_2–correct cardinals. In the case of arbitrary posets against complete Boolean algebras this decision turned out to be a wise one. The canonical kind of forcing notion in the context of Bounded Forcing Axioms seems to consist in complete Boolean algebras since for those there provably exists an equivalent principle of generic absoluteness. However at some point I decided to take a closer look at Axiom A forcing notions and it turned out that—at least according to my experience—the canonical kind of forcing notion seems to be the arbitrary poset. Since I was unable to prove that for every Axiom A poset the corresponding regular open algebra satisfies Axiom A too I decided to introduce the notions of being reasonable, the reasonable hull and Axiom A*—see definition 2.22. This sometimes caused an excess of technicality—such as in corollary 2.39 but dealing with such technical problems in the proof of such a corollary seemed better to me than to look for individual tricky solutions in many other proofs. Finally the creative part of this thesis consists in the formulation of BAAFA in definition 3.7 and the construction of a model of ZFC + BAAFA $+\neg$ BPFA in theorem 6.7. .

1 Reflecting and Σ_n–correct cardinals

We are going to introduce some large cardinal notions—the notion of a reflecting cardinal and the notion of a regular Σ_n–correct cardinal. These large cardinal notions fall between inaccessiblity and Mahloness in the hierarchy of consistency strength. We start with some technical remarks:

1.1. Lemma.

- "$x = H_\kappa$" is a Δ_2-assertion for any cardinal κ.

Since the model relation between a set and a formula can be formulated by restricting the quantifiers of the formula to the set we immediately attain:

- "$H_\kappa \models$ "$\varphi(a)$"" is a Δ_2-assertion for any cardinal κ such that $a \in H_\kappa$.

Proof. First note that $x \in H_\kappa$ is $\Sigma_1\big(\{x, \kappa\}\big)$. This is the case since it can be written as follows:

$$\exists \alpha < \kappa, y \supset x, f \in \text{Func} : f : \alpha \twoheadrightarrow y \tag{1.1}$$

Here the general quantifier expressing f's being onto is bounded by y.

Now we can write $x = H_\kappa$ in a $\Sigma_2\big(\{x, \kappa\}\big)$–fashion as follows:

$$\forall y \in x : y \in H_\kappa \wedge \forall y \big(y \notin H_\kappa \vee y \in x\big) \tag{1.2}$$

The first part of the formula is $\Sigma_1\big(\{x, \kappa\}\big)$ since "$x \in H_\kappa$" is, the second consists of a Π_1–formula preceded by an unbounded general quantifier which renders it Π_1. This shows that it can be conceived of as a $\Sigma_2\big(\{x, \kappa\}\big)$– as well as a $\Pi_2\big(\{x, \kappa\}\big)$–formula. So it is $\Delta_2\big(\{x, \kappa\}\big)$. $\dashv$

We will see soon that the reflecting cardinals are precisely the regular Σ_2–correct cardinals. The main cause for this lies in the following fact which links the sets of hereditarily limited cardinality H_κ with the class of Σ_2–assertions.

1.2. Lemma. If $\kappa \in \mathrm{Card} \setminus \aleph_1$, $a \in H_\kappa$, $\varphi(a)$ is a Σ_2–formula in the language of set theory and $H_\kappa \models$ "$\varphi(a)$" then $\varphi(a)$.

Proof. Let ψ be a Σ_0–formula in the language of set theory, κ a cardinal and suppose that

$$H_\kappa \models \text{“}\exists x \forall y \psi(x, y, a)\text{”}. \tag{1.3}$$

Towards a contradiction assume that

$$\forall x \exists y \neg\psi(x, y, a). \tag{1.4}$$

Choose a witness for the truth of (1.3), i.e. a $b \in H_\kappa$ such that

$$H_\kappa \models \text{“}\forall y \psi(b, y, a)\text{”}. \tag{1.5}$$

By (1.4) we have $\exists y \neg\psi(b, y, a)$. Take a witness c for the truth of this statement then $\neg\psi(b, c, a)$ and clearly $c \notin H_\kappa$. Let $\lambda := \overline{\overline{\mathrm{trcl}(c)}}$ then $\lambda \geqslant \kappa$ and $H_{\lambda^+} \models$ "$\neg\psi(b, c, a)$". Set $\mu := \overline{\overline{\mathrm{trcl}(\{a, b\})}} + \aleph_0$. Since $a, b \in H_\kappa$ we have $\mu < \kappa$. So by the Löwenheim-Skolem theorem one can take an elementary submodel $M \prec H_{\lambda^+}$ such that $M \supset \mathrm{trcl}(\{a, b\})$ and $M \in [H_{\lambda^+}]^\mu$. Now we can form the Mostowski-collapse of M—call it N. Let $\pi : M \longleftrightarrow N$ be the collapsing function. Since $\mathrm{trcl}(\{a, b\}) \subset M$ we have $\pi(a) = a$ and $\pi(b) = b$. But $M \models$ "$\neg\psi(b, c, a)$" so $N \models$ "$\neg\psi\big(b, \pi(c), a\big)$". Furthermore $\pi(c) \in N \subset H_{\mu^+} \subset H_\kappa$ hence $H_\kappa \models$ "$\neg\psi\big(b, \pi(c), a\big)$". This contradicts (1.5). $\dashv$

1.3. Corollary. $\forall \kappa \in \mathrm{Card} : H_\kappa \prec_{\Delta_2} V$

1.4. Definition. A cardinal κ is called *reflecting* if and only if...

- ...it is regular and the following holds:
- Whenever $a \in H_\kappa$, φ is a formula in the language of set theory, λ is a successor cardinal and $H_\lambda \models$ "$\varphi(a)$" there exists a $\mu \in \mathrm{Card} \cap \kappa$ such that $a \in H_\mu$ and $H_\mu \models$ "$\varphi(a)$".

This is **not** the notion of reflecting cardinal which is introduced in [Je 2] on page 697. This becomes clear when one compares the results which are following here with the fact that Jech's notion has a consistency strength well above the one of "There exists a Mahlo cardinal.".

1.5. LEMMA. The following are equivalent for an ordinal κ.

(1) κ is regular and whenever λ is a successor cardinal, $a \in H_\kappa$, φ is a formula in the language of set theory and $H_\lambda \models$ "$\varphi(a)$" there exists a successor cardinal $\mu < \kappa$ such that $a \in H_\mu$ and $H_\mu \models$ "$\varphi(a)$".

(2) κ is reflecting.

(3) κ is regular and whenever $\lambda \in \text{Card}$, $a \in H_\kappa$, φ is a formula in the language of set theory and $H_\lambda \models$ "$\varphi(a)$" there exists a $\mu \in \text{Card} \cap \kappa$ such that $a \in H_\mu$ and $H_\mu \models$ "$\varphi(a)$".

Proof.

- (1) $\Rightarrow$ (2): This is trivial.

- (2) $\Rightarrow$ (3): Let λ be a cardinal, $a \in H_\kappa$ and φ a formula such that $H_\lambda \models$ "$\varphi(a)$". Suppose w.l.o.g. that $\lambda \geqslant \kappa$. We have

$$H_{\lambda^+} \models \text{"}\exists \mu \in \text{Card}\colon H_\mu \models \varphi(a)\text{"}. \tag{1.6}$$

By (2) there is a $\vartheta \in \text{Card} \cap \kappa$ such that

$$H_\vartheta \models \text{"}\exists \mu \in \text{Card}\colon H_\mu \models \varphi(a)\text{"}. \tag{1.7}$$

Let $\mu \in H_\vartheta$ witness this then $a \in H_\mu$, μ is really a cardinal and $H_\vartheta \models$ "$H_\mu \models \varphi(a)$". But then by lemma 1.1 in conjunction with lemma 1.2 $H_\mu \models$ "$\varphi(a)$" really holds true.

- (3) $\Rightarrow$ (1): Let λ be a successor cardinal, $a \in H_\kappa$ and suppose that $H_\lambda \models$ "$\varphi(a)$". Then the following holds true:

$$H_\lambda \models \text{"There exists a largest cardinal and } \varphi(a) \text{ is valid."} \tag{1.8}$$

By (3) there has to be a $\mu \in \text{Card} \cap \kappa$ such that

$$H_\mu \models \text{"There exists a largest cardinal and } \varphi(a) \text{ is valid."}. \tag{1.9}$$

But then μ has to be a successor cardinal.

⊣

1.6. Definition. An ordinal α is called Σ_n*–correct* if and only if $V_\alpha \prec_{\Sigma_n} V$.

1.7. Lemma. A regular cardinal is Σ_1–correct iff it is inaccessible.

Proof.

- ($\Leftarrow$): If κ is inaccessible then in particular it is regular and $H_\kappa = V_\kappa$. Let $a \in V_\kappa$ and $\varphi(a)$ be a Σ_1–formula. Then $\varphi(a), \neg\varphi(a)$ are both Σ_2–formulae. So lemma 1.2 implies that $V_\kappa \models$ "$\varphi(a)$" $\Rightarrow \varphi(a)$ and $V_\kappa \models$ "$\neg\varphi(a)$" $\Rightarrow \neg\varphi(a)$. Hence $V_\kappa \prec_{\Sigma_1} V$.
- ($\Rightarrow$): Let κ be Σ_1–correct and regular. It suffices to show that κ is a strong limit cardinal. To this end let $\lambda \in \mathrm{Card} \cap \kappa$. Since every infinite cardinal is a limit ordinal one can argue as follows:

$$\lambda \subset V_\lambda \tag{1.10}$$
$$\Rightarrow \forall X \subset \lambda \colon X \subset V_\lambda \tag{1.11}$$
$$\Rightarrow \forall X \subset \lambda \colon X \in V_{\lambda+1} \tag{1.12}$$
$$\Leftrightarrow \mathfrak{P}(\lambda) \subset V_{\lambda+1} \tag{1.13}$$
$$\Rightarrow \mathfrak{P}(\lambda) \in V_{\lambda+2} \subset V_\kappa \tag{1.14}$$

Now one considers the following statement which is $\Sigma_1\big(\{\mathfrak{P}(\lambda)\}\big)$ and hence $\Sigma_1(V_\kappa)$:

$$\exists \alpha < \Omega, f \in \mathrm{Func} \colon \alpha \twoheadrightarrow \mathfrak{P}(\lambda). \tag{1.15}$$

κ is Σ_1-correct so there are $\alpha \in \Omega \cap V_\kappa = \kappa$, $f \in \mathrm{Func} \cap V_\kappa$ such that $f : \alpha \twoheadrightarrow \mathfrak{P}(\lambda)$. Clearly $2^\lambda \leqslant \alpha$ so $2^\lambda \in V_\kappa$ and $2^\lambda < \kappa$. So κ is a strong limit cardinal and hence inaccessible.

⊣

1.8. Lemma. A regular cardinal is reflecting iff it is Σ_2–correct.

Proof.

- $(\Rightarrow)$: Let κ be a reflecting cardinal.
  - First we show that κ is Σ_1–correct. So let $a \in V_\kappa$ and suppose $\exists x \varphi(x,a)$ holds where φ is a Σ_0-formula in the language of set theory. Let b be a witness to this, i.e. choose a b such that $\varphi(b,a)$ holds. Define $\lambda := \overline{\overline{\text{trcl}(\{a,b\})}}$. Then $H_{\lambda^+} \models$ "$\exists x \varphi(x,a)$". Since κ is reflecting there exists a cardinal $\mu < \kappa$ such that $H_\mu \models$ "$\exists x \varphi(x,a)$". But then $V_\kappa \models$ "$\exists x \varphi(x,a)$" since $V_\kappa \supset H_\kappa \supset H_\mu$.
  - Since κ is regular and Σ_1–correct it is inaccessible by lemma 1.7 hence $V_\kappa = H_\kappa$. So let $a \in V_\kappa$ and $\varphi(a)$ be a Σ_2–statement. Of course by lemma 1.2 if $V_\kappa \models$ "$\varphi(a)$" then $\varphi(a)$. If on the other hand $\varphi(a)$ the reflection theorem implies that there is a cardinal $\lambda \geqslant \kappa$ such that $H_\lambda \models$ "$\varphi(a)$"—use for example [Ku], IV.7.5 and let Z be the functional relation $\alpha \mapsto H_{\aleph_\alpha}$. Since κ is reflecting there is a cardinal $\mu < \kappa$ such that $H_\mu \models$ "$\varphi(a)$". Because κ is inaccessible one can apply lemma 1.2 within V_κ. This shows that $V_\kappa \models$ "$\varphi(a)$".

- $(\Leftarrow)$: Let κ be regular and Σ_2–correct. Suppose $a \in H_\kappa$, $\lambda \in \text{Card}$ and φ is a formula in the language of set theory such that $H_\lambda \models$ "$\varphi(a)$". Then

$$\exists \mu \in \text{Card}\colon H_\mu \models \text{“}\varphi(a)\text{”} \tag{1.16}$$

  is true and by lemma 1.1 a Σ_2–statement. So it holds in V_κ. Let $\mu \in \text{Card} \cap \kappa$ be a witness to this fact. Then $H_\mu \models$ "$\varphi(a)$" and μ is a cardinal.

$\dashv$

In the following proofs we are going to use the fact that the satisfaction relation is Δ_1–definable—see for example [De 2], Chapter I, section 9. This means there is a Σ_1–formula φ_Σ and a Π_1–formula φ_Π such that for every formula ψ, every set M and every $a \in M$:

$$\text{Sat}(M,a,[\psi]) :\Longleftrightarrow M \models \text{“}\psi(a)\text{”} \Longleftrightarrow \varphi_\Sigma(M,a,[\psi]) \Longleftrightarrow \varphi_\Pi(M,a,[\psi]) \tag{1.17}$$

By $[\psi]$ we refer to the Gödel–number of ψ. In the following proofs we suppose that the satisfaction relation above has been defined for a bijective Gödel–numbering of all Σ_0–formulae by natural numbers.

1.9. Lemma. "α is Σ_n–correct." is Π_n–expressible whenever $n \in \omega \setminus 2$ and "α is regular and Σ_1–correct." is Π_1–expressible. Therefore:

"α is regular and Σ_n–correct." is Π_n–expressible whenever $n \in \omega \setminus 1$.

Proof. Note that trcl(X) is $\Delta_1(X)$–definable and "$x = V_\alpha$" is a $\Pi_1(\{x, \alpha\})$–relation since "rk$(x) < \alpha$" is a $\Delta_1(\{x, \alpha\})$–relation. Then for $n \in \omega \setminus 2$ we can express "α is Σ_n–correct." as

$$\begin{aligned} \forall m < \omega, a\Big(\big(\mathrm{rk}(a) < \alpha \wedge \exists x_1 \dots \forall x_n \varphi_\Pi(\mathrm{trcl}(\{a, x_1, \dots, x_n\}), (a, x_1, \dots, x_n), m)\big) \\ \rightarrow \forall w\big(w = V_\alpha \rightarrow \exists x_1 \in w \forall x_2 \in w \dots \varphi_\Pi(w, (a, x_1, \dots, x_n), m)\big)\Big) \end{aligned} \tag{1.18}$$

if n is even and as

$$\begin{aligned} \forall m < \omega, a\Big(\big(\mathrm{rk}(a) < \alpha \wedge \exists x_1 \dots \exists x_n \varphi_\Sigma(\mathrm{trcl}(\{a, x_1, \dots, x_n\}), (a, x_1, \dots, x_n), m)\big) \\ \rightarrow \forall w\big(w = V_\alpha \rightarrow \exists x_1 \in w \forall x_2 \in w \dots \varphi_\Pi(w, (a, x_1, \dots, x_n), m)\big)\Big) \end{aligned} \tag{1.19}$$

if n is odd. Here the dots stand for alternating (blocks of) quantifiers or for the variables bound by them. Note that the formula is at least Π_2 since "$w = V_\alpha$" in the second line is not Σ_1–definable. This in fact proves the first assertion of the lemma since $\underset{\sim}{\Sigma}_n$–downwards–absoluteness implies $\underset{\sim}{\Pi}_n$–upwards–absoluteness which in turn implies $\underset{\sim}{\Sigma}_{n+1}$–upwards–absoluteness.

For the second one we use lemma 1.7 which allows us to characterize regular Σ_1–correct cardinals as inaccessibles. But being regular is Π_1 and being a strong limit is Π_1 too—consider

$$\forall \beta < \alpha, f \in \mathrm{Func}\,\big((\forall x \in \mathrm{dom}(f) : x \subset \beta) \rightarrow \exists \gamma < \alpha : \gamma \notin \mathrm{ran}(f)\big). \tag{1.20}$$

The third assertion now follows immediately from the the other two when one again considers the fact that being regular is Π_1. ⊣

1.10. Corollary. Let $n \in \omega \setminus 1$. Then: There are unboundedly many Σ_n–correct regular cardinals below every regular Σ_{n+1}–correct cardinal.

Proof. Let $n \in \omega \setminus 1$ and κ be regular and Σ_{n+1}–correct. Then obviously κ is regular and Σ_n–correct and hence for every $\alpha < \kappa$ a witness to the truth of

$$\exists \lambda \in \mathrm{Reg} \setminus \alpha : \text{"}\lambda \text{ is } \Sigma_n\text{–correct."} \tag{1.21}$$

which is a $\Sigma_{n+1}(\{\alpha\})$–assertion by lemma 1.9. So if we choose $\alpha < \kappa$ arbitrarily we get $V_\kappa \models$ "$\exists \lambda \in \mathrm{Reg} \setminus \alpha$: "λ is Σ_n–correct.". If λ is a witness to this then

$$V_\lambda \prec_{\Sigma_n} V_\kappa \prec_{\Sigma_{n+1}} V \tag{1.22}$$

so λ is in fact Σ_n–correct and of course it is regular too since every witness to its singularity would have rank at most λ, meaning—its rank would in particular be smaller than κ. ⊣

1.11. Definition. Levy's scheme is the following collection of formulae:

$$\big\{\forall\kappa \in \mathrm{Card}, (\alpha_\beta)_{\beta<\kappa}\big((\text{"}(\alpha_\beta)_{\beta<\kappa} \text{ is a sequence of ordinals."} \wedge \forall\beta < \kappa : \varphi(\alpha_\beta)) \tag{1.23}$$
$$\rightarrow \varphi(\sup_{\beta<\kappa} \alpha_\beta)\big) \rightarrow \exists\kappa \in \mathrm{Reg} : \varphi(\kappa) \big| \varphi \text{ is a formula in the languaga of set theory.}\big\}.$$

Each formula of this collection is claiming that whenever φ defines a club–class this very class has a regular member. This amounts to the statement that Reg is stationary in Ω, i.e. that Ω is Mahlo.

1.12. Lemma. If $n < \omega$ and Levy's scheme holds then there is a stationary proper class of regular Σ_n–correct cardinals.

Proof. Let C^* be any closed unbounded proper class and $n < \omega$. We have to show that C contains a regular Σ_n–correct cardinal—as element. To this end we again consider the Σ_n–satisfaction relation. More precisely we define an $a \subset \omega$ as follows:

$$a := \big\{n < \omega \big| \exists x \forall y \ldots \mathrm{Sat}\left(\mathrm{trcl}(x, y, \ldots), (x, y, \ldots), n\right)\big\} \tag{1.24}$$

In this formula as well as in the following one—which is clearly true—the dots stand for $n-2$ alternating (blocks of) quantifiers.

$$\forall n < \omega \Big(n \in a \leftrightarrow \exists y \forall y \ldots \mathrm{Sat}\left(\mathrm{trcl}(x, y, \ldots), (x, y, \ldots), n\right)\Big) \tag{1.25}$$

It is a well–known fact that true sentences in the language of set theory reflect down to V_α for a closed unbounded proper class of ordinals α. Let C' be such a closed unbounded proper class. Then $C := C' \cap C^*$ is again a closed unbounded proper class. Since Levy's scheme holds, C has a regular member—call it κ. But then κ is clearly an element of C^* and it is also Σ_n–correct since V_κ reflects the statement (1.25). ⊣

1.13. Corollary. If κ is Mahlo then for every $n < \omega$ the set $\{\lambda < \kappa | V_\kappa \models$ "λ is regular and Σ_n–correct."$\}$ is stationary in κ.

Proof. By definition the set of regular λ below a Mahlo cardinal κ is stationary, i.e. every set which is club in κ contains a regular cardinal. This in particular holds for all closed unbounded subsets of κ which are definable with parameters from V_κ. But these are precisely the closed unbounded proper classes from V_κ's point of view. So Levy's scheme holds true in V_κ. An application of lemma 1.12 within V_κ yields the desired result. $\dashv$

This corollary establishs that the regular Σ_n–correct cardinals lie consistencywise below a Mahlo–cardinal.

Many large cardinal notions relativize to inner models, in order for our account of reflecting cardinals to be complete we prove that this is indeed the case for them too.

1.14. Lemma. Every reflecting cardinal is reflecting in L.

Proof. Suppose κ is a reflecting cardinal. Let $\lambda \in \Omega \setminus \kappa$ be a successor cardinal in L, $a \in H^L_\kappa$, φ a formula in the language of set theory and suppose that $H^L_\lambda \models$ "$\varphi(a)$". We have to find a $\mu \in \mathrm{Card}^L \cap \kappa$ such that $H^L_\mu \models$ "$\varphi(a)$".

To this end let ν be a successor cardinal which is large enough so that $H_\nu \models$ "$H^L_\lambda \models \varphi(a)$". Since κ is reflecting, there exists a cardinal $\chi < \kappa$ such that

$$H_\chi \models \text{“}\exists \mu \in \mathrm{Card}^L : H^L_\mu \models \varphi(a)\text{”}. \tag{1.26}$$

Let $\mu \in H_\chi$ be a witness to this. Now "$\mu \in \mathrm{Card}^L : H^L_\mu \models \varphi(a)$" is a Π_1–assertion. This is the case since it can be written as

$$L_\mu \models \text{“}\varphi(a)\text{”} \wedge \forall \alpha < \Omega : L_\alpha \models \text{“}\mu \in \mathrm{Card}\text{”}. \tag{1.27}$$

and the function $\alpha \mapsto L_\alpha$ is Δ_1 by [Je 2], lemma 13.14. So by lemma 1.2 $\mu \in \mathrm{Card}^L \wedge H^L_\mu \models$ "$\varphi(a)$". Obviously $\mu < \kappa$ so we are finished. $\dashv$

1.15. Lemma. If $n < \omega$ and $0^\#$ exists then all uncountable cardinals are Σ_n–correct in L.

Proof. Suppose φ is a formula in the language of set theory, $\kappa \in \text{Card} \setminus \aleph_1$, $a \in L_\kappa$ and $L \models$ "$\varphi(a)$". By applying the reflection theorem within L we get a $\lambda \in \text{Card} \setminus \aleph_2$ such that $L_\lambda \models$ "$\varphi(a)$". But then by [De 2], theorem V.2.12 $L_\kappa \models$ "$\varphi(a)$". $\dashv$

2 Axiom A and properness

In this chapter we are going to introduce two attributes of forcing notions, the one called Axiom A and the attribute of being proper. Axiom A was introduced by James E. Baumgartner in [Ba 1] while properness is an idea of Saharon Shelah. Both attributes share two nice properties. One is that they are both generalizations both of being countably closed and of satisfying the countable chain condition, the other is that each attribute is preserved under iterations with countable support. In both cases the generalizations are made carefully enough in order to preserve $\aleph_1$ in the generic extension—a property shared by the countably closed notions of forcing as well as by those satisfying the countable chain condition.

We start with an abstract discussion of properness.

2.1 Properness

2.1. Definition. Given a notion of forcing $\mathbb{P}$, the *proper* $(\mathbb{P}, p)$*–game* for $p \in \mathbb{P}$ is the game where in move n, Player I plays a maximal antichain A_n, followed by Player II playing countable subsets of the antichains previously played by her opponent $B_0^n \subset A_0, \dots, B_n^n \subset A_n$.
Player II wins iff $\exists q \leqslant p \, \forall n < \omega : \bigcup_{k \in \omega \setminus n} B_n^k$ is predense below q.

2.2. Definition. We call a notion of forcing $\mathbb{P}$ *proper* if and only if for every $p \in \mathbb{P}$, Player II has a winning strategy in the proper $(\mathbb{P}, p)$–game. The class of all proper notions of forcing will be denoted by $\mathcal{P}_{\text{rop}}$ throughout this thesis.

2.3. Remark. Indeed, looking at it from II's point of view, the rules of the game are formulated in a friendly manner. It would make no difference for example, if II in each move was allowed only to play a singleton instead of a countable set, because she could remember all other elements she would like to play and play them at future moves instead—all II needs is a fixed injection $f : \omega^2 \hookrightarrow \omega$. We will use this fact in some proofs.

2.4. Definition. The *proper* $(\mathbb{P},p)^*$*–game* is played as follows: In the n^{th} move, Player I plays ordinal names $\dot{\alpha}_n$ and Player II plays ordinals β_n. Player II wins iff $\exists q \leqslant p\, q \Vdash_{\mathbb{P}}$ "$\forall m < \omega\, \exists n < \omega : \dot{\alpha}_m = \check{\beta}_n$".

2.5. Lemma. Player II has a winning strategy in the proper $(\mathbb{P},p)^*$–game if she has one in the proper $(\mathbb{P},p)$–game.

Proof. Let $\mathbb{P}$ be a partial order, $p \in \mathbb{P}$ and let there be a play of the proper $(\mathbb{P},p)^*$–game. Player I plays ordinal names $\dot{\alpha}_n$. Associate to each $\dot{\alpha}_n$ a maximal antichain A_n with the property that $\forall n < \omega\, \forall a \in A_n\, \exists \beta_n^a < \Omega : a \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n = \check{\beta}_n^a$" and assume Player I would play these instead of the ordinal names. Since Player II has a winning strategy in the proper $(\mathbb{P},p)$–game, she can play countable sets $B_0^n \subset A_0, \ldots, B_n^n \subset A_n$. Let $f : \omega \longleftrightarrow \omega^3$ be a fixed bijection such that $\forall n < \omega : f(n) = \omega^2 \cdot k + \omega \cdot l + m$ with $k + l \leqslant n$ and let $e_n^m : \omega \longleftrightarrow B_n^m$ be enumerations of the sets B_n^m. Suppose $f(n) = \omega^2 \cdot k + \omega \cdot l + m$. Then let Player II play the ordinal $\beta_n := \beta_k^{e_k^{k+l}(m)}$ in move n.

Now playing this way is a winning strategy for II—Proof:

II wins in the proper $(\mathbb{P},p)$–game, so there is a $q \leqslant_{\mathbb{P}} p$ such that $\forall n < \omega : \bigcup_{m\in\omega\setminus n} B_n^m$ is predense below q. This very q witnesses that II wins the proper $(\mathbb{P};p)^*$–game. Let $n < \omega$. In move n, I played $\dot{\alpha}_n$. Now $\bigcup_{m\in\omega\setminus n} B_n^m$ is predense below q. Let $m \in \omega \setminus n$ and $b \in B_n^m$ be such that $b \|_{\mathbb{P}} q$. Let $l := {e_n^m}^{-1}(b)$ and $k := m - n$. Then $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n = \check{\beta}_{f^{-1}(\omega^2\cdot n+\omega\cdot k+l)} = \check{\beta}_n^{e_n^{n+k}(l)}$".

So we have shown that $\forall n < \omega\, \exists m < \omega : q \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n = \check{\beta}_m$". But then clearly $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n = \check{\beta}_m$". ⊣

We are now going to introduce two alternative characterizations of properness.

2.6. Definition. Let us call $\kappa \in \mathrm{Reg}$ *sufficiently large* for $\mathbb{P}$ iff $\kappa > 2^{\overline{\overline{\mathbb{P}}}}$.

The idea behind this terminology is that we can then consider a H_κ and countable elementary submodels $M \prec H_\kappa$ and analyse $\mathbb{P}$ as well as dense sets, predense sets, dense open sets, antichains and filters (in V) in these M, where we may add parameters to the language of H_κ.

2.7. Lemma. Let $\mathbb{P}$ be a partial order, κ be sufficiently large for $\mathbb{P}$ and $M \prec H_\kappa$. The following assertions are equivalent for $q \in \mathbb{P}$:

(1) For all antichains $A \in M$: $A \cap M$ is predense below q.

(2) For all ordinal names $\dot\alpha \in M$: $\forall p \leqslant_{\mathbb{P}} q\, \exists r \leqslant_{\mathbb{P}} p, \beta \in M : r \Vdash_{\mathbb{P}}$ "$\dot\alpha = \check\beta$".

(3) $q \Vdash_{\mathbb{P}}$ "$\Gamma \cap \check M$ is $\check{\mathbb{P}}$-generic over $\check M$.".

Proof. (1) $\Rightarrow$ (2): Let q be such that (1) is fulfilled, let A be a maximal antichain whose elements decide $\dot\alpha$, i.e. $\forall a \in A\, \exists \beta_a : a \Vdash_{\mathbb{P}}$ "$\dot\alpha = \check\beta_a$". Let $p \leqslant_{\mathbb{P}} q$ be arbitrary. Then $\exists a \in A \cap M : a \|_{\mathbb{P}} p$. $a \Vdash_{\mathbb{P}}$ "$\dot\alpha = \check\beta_a$". Let $s \in \mathbb{P}$ be such that $r \leqslant_{\mathbb{P}} a, p$. Then $r \Vdash_{\mathbb{P}}$ "$\dot\alpha = \check\beta_a$". ⊣

(2) $\Rightarrow$ (3): Suppose $q \nVdash_{\mathbb{P}}$ "$\Gamma \cap \check M$ is $\mathbb{P}$-generic over $\check M$.". Take a maximal antichain $A \in M \cap \mathfrak{P}(\mathbb{P})$ and a $p \leqslant_{\mathbb{P}} q$ such that $p \Vdash_{\mathbb{P}}$ "$\check A \cap \Gamma \cap \check M = \emptyset$". Let $\lambda \in \mathrm{Card}^M$ and $e \in M$ such that $M \models$ "$e : \lambda \longleftrightarrow A$ is an enumeration of A.". Then $\dot\gamma := \big\{(\check\delta, a) \big| e(\delta) = a\big\}$ is an ordinal name in M. Suppose $r \leqslant_{\mathbb{P}} p$ and $r \Vdash_{\mathbb{P}}$ "$\dot\gamma = \check\beta$" for a $\beta \in M$. Then $(\check\beta, a) \in \dot\gamma$ for an $a \geqslant_{\mathbb{P}} r$, so $r \Vdash_{\mathbb{P}}$ "$\check a \in \Gamma \cap \check A$". ↯ ⊣

(3) $\Rightarrow$ (1): Suppose (3) holds and A is a maximal antichain in M. Then clearly $q \Vdash_{\mathbb{P}}$ "$\check A \cap \Gamma \supsetneq \emptyset$". But then there is an $r \leqslant_{\mathbb{P}} q$ and an $a \in A$ such that $r \Vdash_{\mathbb{P}}$ "$\check a \in \Gamma$". But then $r \|_{\mathbb{P}} a$, so we can choose an $s \leqslant_{\mathbb{P}} r, a$. Then $s \leqslant_{\mathbb{P}} a, q$, so (1) holds. ⊣

2.8. Definition. A condition q is called $(M, \mathbb{P})$–generic iff one of the assertions above holds.

2.9. Lemma. If $\mathbb{P}$ is a notion of forcing, κ is sufficiently large for $\mathbb{P}$ and there is a club $C \subset [H_\kappa]^{<\omega_1}$ of models $M \prec H_\kappa$ such that

$$\forall M \in C, p \in M \cap \mathbb{P}\, \exists q \leqslant_{\mathbb{P}} p : q \text{ is } (M, \mathbb{P})\text{–generic}, \tag{2.1}$$

then $\mathbb{P}$ is proper.

Proof. Let $p \in \mathbb{P}$ be arbitrary. The following is a winning strategy for II in the proper $(\mathbb{P}, p)$–game. In move 0, choose a $M_0 \in C$ such that $A_0 \in M_0$. In any move $n \in \omega \setminus 1$, choose a $M_n \in C$ such that $M_n \supset M_{n-1} \cup \{A_n\}$. In every move $n < \omega$, play the countable sets $B_0^n := A_0 \cap M_n, \ldots, B_n^n := A_n \cap M_n$. Define $M_\omega := \bigcup_{n<\omega} M_n$. C is club, so $M_\omega \in C$. Let $q \leqslant_{\mathbb{P}} p$ be $(M_\omega, \mathbb{P})$–generic. $\forall n < \omega : \bigcup_{m \in \omega \setminus n} B_n^m = A_n \cap M_\omega$ and $\forall n < \omega : A_n \cap M_\omega$ is predense below q, so $\forall n < \omega : \bigcup_{m \in \omega \setminus n} B_n^m$ is predense below q, so II won. p was arbitrarily chosen, so $\mathbb{P}$ is proper. ⊣

2.10. Lemma. Being proper is a Σ_2–property.

Proof. Let $\mathbb{P}$ be a notion of forcing The following formula describes the existence of a winning strategy in the proper $(\mathbb{P}, p)$–game. Let $\mathcal{A}^m_{\mathbb{P}}$ denote the set of all maximal antichains in $\mathbb{P}$.

$$\begin{aligned}\exists f \in \text{Func} : \text{dom}(f) = (\mathcal{A}^m_{\mathbb{P}})^{<\omega} \wedge \forall (x,y) \in f, (n, A) \in x \exists (n, b) \in y : \, b \in A \\ \wedge \forall (A_k | k < \omega) \in (\mathcal{A}^m_{\mathbb{P}})^{\omega} \exists q \leqslant p \forall r \in \mathbb{P}_q, m < \omega \exists n \in \omega \setminus m : \\ f((A_k|k < n)) = (b_k | k < n) \wedge b_m \|_{\mathbb{P}} q \end{aligned} \tag{2.2}$$

Here all but the quantifiers at the beginning of one of the first two lines are bounded. So the assertion is Σ_2 in the parameters $\mathbb{P}$ and p. Now if the above formula is preceded by "$\forall p \in \mathbb{P}$" we have a formulation of properness. Since the additional quantifier is bounded, being proper is itself a Σ_2–assertion. ⊣

2.11. Lemma. If $\mathbb{P}$ is a notion of forcing, $p \in \mathbb{P}$ is such that Player II has a winning strategy in the proper $(\mathbb{P}, p)^*$–game, κ is sufficiently large for $\mathbb{P}$ and $M \ni \mathbb{P}, p$ is a countable elementary submodel of H_κ then there is an $(M, \mathbb{P})$–generic $q \leqslant_{\mathbb{P}} p$.

Proof. Since $H_\kappa \models$ "$\mathbb{P}$ is proper." and $M \prec H_\kappa$ there exists a $\sigma_p \in M$ such that $M \models$ "σ_p is a winning strategy for Player II in the proper $(\mathbb{P}, p)^*$–game.". Let $e : \omega \hookrightarrow M^{\mathbb{P}}$ be an enumeration of all ordinal names in M and let I play $e(n)$ in move n. Let II in the n^{th} move play the ordinal β_n according to σ_p and let q witness that II wins. So $q \Vdash_{\mathbb{P}}$ "$\forall m < \omega \, \exists n < \omega : \, \dot{\alpha}_m = \check{\beta}_n$". Let $r \leqslant_{\mathbb{P}} q$ and $\dot{\alpha} \in M^{\mathbb{P}}$ be any ordinal name. Since there is an $m < \omega$ such that $\dot{\alpha} = \dot{\alpha}_m$, $q \Vdash_{\mathbb{P}}$ "$\exists n < \omega : \, \dot{\alpha} = \check{\beta}_n$". So one can take an $s \leqslant_{\mathbb{P}} r$ and an $n < \omega$ such that $s \Vdash_{\mathbb{P}}$ "$\dot{\alpha} = \check{\beta}_n$". So $q \leqslant_{\mathbb{P}} p$ is $(M, \mathbb{P})$–generic. ⊣

2.12. Remark. Of course if κ is sufficiently large for $\mathbb{P}$ and additionally whenever $M \ni \mathbb{P}$ is an elementary submodel of H_κ and $p \in \mathbb{P} \cap M$ there is an $(M, \mathbb{P})$–generic condition $q \leqslant_{\mathbb{P}} p$ then of course there is a club $C \subset [H_\kappa]^{<\omega_1}$ of elementary submodels of H_κ such that (2.1) holds—simply the club $\{M \big| M \in [H_\kappa]^{<\omega_1} \wedge M \prec H_\kappa \wedge \mathbb{P} \in M\}$. So we arrive at the following...

2.13. Resumee. The four last lemmata together with the remark showed us the equivalence of the following four statements for a notion of forcing $\mathbb{P}$ thereby yielding up to now four different possible characterizations of properness:

(1) $\mathbb{P}$ is proper.

(2) $\forall p \in \mathbb{P}$: Player II has a winning strategy in the proper $(\mathbb{P}, p)^*$–game.

(3) For every $\kappa \in$ Card sufficiently large for $\mathbb{P}$, every countable elementary submodel $M \ni \mathbb{P}$ of H_κ and every $p \in M \cap \mathbb{P}$ there exists a $q \leqslant_{\mathbb{P}} p$ that is $(M, \mathbb{P})$–generic.

(4) For every for $\mathbb{P}$ sufficiently large $\kappa \in$ Reg there is a club $C \subset [H_\kappa]^{<\omega_1}$ of models $M \prec H_\kappa$ such that

$$\forall M \in C, p \in \mathbb{P} \cap M \, \exists q \leqslant_{\mathbb{P}} p : \ q \text{ is } (M, \mathbb{P})\text{–generic.} \tag{2.3}$$

Proof.

- (1) $\Rightarrow$ (2): By lemma 2.5.
- (2) $\Rightarrow$ (3): By lemma 2.11.
- (3) $\Rightarrow$ (4): By remark 2.12.
- (4) $\Rightarrow$ (1): By lemma 2.9.

$\dashv$

There still is another possiblity to characterize properness which we just state here—we are not going to use it.

2.14. Theorem. Let $\mathbb{P}$ be any notion of forcing. $\mathbb{P}$ is proper if and only if for every cardinal κ each set stationary in $[\kappa]^{<\omega_1}$ remains stationary in the generic extension.

For a proof as well as details regarding clubs and stationary sets in $[\kappa]^{<\omega_1}$ see [Je 2], page 605, pp. 100 respectively.

2.15. Lemma. Suppose $\mathbb{P}$ is a proper notion of forcing. Then forcing with $\mathbb{P}$ does not collapse $\aleph_1$. In fact, if C^* is a countable set of ordinals in the generic extension $M[G]$, there is a set C, countable in the ground model M such that $C^* \subset C$.

Proof. Suppose that $\mathbb{P}$ is proper, $\dot{C}$ is a $\mathbb{P}$–name and $p \in \mathbb{P}$ is a condition such that $p \Vdash_{\mathbb{P}}$ "$\dot{C}$ is countable.". We can choose a $\mathbb{P}$–name $\dot{f}$ which testifies this, i.e.

$$p \Vdash_{\mathbb{P}} \text{“}\dot{f} : \omega \twoheadrightarrow \dot{C}\text{”}. \tag{2.4}$$

Let us confuse $\dot{f}(n)$ with a name for $\dot{f}^G(n)$ where G is $\mathbb{P}$–generic. Furthermore let us choose a $q \leqslant_{\mathbb{P}} p$ arbitrarily. Our goal is to find a countable set of ordinals C and an $r \leqslant_{\mathbb{P}} q$ such that $r \Vdash_{\mathbb{P}}$ "$\dot{C} \subset \check{C}$". If we consider a play of the proper $(\mathbb{P}, q)^*$–game $\mathbb{P}$'s properness implies that Player II has a winning strategy in this game—let us fix one and call it σ_q. Now in the role of Player I we can force our opponent to deliver C and r.

- In move n Player I plays the ordinal name $\dot{\alpha}_n := \dot{f}(n)$.
- Next II plays according to her strategy, i.e. $\beta_n := \sigma_q(\dot{\alpha}_0, \ldots, \dot{\alpha}_n)$.

Let $C := \{\beta_n | n < \omega\}$. II wins this play of the game, so there exists an $r \leqslant_{\mathbb{P}} q$ such that $r \Vdash_{\mathbb{P}}$ "$\forall m < \omega \exists n < \omega : \dot{\alpha}_m = \check{\beta}_n$"—in other words: $r \Vdash_{\mathbb{P}}$ "$\dot{f}``\omega \subset \check{C}$". Since $r \leqslant_{\mathbb{P}} p$ and by (2.4) this implies that $r \Vdash_{\mathbb{P}}$ "$\dot{C} \subset \check{C}$". ⊣

A property of properness most welcome is its preservation under products.

2.16. Theorem. If $\mathbb{P}_\alpha$ is an iterated forcing construction of length α of proper forcing notions with countable support then $\mathbb{P}_\alpha$ is proper.

For a proof see for example [Je 2], pp. 604 or [Ab], pp. 15.

2.2 Axiom A and a property of classes of forcing notions

We are now going to introduce Axiom A and explain the relationship between c.c.c., σ–closed, Axiom A and proper notions of forcing. Moreover we will define a property of classes of forcing notions which seems to be necessary in order to be able to deal with Axiom A and Boolean algebras at the same time.

2.17. Definition. A notion of forcing $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ satisfies *Axiom A* iff there exists a sequence $(\leqslant_n | n < \omega)$ of ever stronger partial orderings on the set P strengthening $\leqslant_{\mathbb{P}}$, that is we have

$$P \times P \supset \leqslant_{\mathbb{P}} = \leqslant^0_{\mathbb{P}} \supset \leqslant^1_{\mathbb{P}} \supset \leqslant^2_{\mathbb{P}} \supset \cdots \supset \leqslant^n_{\mathbb{P}} \supset \leqslant^{n+1}_{\mathbb{P}} \supset \ldots \tag{2.5}$$

such that

- Whenever $(p_n | n < \omega)$ is a sequence of conditions from P such that $\forall n < \omega : p_{n+1} \leqslant^n_{\mathbb{P}} p_n$ there is a $p_\omega \in P$ such that $\forall n < \omega : p_\omega \leqslant^n_{\mathbb{P}} p_n$.

- For any ordinal $\mathbb{P}$-name $\dot{\alpha}$, any condition $p \in P$ and any $n < \omega$ there exists a condition $q \in P$ and a countable set of ordinals B such that $q \leqslant_{\mathbb{P}}^n p$ and $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha} \in \check{B}$".

2.18. Lemma. There are (at least) two possibilities to rephrase the second condition in the definition of Axiom A forcing notions $\mathbb{P} = (\mathbb{P}, \leqslant_{\mathbb{P}})$. Equivalent are:

(1) If $p \in P$, A is an antichain maximal below p and $n < \omega$ then there exists a $q \leqslant_{\mathbb{P}}^n p$ such that $\overline{\overline{\{a | a \in A \wedge a \|_{\mathbb{P}} q\}}} < \aleph_1$.

(2) If $p \in P$, $D \subset P$ is dense below p and $n < \omega$ then there exists a $B \in [D]^{<\omega_1}$ and a $q \leqslant_{\mathbb{P}}^n p$ such that B is predense below q.

(3) If $\dot{\alpha}$ is a $\mathbb{P}$-name, $p \Vdash_{\mathbb{P}}$ "$\dot{\alpha} < \Omega$" and $n < \omega$ then there exists a $q \leqslant_{\mathbb{P}}^n p$ and a countable set of ordinals C such that $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha} \in \check{C}$".

Proof. We assume throughout that $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ is a notion of forcing and $P \times P \supset \leqslant_{\mathbb{P}} = \leqslant_{\mathbb{P}}^0 \supset \leqslant_{\mathbb{P}}^1 \supset \dots$ is a sequence for which the first condition in definition of Axiom A forcing notions holds true.

(1) $\Rightarrow$ (2): Let p, n and D be as above. Choose an antichain $A \subset D$ which is maximal under all antichains which are subsets of D. Then of course it is also maximal below p since if it was not and b was a witness for this, i.e. $\forall a \in A\colon a \perp_{\mathbb{P}} b$ then by density of D one could take a $c \in D$ such that $c \leqslant b$. Then $\forall a \in A\colon a \perp_{\mathbb{P}} c$—$\lightning$.
Let q be as in (1) and define $B := \{a | a \in A \wedge a \|_{\mathbb{P}} q\}$. By (1) B is countable. But B is also predense below q. For if $r \leqslant q$ is arbitrarily chosen since A is maximal below p there is an $a \in A$ such that $a \|_{\mathbb{P}} r$. Then $a \|_{\mathbb{P}} q$ so $a \in B$. $\dashv$ ((1) $\Rightarrow$ (2))

(2) $\Rightarrow$ (3): Let p, n and $\dot{\alpha}$ be given. Let $D := \{q | q \in P \wedge \exists \alpha < \Omega\colon q \Vdash_{\mathbb{P}} \text{"}\dot{\alpha} = \alpha\text{"}\}$. D is dense. Then there exists a $q \leqslant^n p$ and a $B \in [D]^{<\omega_1}$ which is predense below q. Define $C := \{\bigcup\{\alpha | b \Vdash_{\mathbb{P}} \text{"}\dot{\alpha} = \alpha\text{"}\} | b \in B\}$. Then $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha} \in \check{C}$". $\dashv$ ((2) $\Rightarrow$ (3))

(3) $\Rightarrow$ (1): Let p, n and A be as above. Let $\kappa := \overline{\overline{A}}$ and $e : \kappa \longleftrightarrow A$ be an enumeration. Then $p \Vdash_{\mathbb{P}}$ "$e < \check{\kappa}$". By (3) there is a countable set of ordinals C and a $q \leqslant^n p$ such that $q \Vdash_{\mathbb{P}}$ "$e \in \check{C}$". But $\{a | a \in A \wedge a \|_{\mathbb{P}} q\} \subset e``C \in [A]^{<\omega_1}$. $\dashv$ ((3) $\Rightarrow$ (1)) $\dashv$

2.19. Lemma. Every notion of forcing that satisfies the c.c.c. also satisfies Axiom A.

Proof. Let $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ be a c.c.c. forcing notion. Define $\leqslant_{\mathbb{P}}^n := \mathrm{id}_P$ for every $n < \omega$. Then clearly the first condition is fulfilled. But the second one is fulfilled too for if $p \in P$

and an ordinal $\mathbb{P}$-name $\dot\alpha$ are given one can choose an antichain A deciding $\dot\alpha$ which is maximal in below p. $\mathbb{P}$ satisfies the c.c.c. so $\overline{\overline{A}} < \aleph_1$. Let $B \in [\Omega]^{<\omega_1}$ be such that $\forall q \in A\, \exists \beta \in B : q \Vdash_{\mathbb{P}}$ "$\dot\alpha = \check\beta$". Then $\left\{q \leqslant_{\mathbb{P}} p \middle| q \Vdash_{\mathbb{P}} \text{"}\dot\alpha \in \check B\text{"}\right\}$ is dense below p. Hence $p \Vdash_{\mathbb{P}}$ "$\dot\alpha \in \check B$". ⊣

2.20. Lemma. Every countably closed notion of forcing satisfies Axiom A.

Proof. Let $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ be a countably closed notion of forcing. Define $\leqslant^n_{\mathbb{P}} := \leqslant_{\mathbb{P}}$ for every $n < \omega$. Then the first condition is fulfilled by countable closedness. If $p \in P$ and an ordinal $\mathbb{P}$-name $\dot\alpha$ are given one simply chooses a condition $q \leqslant_{\mathbb{P}} p$ which decides $\dot\alpha$. Then of course the second condition is fulfilled. ⊣

Now we call to mind Kunen's notion of dense embedding—see [Ku], Definition VII.7.7.

2.21. Definition. Suppose $\mathbb{P}$, $\mathbb{Q}$ are partially ordered sets. Then $\delta : \mathbb{P} \longrightarrow \mathbb{Q}$ is a *dense embedding* if and only if

(1) $\forall p, q \in \mathbb{P} : \delta(p) \leqslant_{\mathbb{Q}} \delta(q)$,

(2) $\forall p, q \in \mathbb{P} : \delta(p) \perp_{\mathbb{Q}} \delta(q)$,

(3) $\delta``\mathbb{P}$ is dense in $\mathbb{Q}$.

2.22. Definition.

- Let us call a class of forcing notions $\mathcal{C}$ *reasonable* by definition if and only if the following holds:

  Suppose $\mathbb{P}$ is a forcing notion in $\mathcal{C}, \mathbb{Q}$ is any forcing notion, there exist a complete Boolean algebra $\mathbb{B}$ and dense embeddings $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}, \delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$. Then $\mathbb{Q} \in \mathcal{C}$. (2.6)

- Furthermore for a class of forcing notions $\mathcal{C}$ let us define the *reasonable hull* $\mathfrak{rh}(\mathcal{C})$ as follows:

$$\mathfrak{rh}(\mathcal{C}) := \big\{\mathbb{P} \big| \mathbb{P} \text{ is a forcing notion and there exists a forcing notion } \mathbb{Q} \in \mathcal{C}, \text{ a complete Boolean algebra and dense embeddings } \delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}, \delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}\big\}. \tag{2.7}$$

- We call $\mathcal{A}\!\!\mathcal{A}^* := \mathfrak{th}(\mathcal{A}\!\!\mathcal{A})$ the class of forcing notions *satisfying Axiom A**.

[Ku], II.3.3. says that for any poset $\mathbb{P}$ there is at least one complete Boolean algebra in which $\mathbb{P}$ can be densely embedded. The following lemma shows that this complete Boolean algebra is unique up to isomorphism.

2.23. Lemma. Let $\mathbb{P}$ be a poset and $\mathbb{B}, \mathbb{E}$ complete Boolean algebras. If there are dense embeddings $\delta_{\mathbb{B}} : \mathbb{P} \longrightarrow \mathbb{B}^+, \delta_{\mathbb{E}} : \mathbb{P} \longrightarrow \mathbb{E}^+$ then $\mathbb{B}$ and $\mathbb{E}$ are isomorphic.

Proof. Let $\mathbb{P}, \mathbb{B}, \mathbb{E}, \delta_{\mathbb{B}}$ and $\delta_{\mathbb{E}}$ be as above. Then the following is an isomorphism between $\mathbb{B}$ and $\mathbb{E}$.

$$\begin{aligned} \varphi : \mathbb{B} &\simeq \mathbb{E} \\ b &\longmapsto \bigvee \left\{ \delta_{\mathbb{E}}(p) \middle| p \in \mathbb{P} \wedge \delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} b \right\}. \end{aligned} \tag{2.8}$$

(1) $\varphi(\mathbb{0}_{\mathbb{B}}) = \mathbb{0}_{\mathbb{E}}$. This follows easily from $\varphi(\mathbb{0}_{\mathbb{B}}) = \bigvee \emptyset$ and $\bigvee \emptyset = \mathbb{0}_{\mathbb{E}}$.

(2) $\forall a, b \in \mathbb{B} : \varphi(a) \leqslant_{\mathbb{E}} \varphi(b)$. Suppose that $a \leqslant_{\mathbb{B}} b$. Then $\left\{ \delta_{\mathbb{E}}(p) \middle| p \in \mathbb{P} \wedge \delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} a \right\} \subset \left\{ \delta_{\mathbb{E}}(p) \middle| p \in \mathbb{P} \wedge \delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} b \right\}$ and hence $\varphi(a) \leqslant_{\mathbb{E}} \varphi(b)$.

(3) $a \not\leqslant_{\mathbb{B}} b \Rightarrow \varphi(a) \not\leqslant_{\mathbb{E}} \varphi(b)$. Suppose towards a contradiction that $a \not\leqslant_{\mathbb{B}} b$ but $\varphi(a) \leqslant \varphi(b)$. By separativity there exists a $c \in \mathbb{B}$ such that $c \leqslant_{\mathbb{B}} a$ but $c \perp_{\mathbb{B}} b$. Let $p \in \mathbb{P}$ be such that $\delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} c$ then $\delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} a$ but $\delta_{\mathbb{B}}(p) \perp_{\mathbb{B}} b$. We have

$$\forall q \in \mathbb{P}\big(\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} b \rightarrow \delta_{\mathbb{B}}(p) \perp_{\mathbb{B}} \delta_{\mathbb{B}}(q)\big) \tag{2.9}$$
$$\Longrightarrow \forall q \in \mathbb{P}\big(\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} b \rightarrow p \perp_{\mathbb{P}} q\big) \tag{2.10}$$
$$\Longrightarrow \forall q \in \mathbb{P}\big(\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} b \rightarrow \delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \delta_{\mathbb{E}}(q)\big). \tag{2.11}$$

This shows that $\varphi(b) \leqslant_{\mathbb{E}} \neg\delta_{\mathbb{E}}(p)$. But $\delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} a$ implies that $\delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} \varphi(a)$. So $\delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} \neg\delta_{\mathbb{E}}(p)$. But then $\delta_{\mathbb{E}}(p) = \mathbb{0}_{\mathbb{E}}$—$\lightning$

(4) Surjectivity. Let $e \in \mathbb{E}$ be given. Define $b := \bigvee_{\mathbb{B}} \left\{ \delta_{\mathbb{B}}(p) \middle| p \in \mathbb{P} \wedge \delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} e \right\}$. We show $\varphi(b) = e$.

  - Suppose $\varphi(b) \not\leqslant e$. By separativity there is an element in $\mathbb{E}$ at least as strong as $\varphi(b)$ but incompatible with e. Since $\delta_{\mathbb{E}}``\mathbb{P}$ is dense in $\mathbb{E}$ there is also such an element in $\delta_{\mathbb{E}}``\mathbb{P}$. So there is a $p \in \mathbb{P}$ such that $\delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} \varphi(b)$ but $\delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} e$. By definition of φ there is a $q \in \mathbb{P}$ such that $\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{E}} b$ and $\delta_{\mathbb{E}}(q) \|_{\mathbb{E}} \delta_{\mathbb{E}}(p)$. Then $p \|_{\mathbb{P}} q$—let $r \leqslant_{\mathbb{P}} p, q$. In particular we have $\delta_{\mathbb{B}}(r) \leqslant_{\mathbb{B}} b$. Now by definition of b there is an $s \in \mathbb{P}$ such that $\delta_{\mathbb{E}}(s) \leqslant_{\mathbb{E}} e$ and $\delta_{\mathbb{B}}(s) \|_{\mathbb{B}} \delta_{\mathbb{B}}(r)$. Then $s \|_{\mathbb{P}} r$—let $t \leqslant_{\mathbb{P}} r, s$. Then $t \leqslant_{\mathbb{P}} r \leqslant_{\mathbb{P}} p$ so $\delta_{\mathbb{E}}(t) \leqslant_{\mathbb{E}} \delta_{\mathbb{E}}(p)$ hence $\delta_{\mathbb{E}}(t) \perp e$. But also $t \leqslant_{\mathbb{P}} s$ so $\delta_{\mathbb{E}}(t) \leqslant_{\mathbb{E}} \delta_{\mathbb{E}}(s) \leqslant_{\mathbb{E}} e$—$\lightning$

- Suppose $e \not\leqslant \varphi(b)$. As above there is a $p \in \mathbb{P}$ such that $\delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} e$ but $\delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \varphi(b)$. By definition of b the first assertion implies that $\delta_{\mathbb{B}}(p) \leqslant_{\mathbb{B}} b$. By definition of φ the second assertion yields $\forall q \in \mathbb{P}\big(\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} b \rightarrow \delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \delta_{\mathbb{E}}(q)\big)$. So $\delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \delta_{\mathbb{E}}(p)$—$\lightning$

(5) $\forall B \subset \mathbb{B}\colon \bigvee_{b \in B} \varphi(b) = \varphi\big(\bigvee_{b \in B} b\big)$.
In order to prove this let $B \subset \mathbb{B}$ be given.

- Since by (2) $\forall b \in B\colon \varphi(b) \leqslant_{\mathbb{E}} \varphi(\bigvee_{b \in B} b)$ clearly $\bigvee_{b \in B} \varphi(b) \leqslant_{\mathbb{E}} \varphi(\bigvee_{b \in B} b)$.
- Suppose that $\varphi(\bigvee_{b \in B} b) \not\leqslant_{\mathbb{E}} \bigvee_{b \in B} \varphi(b)$. Using separativity of $\leqslant_{\mathbb{E}}$ and density of $\delta_{\mathbb{E}}``\mathbb{P}$ in $\mathbb{E}$ choose a $p \in \mathbb{P}$ such that

$$\delta_{\mathbb{E}}(p) \leqslant_{\mathbb{E}} \varphi\big(\bigvee_{b \in B} b\big), \tag{2.12}$$

$$\text{but } \delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \bigvee_{b \in B} \varphi(b). \tag{2.13}$$

By definition of φ there is a $q \in \mathbb{P}$ such that $\delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} \bigvee_{b \in B} b$ and $\delta_{\mathbb{B}}(q) \|_{\mathbb{B}} \delta_{\mathbb{B}}(p)$. The last assertion implies $p \|_{\mathbb{P}} q$. Let $r \leqslant_{\mathbb{P}} p, q$ witness this. Since $\delta_{\mathbb{B}}(r) \leqslant_{\mathbb{B}} \delta_{\mathbb{B}}(q) \leqslant_{\mathbb{B}} \bigvee_{b \in B} b$ there is a $b \in B$ such that $\delta_{\mathbb{B}}(r) \|_{\mathbb{B}} b$. Let $s \in \mathbb{P}$ be such that $\delta_{\mathbb{B}}(s) \leqslant_{\mathbb{B}} b$ and $\delta_{\mathbb{B}}(s) \|_{\mathbb{B}} \delta_{\mathbb{B}}(r)$. Then $r \|_{\mathbb{P}} s$ so choose a $t \leqslant_{\mathbb{P}} r, s$. $\delta_{\mathbb{B}}(t) \leqslant_{\mathbb{B}} \delta_{\mathbb{B}}(s) \leqslant_{\mathbb{B}} b$ follows so $\delta_{\mathbb{E}}(t) \leqslant_{\mathbb{E}} \varphi(b) \leqslant_{\mathbb{E}} \bigvee_{b \in B} \varphi(b)$ by definition of φ. On the other hand $t \leqslant_{\mathbb{P}} r \leqslant_{\mathbb{P}} p$ so $\delta_{\mathbb{E}}(t) \leqslant_{\mathbb{E}} \delta_{\mathbb{E}}(p) \perp_{\mathbb{E}} \bigvee_{b \in B} \varphi(b)$—$\lightning$

(6) $\varphi(\neg b) = \neg\varphi(b)$. This follows from (1)–(5) and the fact that $\neg b = \bigvee \big\{a \big| \forall c(c \leqslant \mathbb{0} \vee c \not\leqslant a \vee c \not\leqslant b)\big\}$.

(7) $\varphi(\mathbb{1}_{\mathbb{B}}) = \mathbb{1}_{\mathbb{E}}$. This is an easy consequence of (1) and (6).

(8) $\bigwedge_{b \in B} \varphi(b) = \varphi\big(\bigwedge_{b \in B} b\big)$. This follows from (5) and (6).

(9) Injectivity. Follows from (3) and the fact that $\leqslant_{\mathbb{B}}$ is antisymmetric.

$\dashv$

We are denoting this uniquely determined complete Boolean algebra by ro($\mathbb{P}$). The functional relation ro is denoted as here because one attains the algebra by considering all **r**egular **o**pen subsets of $\mathbb{P}$ in the canonical topology on $\mathbb{P}$, i.e. the topology with base clopen sets $U_p := \{q | q \in \mathbb{P} \wedge q \leqslant_{\mathbb{P}} p\}$. An important point here is of course that forcing with ro($\mathbb{P}$) always yields the same generic extension as forcing with $\mathbb{P}$. See for example [Je 2], pp. 204 or [Ku], VII.7.11. .

2.24. Lemma. The classes of c.c.c., the class of forcing notions which preserve stationary subsets of $\aleph_1$ and $\mathcal{P}_{\text{rop}}$ are reasonable.

Proof.

- c.c.c.: Suppose $\mathbb{P}$ and $\mathbb{Q}$ are posets, $\mathbb{B}$ is a complete Boolean algebra and $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$ as well as $\delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$ are dense embeddings. Let A be an uncountable antichain in $\mathbb{Q}$. Then $\delta_{\mathbb{Q}}``A$ is an uncountable antichain in $\mathbb{B}$. Since $\delta_{\mathbb{P}}``\mathbb{P}$ is dense in $\mathbb{B}$ there is an antichain $A' \subset \delta_{\mathbb{P}}``\mathbb{P}$ which is a refinement of $A$. Of course $A'$ is also uncountable. But then by the first condition in the definition of the notion of dense embeddings, $\delta^{-1}``A'$ is an uncountable antichain in $\mathbb{P}$. By contraposition we see that $\mathbb{Q}$ satisfies the c.c.c. if $\mathbb{P}$ does.

- stat.pres.: If $\mathbb{P}$ and $\mathbb{Q}$ are posets, $\mathbb{B}$ is a complete Boolean algebra and $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$, $\delta_{\mathbb{Q}} \longrightarrow \mathbb{B}$ are dense embeddings then by [Ku], theorem 7.11 $\mathbb{P}$ and $\mathbb{B}$ yield the same generic extensions and $\mathbb{Q}$ and $\mathbb{B}$ yield the same generic extensions. So $\mathbb{P}$ and $\mathbb{Q}$ yield the same generic extensions. Since to preserve stationary subsets of $\aleph_1$ is a property defined via the generic extension, the class of forcing notions which preserve stationary subsets of $\aleph_1$ is resonable.

- $\mathcal{P}_{\text{rop}}$: One can argue just as above using theorem 2.14 which allows one to define properness via attributes of the generic extension. Since we did not prove this theorem we nevertheless give a direct proof here.

  Let $\mathbb{P}$ be a proper notion of forcing, $\mathbb{Q}$ any notion of forcing, $\mathbb{B}$ a complete Boolean algebra and $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$, $\delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$ dense embeddings. We want to show that $\mathbb{Q}$ is proper too. Let $q \in \mathbb{Q}$ be given. By density of $\delta_{\mathbb{P}}``\mathbb{P}$ choose a $p \in \mathbb{P}$ such that $\delta_{\mathbb{P}}(p) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q)$. We are now going to use II's winning strategy in the proper $(\mathbb{P}, p)$–game to define a winning strategy for her in the proper $(\mathbb{Q}, q)$–game. In order to prove this strategy to be winning Player I and II will simultaneously play a proper $(\mathbb{P}, p)$–game.

- In move n Player I plays a maximal antichain $A_n \subset \mathbb{Q}$. Then because $\delta_{\mathbb{Q}}$ is a dense embedding $\delta_{\mathbb{Q}}``A_n$ is a maximal antichain in $\mathbb{B}$. Since $\delta_{\mathbb{P}}``\mathbb{P}$ is dense in $\mathbb{B}$ we can choose an antichain $B_n \subset \delta_{\mathbb{P}}``\mathbb{P}$ refining $\delta_{\mathbb{Q}}``A_n$. If c_n is a choice function with domain $D_n := \{\delta_{\mathbb{P}}^{-1}``\{b\} \mid b \in B_n\}$ then $E_n := c_n``D_n$ is a maximal antichain in $\mathbb{P}$ and $\delta_{\mathbb{P}}``E_n = B_n$. This E_n is the antichain played by I in the proper $(\mathbb{P}, p)$–game. Now by her winning strategy in the proper $(\mathbb{P}, p)$–game Player II plays $C_m^n \in [E_m]^{<\omega_1}$ for $m \leqslant n$. Back in the proper $(\mathbb{Q}, q)$–game she plays $F_m^n := \{a \in A_m \mid \exists c \in C_m^n : \delta_{\mathbb{P}}(c) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(a)\}$. For every $m \leqslant n$ since C_m^n is countable and A_m is an antichain this subset of A_m is clearly countable.

Now suppose that these simultaneous plays have been finished. We know that Player II won the proper $(\mathbb{P}, p)$–game, i.e. there exists an $r \leqslant_{\mathbb{P}} p$ such that for all $m < \omega$ $\bigcup_{n\in\omega\setminus m} C_m^n$ is predense below r. Now choose—in the role of Player II—an $s \leqslant_{\mathbb{Q}} q$ such that $\delta_{\mathbb{Q}}(s) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}}(r)$—this can be achieved by first choosing a $u \in \mathbb{Q}$ such that $\delta_{\mathbb{Q}}(u) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}}(r) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}}(p) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q)$. Then u and q are compatible so one can take an $s \leqslant_{\mathbb{Q}} q, u$ which provides what was demanded.

We finally show that this implies that for all $m < \omega$ that $\bigcup_{n\in\omega\setminus m} F_m^n$ is predense below s. To this end let $m < \omega$ and $u \leqslant_{\mathbb{Q}} s$ be arbitrarily chosen. By the argument above take a $t \leqslant_{\mathbb{P}} r$ such that $\delta_{\mathbb{P}}(t) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(u)$. Since $\bigcup_{n\in\omega\setminus m} C_m^n$ is predense below r there is an $n \in \omega \setminus m$ and a $c \in C_m^n$ such that $t \|_{\mathbb{P}} c$. But then $\delta_{\mathbb{P}}(t) \|_{\mathbb{B}} \delta_{\mathbb{P}}(c)$. Since $\delta_{\mathbb{P}}``E_m$ was a refinement of $\delta_{\mathbb{Q}}``A_m$ there is an $a \in A_m$—and hence $\in F_m^n$—such that $\delta_{\mathbb{Q}}(a) \geqslant_{\mathbb{B}} \delta_{\mathbb{P}}(c)$. So $\delta_{\mathbb{Q}}(a) \|_{\mathbb{B}} \delta_{\mathbb{Q}}(u)$ which implies $a \|_{\mathbb{Q}} u$.

$\dashv$

But not all proper forcing notions satisfy Axiom A*. We now give an example of a forcing notion $\mathbb{P} \in \mathcal{P}_{\mathrm{rop}} \setminus \mathcal{A\!A}^*$:

2.25. Example. Consider the forcing notion $\mathbb{P}$ which adds a club subset of $\aleph_1$ with finite conditions. $\mathbb{P} := (P, \leqslant_{\mathbb{P}})$ where

$$P := \{p \in \mathrm{Func} \mid \mathrm{dom}(p) \in [\aleph_1]^{<\omega} \wedge \mathrm{ran}(p) \subset \aleph_1 \wedge \exists f \supset p \colon f \text{ is a normal function.}\} \tag{2.14}$$

and $p \leqslant_{\mathbb{P}} q :\Longleftrightarrow p \supset q$.

In order to prove that this forcing notion is indeed proper, but fails to satisfy Axiom A* we will now introduce another game.

2.26. Definition. Let $\mathbb{P}$ be any notion of forcing and $p \in \mathbb{P}$. The *strengthened proper* $(\mathbb{P}, p)$*–game* is defined as follows: Let $n < \omega$. In move n Player I plays an ordinal $\mathbb{P}$-name $\dot{\alpha}_n$, i.e. a $\mathbb{P}$-name such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\dot{\alpha} < \Omega$" while Player II responds by playing a $B_n \in [\Omega]^{<\omega_1}$. Player II wins if and only if $\exists q \leqslant p\colon q \Vdash_{\mathbb{P}}$ "$\forall n < \omega\colon \dot{\alpha}_n \in \check{B}_n$".

2.27. Lemma.

(1) Let $\mathbb{P} \in \mathcal{AA}^*$. Then for every $p \in \mathbb{P}$ Player II has a winning strategy in the strengthened proper $(\mathbb{P}, p)$–game.

(2) If $\mathbb{P}$ is a notion of forcing, $p \in \mathbb{P}$ and Player II has a winning strategy in the strengthened proper $(\mathbb{P}, p)$–game then she has one in the proper $(\mathbb{P}, p)^*$–game.

Proof.

- Ad (1): Let $\mathbb{P}$ be any forcing satisfying Axiom A*. Then there is a forcing notion $\mathbb{Q} \in \mathcal{AA}$, a complete Boolean algebra $\mathbb{B}$ and dense embeddings $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$ and $\delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$. Now let $p \in \mathbb{P}$ be arbitrarily chosen and choose a $q_0 \in \mathbb{Q}$ such that $\delta_{\mathbb{Q}}(q_0) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}}(p)$—this is possible because $\delta``\mathbb{Q}$ is dense in $\mathbb{B}$. We start a play of the strengthened proper $(\mathbb{P}, p)$–game. The following is a winning strategy for II: Construct a descending sequence of conditions $(q_n | n < \omega)$ in $\mathbb{Q}$ like this:
  - In move n Player I plays an ordinal $\mathbb{P}$-name $\dot{\alpha}_n$.
  - After that in the role of Player II the existence of the dense embeddings $\delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}$, $\delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B}$ allows us to choose a $\mathbb{Q}$-name $\dot{\gamma}_n$ which names the same object as $\dot{\alpha}_n$—this is possible by [Ku], VII.7.12, VII.7.13. By definition of Axiom A forcing we can choose a $q_{n+1} \leqslant^n_{\mathbb{Q}} q_n$ and a $B_n \in [\Omega]^{<\omega_1}$ such that

$$q_{n+1} \Vdash_{\mathbb{Q}} \text{“}\dot{\gamma}_n \in \check{B}_n\text{”}. \tag{2.15}$$

  At the end of the game we have defined a sequence $q_0 \geqslant^0_{\mathbb{Q}} q_1 \geqslant^1_{\mathbb{Q}} \dots$ so by definition of Axiom A forcing there is a $q_\omega \in \mathbb{Q}$ such that

$$\forall n < \omega\colon q_\omega \leqslant^n_{\mathbb{Q}} q_n \text{ and thus} \tag{2.16}$$

$$q_\omega \Vdash_{\mathbb{Q}} \text{“}\forall n < \omega\colon \dot{\gamma}_n \in \check{B}_n\text{”}. \tag{2.17}$$

  $q_\omega \leqslant_{\mathbb{Q}} q_0$ implies $\delta_{\mathbb{Q}}(q_\omega) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q_0)$. Choose an $r \in \mathbb{P}$ such that $\delta_{\mathbb{P}}(r) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q_\omega)$. Since $\delta_{\mathbb{P}}(r) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q_\omega) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q_0) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}}(p)$ we have in particular $\delta_{\mathbb{P}}(r) \|_{\mathbb{B}} \delta_{\mathbb{P}}(p)$. By definition of dense embeddings it follows that $r \|_{\mathbb{P}} p$. Choose a witness for this, i.e. an $s \leqslant_{\mathbb{P}} p, r$. Then $\delta_{\mathbb{P}}(s) \leqslant_{\mathbb{B}} \delta_{\mathbb{Q}}(q_\omega)$ and thus

$$s \Vdash_{\mathbb{P}} \text{“}\forall n < \omega \colon \dot{\alpha}_n \in \check{B}_n\text{”}. \tag{2.18}$$

So obviously playing like this is a winning strategy for Player II.

- Ad (2): Suppose $\mathbb{P}$ is a notion of forcing, $p \in \mathbb{P}$ and Player II has a winning strategy in the strengthened proper $(\mathbb{P}, p)$–game. Let $f : \omega \longleftrightarrow \omega^2$ be any bijection such that $\forall n < \omega \colon f(n) \leqslant \omega \cdot n$. Now let there be a play of the proper $(\mathbb{P}, p)^*$–game.
  In move n Player I plays an ordinal $\mathbb{P}$-name $\dot{\alpha}_n$. Let B_n be the set II would play according to her winning strategy in the strengthened proper $(\mathbb{P}, p)$–game. Choose an enumeration $e_n : \omega \longleftrightarrow B_n$ of this set. Then if $f(n) = \omega \cdot k + m$ II plays the ordinal $\beta_n := e_k(m)$.
  Playing like this is a winning strategy for II. In order to see this let $q \leqslant_{\mathbb{P}} p$ be such that $q \Vdash_{\mathbb{P}}$ “$\forall n < \omega \colon \dot{\alpha}_n \in \check{B}_n$”. This q is our witness. Suppose $q \not\Vdash_{\mathbb{P}}$ “$\forall m < \omega\, \exists n < \omega \colon \dot{\alpha}_m = \check{\beta}_n$”. Then there would be an $r \leqslant_{\mathbb{P}} q$ and an $n < \omega$ such that $r \Vdash_{\mathbb{P}}$ “$\dot{\alpha}_n \notin \bigcup_{m<\omega} \check{\beta}_m$”. But $r \leqslant_{\mathbb{P}} q, q \Vdash_{\mathbb{P}}$ “$\dot{\alpha}_n \in \check{B}_m$” and $B_m \subset \bigcup_{m<\omega} \beta_m$—↯

⊣

2.28. Corollary. $\mathcal{AA}^* \subset \mathcal{P}_{\text{rop}}$.

2.29. Lemma. Let $\mathbb{P}$ be the forcing from example 2.25. $\mathbb{P} \in \mathcal{P}_{\text{rop}} \setminus \mathcal{AA}^*$.

Proof.

- Choose a $p \in \mathbb{P}$ arbitrarily. We will define a winning strategy in the proper $(\mathbb{P}, p)$–game, thereby showing $\mathbb{P}$ to be proper. So let there be a play of the proper $(\mathbb{P}, p)$–game and $\alpha_0 := \max\big(\operatorname{ran}(p)\big)$. Suppose in move n Player I just played a maximal antichain A_n. Then Player II chooses an indecomposable ordinal $\alpha_{n+1} \in \aleph_1 \setminus (\alpha_n + 1)$ such that

$$\forall k \leqslant n, \beta < \alpha_{n+1} \exists \gamma < \alpha_{n+1} \forall p \subset \beta \times \beta \exists q \in A_k \cap \mathfrak{P}(\gamma \times \gamma) \colon p \|_{\mathbb{P}} q. \tag{2.19}$$

  Then she plays the sets $B_k^n := \big\{p \in A_k \big| p \subset \alpha_{n+1} \times \alpha_{n+1}\big\}$ for $k \leqslant n$. Obviously these sets are countable.

Now we want to see that this is possible at all. We are going to define an ascending sequence of ordinals $(\eta_m | m < \omega)$ as follows:

- $\eta_0 := \min\left\{\zeta < \aleph_1 \middle| \omega^\zeta \geqslant \alpha_n\right\}$,
- $\eta_{m+1} := \max_{k \leqslant n}\Big(\min\left\{\zeta \in \aleph_1 \setminus (\eta_m + 1) \middle| \forall p \in \omega^{\eta_m} \times \omega^{\eta_m} \exists q \in A_k \cap \mathfrak{P}(\omega^\zeta \times \omega^\zeta) : p \|_{\mathbb{P}} q\right\}\Big)$.

Set $\alpha_{n+1} := \omega^{\sup_{m<\omega} \eta_m}$. Clearly α_{n+1} is an indecomposable ordinal smaller than $\aleph_1$. In order to see that (2.19) holds let $k \leqslant n$ and $\beta < \alpha_{n+1}$ be given. There is an $m < \omega$ with $\omega^{\eta_m} > \beta$. Then by definition of the η_m we get $\forall p \subset \beta \times \beta \exists q \in A_k \cap \mathfrak{P}(\omega^{\eta_{m+1}} \times \omega^{\eta_{m+1}}) : p \|_{\mathbb{P}} q$. But $\omega^{\eta_{m+1}} < \alpha_{n+1}$.

Now we have to prove that in fact this is a winning strategy for Player II. To this end set $\alpha_\omega := \sup_{n<\omega} \alpha_n$ and $q := p \cup \{(\alpha_\omega, \alpha_\omega)\}$. q is a condition in $\mathbb{P}$. In order to show this let $f : \aleph_1 \longrightarrow \aleph_1$ be a normal function extending p. Then one can define

$$g : \aleph_1 \longrightarrow \aleph_1 \qquad (2.20)$$
$$\beta \longmapsto \begin{cases} f(\beta) & \text{iff } f(\beta) \leqslant \alpha_0 \\ \alpha_0 + \big(\beta - f^{-1}(\alpha_0)\big) & \text{otherwise.} \end{cases}$$

g is a normal function extending q. By definition it is a normal function. From the fact that α_ω is indecomposable we get that $\alpha_\omega - f^{-1}(\alpha_0) = \alpha_\omega$. Again because of α_ω's indecomposablility $g(\alpha_\omega) = \alpha_\omega$ follows.

Choose now any $r \leqslant_{\mathbb{P}} q$ and an $n < \omega$. Set $s := r \cap (\alpha_\omega \times \alpha_\omega)$. Since r is finite we have that $s \subset \alpha_m \times \alpha_m$ for some $m \in \omega \setminus n$ already. To be even more specific we have $s \subset \beta \times \beta$ for some $\beta < \alpha_m$. By definition of the α_k it follows that there is a $\gamma < \alpha_m$ and a $t \in A_n \cap \mathfrak{P}(\gamma \times \gamma)$ such that $s \|_{\mathbb{P}} t$. Now $r \cup t \leqslant_{\mathbb{P}} r, t$ witnesses that $\bigcup_{k \in \omega \setminus n} B_n^k$ is predense below q since $t \in B_n^{m-1}$ and $r \cup t \in \mathbb{P}$. Towards showing the latter note that $r \cup t = s \cup t \cup (r \setminus s)$. $s \cup t$ is a condition since $s \|_{\mathbb{P}} t$. So let $f \supset s \cup t, g \supset r$ be normal functions and let $\gamma := \max\big(\mathrm{dom}(s \cup t)\big)$. Then the following is a normal function extending $r \cup t$:

$$h : \aleph_1 \longrightarrow \aleph_1 \qquad (2.21)$$
$$\beta \longmapsto \begin{cases} f(\beta) & \text{iff } \beta < \gamma \\ f(\gamma) + (\beta - \gamma) & \text{iff } \beta \in \alpha_\omega \setminus \gamma \\ g(\beta) & \text{otherwise.} \end{cases}$$

This shows $\mathbb{P} \in \mathcal{P}_{\text{rop}}$.

- Now we will show that Player I has a winning strategy in the strengthened proper $(\mathbb{P}, p)$–game. Together with lemma 2.27 this implies $\mathbb{P} \notin \mathcal{A}^*$.

  So let there be a play of the strengthened proper $(\mathbb{P}, p)$–game where in move n I plays ordinal names $\dot{\alpha}_n$ and II plays countable sets of ordinals B_n . The following is a winning strategy for I:

  - In move 0 she plays $\dot{\alpha}_0 := 0$ and sets $\beta_0 := \omega$.
  - In move $n+1$ she chooses an indecomposable ordinal β_{n+1} from $\aleph_1 \setminus \big(\sup(B_n) + \beta_n + 1\big)$ and plays the following name:

  $$\dot{\alpha}_{n+1} := \big\{\big(\gamma, \{(\beta_{n+1}, \gamma)\}\big) \big| \gamma \in \aleph_1 \setminus \beta_{n+1}\big\}. \tag{2.22}$$

  Choose now an arbitrary $q \leqslant_{\mathbb{P}} p$. We have to show that there are $n < \omega$, $r \leqslant_{\mathbb{P}} q$ such that $r \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n \notin \check{B}_n$". So let $\beta_\omega := \sup_{n<\omega} \beta_n$ and $\gamma := \max\big(\mathrm{dom}(q) \cap \beta_\omega\big)$ where we assume w.l.o.g. that $q \supsetneq \emptyset$. Take an $n < \omega$ such that $\beta_n > \gamma$. Again w.l.o.g. we may assume that $q(\gamma) \leqslant \sup(B_n)$ since otherwise we already have $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n \notin \check{B}_n$". Define $r := q \cup \big\{\big(\beta_n, \sup(B_n) + \beta_n\big)\big\}$. In order to see that r is a condition suppose f is a normal function extending q. Then the following is a normal function extending r.

  $$\begin{aligned} g : \aleph_1 &\longrightarrow \aleph_1 \\ \delta &\longmapsto \begin{cases} \sup(B_n) + 1 + \delta & \text{iff } \delta \in (\beta_{n+1} + 1) \setminus (\gamma + 1) \\ f(\delta) & \text{otherwise.} \end{cases} \end{aligned} \tag{2.23}$$

  The important points here are that on every limit ordinal g is defined by the same clause as its immediate predecessors, both clauses define segmentwise continuous functions, $\sup(B_n) > q(\gamma) = f(\gamma)$ and finally β_{n+1} is indecomposable and greater than $\sup(B_n)$ by II's strategy hence $g(\beta_{n+1}+1) = f(\beta_{n+1}+1) \geqslant \beta_{n+1}+1 > \beta_{n+1} = g(\beta_{n+1})$.

  But clearly $r \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n = \sup(\check{B}_n) + \check{\beta}_n$" so in particular $r \Vdash_{\mathbb{P}}$ "$\dot{\alpha}_n \notin \check{B}_n$". This concludes the proof.

$\dashv$

We have seen that the following holds:

$$\begin{array}{ccccc} \text{c.c.c.} & \subsetneq & \mathcal{AA} & \supsetneq & \sigma\text{cl} \\ & & \cap & & \\ & & \mathcal{AA}^* & & \\ & & \not\cap & & \\ & & \mathcal{P}_{\text{rop}} & & \end{array} \tag{2.24}$$

Of course we did not show officially that $\mathcal{AA} \not\subset$ c.c.c. or that $\mathcal{AA} \not\subset \sigma\text{cl}$ but this follows easily from c.c.c., $\sigma\text{cl} \subset \mathcal{AA}$ and the fact that neither c.c.c. $\subset \sigma\text{cl}$ nor $\sigma\text{cl} \subset$ c.c.c..

2.3 Preservation of Axiom A

The goal of this section is the preservation of Axiom A in iterations with countable support. We will however start by stating a fact which was clear from the very beginning of the analysis of the notion of "Axiom A".

2.30. Lemma. If $\mathbb{P}$ is a forcing notion satisfying Axiom A and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "π is a forcing notion satisfying Axiom A." then $\mathbb{P} \star \pi$ is a forcing notion satisfying Axiom A. Moreover if $p_\omega \in \mathbb{P}$ is a witness for the truth of the first condition defining Axiom A forcings for $\mathbb{P}$ then there is a $\mathbb{P}$-name σ_ω such that $(p_\omega, \sigma_\omega)$ is such a witness for $\mathbb{P} \star \pi$.

Proof. We define:

$$(p, \sigma) \leqslant^n_{\mathbb{P}\star\pi} (q, \tau) :\Longleftrightarrow p \leqslant^n_{\mathbb{P}} q \wedge p \Vdash_{\mathbb{P}} \text{“}\sigma \preccurlyeq^n_\pi \tau\text{”}. \tag{2.25}$$

We will now check that this definition yields the sequence of partial orders sought.

- Let $\big((p_n, \sigma_n)|n < \omega\big)$ be a sequence of conditions from $\mathbb{P} \star \pi$ such that

$$\forall n < \omega : \, (p_{n+1}, \sigma_{n+1}) \leqslant^n_{\mathbb{P}\star\pi} (p_n, \sigma_n). \tag{2.26}$$

Let p_ω be such that $\forall n<\omega:\ p_\omega \leqslant^n_{\mathbb{P}} p_n$. Then we obviously have that $\forall n<\omega:$ $p_\omega \Vdash_{\mathbb{P}}$ "$\sigma_{n+1} \preccurlyeq^n_\pi \sigma_n$" and

$$\begin{aligned} p_\omega \Vdash_{\mathbb{P}} \text{“}(\sigma_n|n<\omega) &\text{ is a sequence of length } \omega \text{ of conditions} \\ &\text{from } \pi \text{ such that } \forall n<\omega:\ \sigma_{n+1} \preccurlyeq^n_\pi \sigma_n.\text{”}. \end{aligned} \tag{2.27}$$

Here $(\sigma_n|n<\omega)$ has to be understood as a name for the sequence of the interpretations of the σ_n. Such a name can be easily constructed depending on the particular coding of ordered pairs and sequences employed. Since π was forced to satisfy Axiom A and by the maximal principle we know of the existence of a $\mathbb{P}$-name σ_ω such that $p_\omega \Vdash_{\mathbb{P}}$ "$\forall n<\omega:\ \sigma_\omega \preccurlyeq^n_\pi \sigma_n$". But then

$$\forall n<\omega:\ (p_\omega,\sigma_\omega) \leqslant^n_{\mathbb{P}\star\pi} (p_n,\sigma_n). \tag{2.28}$$

- Let $(p_0,\sigma)\in\mathbb{P}\star\pi, n<\omega$ be arbitrarily chosen and $\dot\alpha$ be any ordinal $\mathbb{P}\star\pi$-name. One can conceive of $\dot\alpha$ as a $\mathbb{P}$-name for an ordinal π-name via the following recursively defined morphism:

$$\begin{aligned} \varphi: V^{\mathbb{P}\star\pi} &\longrightarrow V^{\mathbb{P}} \\ \big\{\big(\eta_i,(q_i,\tau_i)\big)\big|i\in I\big\} &\longmapsto \big\{(\nu_i,q_i)\big|i\in I\big\}. \end{aligned} \tag{2.29}$$

where ν_i is a canonical name for the ordered pair $\big(\varphi(\eta_i)^G,\tau_i^G\big)$ where G is $\mathbb{P}$–generic. Now we argue as follows:
π is forced to satisfy Axiom A so we have

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\exists\tau\preccurlyeq_\pi\sigma, B:\ \tau\Vdash_\pi \text{“}\varphi(\dot\alpha)\in\check B\text{”} \wedge \overline{\overline{B}}<\aleph_1\text{”}. \tag{2.30}$$

By the maximal principle we can choose $\mathbb{P}$–names τ,β such that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\tau\preccurlyeq_\pi\sigma\wedge\tau\Vdash_\pi \text{“}\varphi(\dot\alpha)\in\check\beta\wedge\overline{\overline{\check\beta}}<\aleph_1\text{””}. \tag{2.31}$$

We can also find a $\mathbb{P}$–name η for an enumeration of the set named by β:

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\tau\preccurlyeq_\pi\sigma\wedge\eta \text{ is a function with } \operatorname{dom}(\eta)=\omega\wedge\tau\Vdash_\pi \text{“}\varphi(\dot\alpha)\in\check\eta\text{“}\omega\text{””} \tag{2.32}$$

Now construct by cisfinite induction a descending chain of conditions in $\mathbb{P}$ as follows—let $p_{m+1}\leqslant^{m+n}_{\mathbb{P}} p_m$ and $B_m\in[\Omega]^{<\omega_1}$ be such that $p_{m+1}\Vdash_{\mathbb{P}}$ "$\eta(m)\in B_m$"

for all $m < \omega$. Then let p_ω be such that $\forall m < \omega : p_\omega \leqslant_{\mathbb{P}}^{m+n} p_m$ and $B_\omega := \bigcup_{m<\omega} B_m$. Now from

$$p_\omega \Vdash_{\mathbb{P}} \text{“}\tau \Vdash_\pi \text{“}\varphi(\dot\alpha) \in \check{B}_\omega\text{””} \tag{2.33}$$

it follows that

$$(p_\omega, \tau) \Vdash_{\mathbb{P}\star\pi} \text{“}\dot\alpha \in \check{B}_\omega\text{”}. \tag{2.34}$$

⊣

2.31. Corollary. Any finite iteration of Axiom A forcings satisfies Axiom A.

Here some comment is in order. Note that here we employ the definition of iterated forcing of [Ba 1] or [Je 2]—not of [Ku]. Kunen requires the second components of the forcing conditions in two-step-iterations to be elements of $\mathrm{dom}(\pi)$. With this definition one could indeed prove that the iteration of countably closed forcings with Axiom A forcings or the iteration of Axiom A forcings with c.c.c. forcings each satisfy Axiom A. The general proof would at least not be possible like this since one would require the first component to decide the second one within $\mathrm{dom}(\pi)$. Nevertheless we are following Kunen to the extent that we do not require the partial orders to be antisymmetric. Why we do this will become obvious in the proof of the fact that CS–Iterations preserve Axiom A.

The following definition will be utilized to prove that CS–Iterations preserve Axiom A.

2.32. Definition. Let $\mathbb{P}_\alpha = \Big(\big((\mathbb{P}_\beta, \leqslant_\beta, \mathbb{1}_\beta), (\pi_\beta, \preccurlyeq_\beta, \varepsilon_\beta)\big) | \beta < \alpha\Big)$ be an iterated forcing construction with countable support such that

$$\forall \beta < \alpha : \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta} \text{“}\pi_\beta \text{ satisfies Axiom A.”} \tag{2.35}$$

- If $p, q \in \mathbb{P}_\alpha$, $n < \omega$ and $F \in [\alpha]^{<\omega}$ define

$$p \leqslant^{F,n} q :\Longleftrightarrow p \leqslant q \wedge \forall \beta \in F : p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}p(\beta) \preccurlyeq_\beta^n q(\beta)\text{”}. \tag{2.36}$$

- Furthermore call $\big((p_n, F_n)|n < \omega\big)$ a *fusion sequence* iff

$$\forall n < \omega : p_{n+1} \leqslant^{F_n,n} p_n, \tag{2.37}$$

$$\forall n < \omega : F_n \subset F_{n+1}, \tag{2.38}$$

$$\bigcup_{n<\omega} F_n = \bigcup_{n<\omega} \operatorname{supt}(p_n). \tag{2.39}$$

The following, due to Baumgartner is known as the *fusion lemma*:

2.33. Lemma. If $\mathbb{P}_\alpha = \Big(\big((\mathbb{P}_\beta, \leqslant_\beta, \mathbb{1}_\beta), (\pi_\beta, \preccurlyeq_\beta, \varepsilon_\beta)\big)|\beta < \alpha\Big)$ is an iterated forcing construction with countable support and $\big((p_n, F_n)|n < \omega)\big)$ is a fusion sequence then there exists a $p_\omega \in \mathbb{P}_\alpha$ such that

$$\forall n < \omega : \; p_\omega \leqslant^{F_n,n} p_n. \tag{2.40}$$

Proof. We wil define p_ω by defining $p_\omega(\beta)$ by induction on β.

- Suppose $\beta = \gamma + 1$ for some $\gamma < \alpha$.

$$\text{If } \gamma \notin \bigcup_{n<\omega} \operatorname{supt}(p_n) \text{ let } p_\omega(\gamma) := \varepsilon_\gamma. \tag{2.41}$$

Otherwise the final condition in the definition of fusion sequences tells us that $\exists n < \omega : \; \gamma \in F_n$. So one can define $n := \min \big\{n \big| n < \omega \wedge \gamma \in F_n\big\}$ and thus a sequence $(\nu_n|n < \omega)$ of $\mathbb{P}_\gamma$-names by setting $\nu_m := p_{m+n}(\gamma)$ for all $m < \omega$. Then

$$p_\omega \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\nu_0 \succcurlyeq^0 \nu_1 \succcurlyeq^1 \nu_2 \succcurlyeq^2 \ldots\text{”} \tag{2.42}$$

since $\forall n < \omega : \; p_\omega \restriction \gamma \leqslant p_n \restriction \gamma$ by the inductive hypothesis and $p_{m+n+1} \Vdash_{\mathbb{P}_\gamma}$ “$\nu_{m+1} \preccurlyeq \nu_m$” for $m < \omega$ by the first condition in the definition of fusion sequences.

So since $\mathbb{1}_\gamma \Vdash_{\mathbb{P}_\gamma}$ “π_γ satisfies Axiom A.” and by the maximal principle there exists a $\mathbb{P}_\gamma$-name μ such that

$$p_\omega \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \text{“}\forall m < \omega : \; \mu \preccurlyeq^m \nu_m\text{”}. \tag{2.43}$$

Let $p(\gamma) := \mu$.

- Suppose $\beta \in \alpha \cap \mathrm{Lim}$. Define

$$p \restriction \beta := \bigcup_{\gamma<\beta} (p \restriction \gamma). \tag{2.44}$$

p_ω provides what was demanded. $\dashv$

2.34. Lemma. Suppose $p \in \mathbb{P}_\alpha, n < \omega, F \in [\alpha]^{<\omega}$ and $\dot{\alpha}$ is a $\mathbb{P}$-name such that $p \Vdash_{\mathbb{P}}$ "$\dot{\alpha} < \Omega$". Then there exists a $q \leqslant^{F,n} p$ and a countable set of ordinals C such that $q \Vdash_{\mathbb{P}}$ "$\dot{\alpha} \in \Omega$".

Proof. Let $\alpha < \Omega, F \in [\alpha]^{<\omega}, n < \omega, p \in \mathbb{P}_\alpha$ and a $\mathbb{P}_\alpha$-name $\dot{\alpha}$ be given such that $p \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\alpha} < \Omega$". We will prove the lemma by induction on α.

- $\alpha = \beta + 1$. In this case we can interpret $\dot{\alpha}$ as a $\mathbb{P}_\beta$-name for a π_β-name. Doing this one gets the following:

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}p(\beta) \Vdash_{\pi_\beta} \text{“}\dot{\alpha} < \Omega\text{””}. \tag{2.45}$$

Now one can use the fact that $\mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "π_β satisfies Axiom A.". This yields

$$p \restriction \beta \Vdash_{\mathbb{P}_\beta} \text{“}\exists q \preccurlyeq^n_\beta p(\beta), C \in [\Omega]^{<\omega_1} : q \Vdash_{\pi_\beta} \text{“}\dot{\alpha} \in C\text{””}. \tag{2.46}$$

By the maximal principle there are $\mathbb{P}_\beta$-names η, ζ such that $p \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$\eta \preccurlyeq^n_\beta p(\beta) \wedge \zeta \in \mathrm{Func} \wedge \mathrm{dom}(\zeta) = \omega \wedge \eta \Vdash_{\pi_\beta}$ "$\exists m < \omega : \dot{\alpha} = \zeta(m)$"". Now we can use the inductive hypothesis to define by a second induction—this time on $m < \omega$—a fusion sequence $\big((q_m, F_m) | m < \omega\big)$ together with a sequence $(C_m | m < \omega)$ such that $\forall m < \omega \colon C_m \in [\Omega]^{<\omega_1}$. In what follows, by $\zeta(m)$ we mean a $\mathbb{P}_\beta$-name for the m^{th} value of the interpretation of ζ. Suppose $m < \omega$ is given.

  - Let $q_m := \begin{cases} p \restriction \beta & \text{iff } m \leqslant n \\ \text{some } q \leqslant^{F_{m-1}, m-1}_\beta q_{m-1} \text{ such that} & \\ q \Vdash_{\mathbb{P}_\beta} \text{“}\zeta(m) \in C_{m-(n+1)}\text{”} & \\ \text{for some } C_{m-(n+1)} \in [\Omega]^{<\omega_1} & \text{otherwise.} \end{cases}$
  - Choose an enumeration $e_m : \omega \longleftrightarrow \mathrm{supt}(q_m)$.
  - Define $F_m := \begin{cases} F \cap \beta & \text{iff } m \leqslant n \\ (F \cap \beta) \cup \bigcup_{k<m} e_k``m & \text{otherwise.} \end{cases}$

Now $\big((q_m, F_m)|m < \omega\big)$ is clearly a fusion sequence. By the fusion lemma there exists a $q \in \mathbb{P}_\beta$ such that $\forall m < \omega\colon q \leqslant_\beta^{F_m,m} q_m$. So in particular $q \leqslant_\beta^{F\cap\beta,n} p$. Also since $\forall m < \omega\colon q_{m+n+1} \Vdash_{\mathbb{P}_\beta}$ "$\zeta(m) \in C_m$" we have that $\forall m < \omega\colon q \Vdash_{\mathbb{P}_\beta}$ "$\zeta(m) \in C_m$". Define $C := \bigcup_{m<\omega} C_m$. By $p\restriction\beta \Vdash_{\mathbb{P}_\beta}$ "$\eta \Vdash_{\pi_\beta}$ "$\exists m < \omega\colon \dot{\alpha} = \zeta(m)$"" we get:

$$q \Vdash_{\mathbb{P}_\beta} \text{“}\eta \Vdash_{\pi_\beta} \text{“}\dot{\alpha} \in \check{C}\text{””} \tag{2.47}$$

and so in the normal interpretation of $\dot{\alpha}$:

$$q^\frown(\eta) \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\alpha} \in \check{C}\text{”}. \tag{2.48}$$

But $q \leqslant_\beta^{F\cap\beta,n} p\restriction\beta$ and $p\restriction\beta \Vdash_{\mathbb{P}_\beta}$ "$\eta \preccurlyeq_\beta^n p(\beta)$". So regardless of whether or not $\beta \in F$ we have that

$$q^\frown(\eta) \leqslant_\alpha^{F,n} p. \tag{2.49}$$

This concludes the proof of the successor step.

- $\alpha \in \mathrm{Lim}$. Set $\beta := \max(F) + 1$. Since $\overline{\overline{F}} < \omega$ and $\alpha \in \mathrm{Lim}$ it follows that $\beta < \alpha$. We now can conceive of $\dot{\alpha}$ as a $\mathbb{P}_\beta$-name for an ordinal $\frac{\mathbb{P}_\alpha}{G_\beta}$-name, then we get

$$p\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}p\restriction(\alpha\setminus\beta) \Vdash_{\frac{\check{\mathbb{P}}_\alpha}{\Gamma_\beta}} \text{“}\dot{\alpha} < \Omega\text{””}. \tag{2.50}$$

Then there is a $\mathbb{P}_\beta$-name η for an element of $\frac{\mathbb{P}_\alpha}{G_\beta}$ and a $\mathbb{P}_\beta$-name ζ for an ordinal such that

$$p\restriction\beta \Vdash_{\mathbb{P}_\beta} \text{“}\eta \preccurlyeq_{\frac{\check{\mathbb{P}}_\alpha}{\Gamma_\beta}} p\restriction(\alpha\setminus\beta) \wedge \eta \Vdash_{\frac{\check{\mathbb{P}}_\alpha}{\Gamma_\beta}} \text{“}\dot{\alpha} = \zeta\text{””}. \tag{2.51}$$

By the inductive hypothesis we know that there is a $q \leqslant_\beta^n p\restriction\beta$ and a countable set of ordinals C such that $q \Vdash_{\mathbb{P}_\beta}$ "$\zeta \in C$". But then

$$q \Vdash_{\mathbb{P}_\beta} \text{“}\eta \Vdash_{\frac{\check{\mathbb{P}}_\alpha}{\Gamma_\beta}} \text{“}\dot{\alpha} \in \check{C}\text{””} \tag{2.52}$$

which in turn means with the normal interpretation of $\dot{\alpha}$

$$q^\frown\eta \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\alpha} \in \check{C}\text{”}. \tag{2.53}$$

Since $F \subset \beta$ we also immediately get $q^\frown(\eta) \leqslant_\alpha^{F,n} p$. This concludes the proof of the limit step and thereby the whole proof.

⊣

2.35. Lemma. Whenever $\mathbb{P}$ is a nontrivial forcing notion and $\nu \supsetneq \emptyset$ is a $\mathbb{P}$-name there are at least $\aleph_0$ $\mathbb{P}$-names $\nu_n, (n < \omega)$ such that

$$\forall n < \omega : \mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\nu = \nu_n\text{”}. \tag{2.54}$$

Proof. By induction on n.

- Choose an arbitrary $(p_0, \sigma) \in \nu$ and let $\nu_0 := \nu$.
- If ν_n, p_n are given let A_n be a nontrivial—meaning A_n shall have at least two elements—antichain in $\mathbb{P}$ maximal below p_n and let $\nu_{n+1} := \nu_n \cup \{(p, \sigma) \big| p \in A_n\} \setminus \{(p_n, \sigma)\}$. Finally choose an arbitrary $p_{n+1} \in A_n$.

Obviously this inductive definition provides the necessary $\mathbb{P}$–names. ⊣

We are now going to define a function which together with the preceding lemmata allows us to define the necessary partial orders to prove the theorem below.

2.36. Definition. By lemma 2.35 for every $\beta < \alpha$ there exists a function

$$F_\beta : \{\nu \big| \nu \text{ is a } \mathbb{P}_\beta\text{-name and } \exists p \in \mathbb{P}_\beta : p \Vdash_{\mathbb{P}_\beta} \text{“}\nu \in \pi_\beta\text{”}\} \longrightarrow \omega \tag{2.55}$$

such that for every $\mathbb{P}_\beta$-name ν with $\exists p \in \mathbb{P}_\beta : p \Vdash_{\mathbb{P}_\beta} \text{“}\nu \in \pi_\beta\text{”}$:

$$F_\beta{}^{\text{“}}\{\mu \big| \mu \text{ is a } \mathbb{P}_\beta\text{-name and } \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta} \text{“}\mu = \nu\text{”}\} = \omega. \tag{2.56}$$

Fix furthermore for every $X \in [\alpha]^\omega$ a bijection

$$G_X : {}^X\omega \longleftrightarrow \{f \big| f \in {}^X\omega \wedge \forall n < \omega : \overline{\overline{f^{-1}\text{“}\{n\}}} < \omega\}. \tag{2.57}$$

Then for every $p \in \mathbb{P}$ we can define a function f_p as follows:

$$f_p : \operatorname{supt}(p) \longrightarrow \omega \tag{2.58}$$
$$\beta \longmapsto$$

$$\begin{cases} 0 & \text{iff } \beta = 0 \\ F_\beta\big(p(\beta)\big) & \text{iff } \beta \in \Omega \setminus 1 \wedge \overline{\overline{\operatorname{supt}(p)}} < \omega \\ G_{\operatorname{supt}(p)}\Big(\big\{(\gamma, F_\gamma\big(p(\gamma)\big) \big| \gamma \in \operatorname{supt}(p)\big\}\Big)(\beta) & \text{iff } \beta \in \Omega \setminus 1 \wedge \overline{\overline{\operatorname{supt}(p)}} \geqslant \omega \end{cases}$$

With the help of these functions we can now define the partial orders $\leqslant_n^\alpha$ for $n < \omega$.

- Let $p \leqslant_\alpha^0 q$ iff $p \leqslant_\alpha q$ for any $p, q \in \mathbb{P}_\alpha$.

- Let for $n \in \omega \setminus 1$
$$p \leqslant_\alpha^n q \text{ iff } \begin{cases} p \leqslant^{f_p^{-1}``(n+1),n} q & , \\ f_p \supset f_q & \text{and} \\ f_p``(n+1) = f_q``(n+1). \end{cases}$$

2.37. Lemma. Whenever $q \in \mathbb{P}_\alpha$ and $f \in {}^{\operatorname{supt}(q)}\omega$ such that $f(0) = 0$ and $\forall n < \omega : \overline{\overline{f^{-1}``n}} < \omega$ hold there exists an $q_f \in \mathbb{P}_\alpha$ such that $f = f_{q_f}$ and

$$\forall F \in [\operatorname{supt}(q)]^{<\omega}, n < \omega, p \in \mathbb{P}_\alpha(q \leqslant^{F,n} p \leftrightarrow q_f \leqslant^{F,n} p). \tag{2.59}$$

Proof. We distinguish two cases:

- $\overline{\overline{\operatorname{supt}(q)}} < \omega$. In this case whenever $\beta \in \operatorname{supt}(q)$ let ν_β be a $\mathbb{P}_\beta$-name such that $\mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) = \nu_\beta$" and $F_\beta(\nu_\beta) = f(\beta)$.

- $\overline{\overline{\operatorname{supt}(q)}} = \omega$. Then if $\beta \in \operatorname{supt}(q)$ let ν_β be a $\mathbb{P}_\beta$-name such that $\mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) = \nu_\beta$" and $G^{-1}_{\operatorname{supt}(q)} \circ f(\beta) = F_\beta(\nu_\beta)$.

In both cases this is possible because of the choice of the functions F_β. Now define
$$q_f(\beta) := \begin{cases} \nu_\beta & \text{iff } \beta \in \operatorname{supt}(q) \\ \mathbb{1}_\beta & \text{otherwise.} \end{cases}$$

A close look at $q_f's$ definition reveals that $\forall\beta < \alpha : \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) = q_f(\beta)$" and $f = f_{q_f}$. So we have for any $F \in [\alpha]^{<\omega}$:

$$q \leqslant^{F,n} p \tag{2.60}$$
$$\Rightarrow q \leqslant p \wedge \forall\beta \in F : \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta} \text{“}q \preccurlyeq^n_\beta p\text{”} \tag{2.61}$$
$$\Rightarrow q_f \leqslant p \wedge \forall\beta \in F : \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta} \text{“}q_f \preccurlyeq^n_\beta p\text{”} \tag{2.62}$$
$$\Rightarrow q_f \leqslant^{F,n} p. \tag{2.63}$$

This of course uses that $q \leqslant p \Rightarrow q_f \leqslant p$. In order to see that this is true assume the contrary, i.e. assume there would exist a $p \in \mathbb{P}_\alpha$ such that $q \leqslant p$ while $q_f \nleqslant p$. Then let $\beta < \alpha$ be minimal such that $q_f \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q_f(\beta) \npreccurlyeq_\beta p(\beta)$". But $\mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "$q_f(\beta) = q(\beta)$" so $q_f \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) \npreccurlyeq_\beta p(\beta)$"—↯ ⊣

2.38. THEOREM. (Piotr Koszmider, 1993) If $\mathbb{P}_\alpha = \Big(((\mathbb{P}_\beta, \leqslant_\beta, \mathbb{1}_\beta), (\pi_\beta, \preccurlyeq_\beta, \varepsilon_\beta)) | \beta < \alpha\Big)$ is an iterated forcing construction with countable support such that

$$\forall\beta < \alpha : \mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta} \text{“}\pi_\beta \text{ satisfies Axiom A.”} \tag{2.64}$$

then the partial orders defined above witness that $\mathbb{P}_\alpha$ satisfies Axiom A too.

The following proof is due to Piotr Koszmider and was first given in [Ko]—interestingly enough—more than nine years after James Baumgartner introduced Axiom A in [Ba 1].

Proof.

- We have that $p \leqslant^0_\alpha q \iff p \leqslant_\alpha q$ for any $p, q \in \mathbb{P}_\alpha$ by definition.
- Suppose that $p \leqslant^{n+1}_\alpha q$ for some $p, q \in \mathbb{P}_\alpha$ and an $n < \omega$.
  - $p \leqslant_\alpha^{f_p^{-1}\text{“}(n+2), n+1} q$ implies $p \leqslant_\alpha^{f_p^{-1}\text{“}(n+1), n} q$ since $f_p^{-1}\text{“}(n+2) \supset f_p^{-1}\text{“}(n+1)$ and $\mathbb{1}_\beta \Vdash_{\mathbb{P}_\beta}$ "π_β satisfies Axiom A.".
  - If $f_p \supset f_q$ then clearly $f_p \supset f_q$.
  - $f_p \supset f_q$ hence $f_p^{-1}\text{“}(n+1) \supset f_q^{-1}\text{“}(n+1)$. In order to see that also $f_p^{-1}\text{“}(n+1) \subset f_q^{-1}\text{“}(n+1)$ let $\beta \in \mathrm{supt}(p)$ be such that $f_p(\beta) \leqslant n$. Since $f_p^{-1}\text{“}(n+2) = f_q^{-1}\text{“}(n+2)$ we see that $f_q(\beta) < n+2$. But if $f_q(\beta) = n+1$ then by $f_p \supset f_q$ we would have $f_p(\beta) = n+1$—↯ So $\beta \in f_q^{-1}\text{“}(n+1)$ after all—hence $f_p^{-1}\text{“}(n+1) = f_q^{-1}\text{“}(n+1)$.

- Suppose that we have been given a sequence $p_0 \geqslant^0 p_1 \geqslant^1 p_2 \geqslant^2 \dots$. Define
$$F_n := \begin{cases} \emptyset & \text{iff } n = 0 \\ f^{-1}_{p_{n+1}}``(n+1) & \text{iff } n > 0. \end{cases}$$
Now one is able to check that $p_1 \geqslant^{F_0,0} p_1 \geqslant^{F_1,1} p_2 \geqslant^{F_2,2} \dots$ is a fusion sequence:
    - $p_{n+1} \leqslant^{F_n,n} p_n$ holds for every $n < \omega$ by definition—note here that $p_1 \geqslant^{F_0,0} p_1$ means the same as $p_1 \geqslant p_1$ which is in turn implied by $p_1 = p_1$.
    - Since $f_{p_{n+1}} \supset f_{p_n}$ for every $n < \omega$ we have that $F_n = f^{-1}_{p_{n+1}}``(n+1) \subset f^{-1}_{p_{n+2}}``(n+2) = F_{n+1}$.
    - Finally one has to prove $\bigcup\limits_{n<\omega} F_n = \bigcup\limits_{n<\omega} \operatorname{supt}(p_n)$.
        * " $\subset$ " : This is simple—if $\beta \in \bigcup_{n<\omega} F_n = \bigcup_{n<\omega} f^{-1}_{p_{n+1}}``(n+1)$ then $\beta \in f^{-1}_{p_{n+1}}``(n+1)$ for some $n < \omega$. But then $\beta \in \operatorname{supt}(p_{n+1})$ for this very n by definition of the f_p.
        * " $\supset$ " : Let $\beta \in \bigcup_{n<\omega} \operatorname{supt}(p_n)$. Then there is an $n < \omega$ such that $\beta \in \operatorname{supt}(p_n)$ so since $\operatorname{ran}(f_{p_n}) = \omega$ there is an $m < \omega$ with the property $\beta \in f^{-1}_{p_n}``(m+1)$. One can assume w.l.o.g. that $m \geqslant n$. That means $(\beta, k) \in f_{p_n}$ for some $k \leqslant m$. But $f_{p_n} \subset \dots \subset f_{p_m} \subset f_{p_{m+1}}$ so $(\beta, k) \in f_{p_{m+1}}$ and hence $\beta \in f^{-1}_{p_{m+1}}``\{k\} \subset f^{-1}_{p_{m+1}}``(m+1) = F_m \subset \bigcup_{k<\omega} F_k$.

By lemma 2.33 we know of the existence of a $q \in \mathbb{P}_\alpha$ such that $\forall n < \omega : q \leqslant^{F_n,n} p_n$
Now define $f := \bigcup\limits_{n\in\omega\setminus 1} f_{p_n}$.

- f is a function since if $(\beta, m), (\beta, n) \in f$ there is a $k < \omega$ such that $(\beta, m), (\beta, n) \in f_{p_k}$ but f_{p_k} is a function so $m = n$. Hence f is a function.
- $f(0) = 0$ since $f_{p_1}(0) = 0$.
- $\forall n < \omega : \overline{\overline{f^{-1}``\{n\}}} < \omega$ since $\forall m, n < \omega : \overline{\overline{f^{-1}_{p_m}``\{n\}}} < \omega$ and $\forall n < \omega : f^{-1}``(n+1) = f^{-1}_{p_n}``(n+1)$. Towards proving the latter statement let $n < \omega$ be arbitrarily chosen. That $f^{-1}``(n+1) \supset f^{-1}_{p_n}``(n+1)$ is trivial so let $\beta \in f^{-1}``(n+1)$. Then there exist $m < \omega, k \leqslant n$ such that $(\beta, k) \in f_{p_m}$ where one can assume w.l.o.g. that $m \geqslant n$. Since $\forall l \in m \setminus n : f^{-1}_{p_{l+1}}``(l+1) = f^{-1}_{p_l}``(l+1)$ it follows by a finite induction that $(\beta, k) \in f_{p_n}$ so $\beta \in f^{-1}_{p_n}``(n+1)$.

Our p_ω sought will be the q_f of lemma 2.37 with respect to the recently defined condition q and the very recently defined function f. So $p_\omega := q_f$. We want to see that $\forall n < \omega : p_\omega \leqslant_n p_n$. Once again we distinguish two cases:

- $n = 0$: We know that $p_\omega \leqslant^{F_1,1} p_1$ so in particular $p_\omega \leqslant p_1$. Also we know that $p_1 \leqslant_0 p_0$, i.e. $p_1 \leqslant p_0$. So by transitivity $p_\omega \leqslant p_0$, i.e. $p_\omega \leqslant_0 p_0$ by definition.
- $n \in \omega \setminus 1$:
  * $f_{p_\omega} \supset f_{p_n}$ since $f_{p_\omega} = f$, $f = \bigcup_{m<\omega} f_{p_m}$ and $\bigcup_{m<\omega} f_{p_m} \supset f_{p_n}$.
  * $f_{p_\omega}^{-1}``(n+1) = f_{q_f}^{-1}``(n+1) = f^{-1}``(n+1) = f_{p_n}^{-1}``(n+1)$ as it was proved earlier.
  * $p_\omega \leqslant^{f^{-1}``(n+1),n} p_n$ since
    · $q \leqslant^{F_n,n} p_n$ so $q \leqslant p_n$ hence $\forall \beta < \alpha : q \restriction \beta \leqslant_\beta p_n \restriction \beta \wedge q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) \preccurlyeq p_n(\beta)$". One can see by an induction on β that this implies $\forall \beta < \alpha : q_f \restriction \beta \leqslant p_n \restriction \beta \wedge q_f \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q_f(\beta) \preccurlyeq p_n(\beta)$". But then $p_\omega = q_f \leqslant p_n$.
    · Let $\beta \in f_{p_\omega}^{-1}``(n+1) = f^{-1}``(n+1) = f_{p_n}^{-1}``(n+1)$. From $q \leqslant^{F_n,n} p_n$ we get $q \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q(\beta) \preccurlyeq p_n(\beta)$". Again by an induction on β one can see that $q_f \restriction \beta \leqslant_\beta q \restriction \beta$. This yields $q_f \restriction \beta \Vdash_{\mathbb{P}_\beta}$ "$q_f(\beta) \preccurlyeq p_n(\beta)$".

• Let $p \in \mathbb{P}_\alpha, n < \omega$ and $\dot{\alpha}$ be a $\mathbb{P}_\alpha$-name such that $p \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\alpha} < \Omega$". By lemma 2.34 there is a $q \leqslant_\alpha^{f_p^{-1}``(n+1),n} p$ and a countable set of ordinals C such that $q \Vdash_{\mathbb{P}_\alpha}$ "$\dot{\alpha} \in \check{C}$". Now let $e : \omega \longleftrightarrow \mathrm{supt}(q)$ be an enumeration and define a function f as follows:

$$\begin{aligned} f : \mathrm{supt}(q) &\longrightarrow \omega \\ \beta &\longmapsto \begin{cases} f_p(\beta) & \text{iff } \beta \in \mathrm{supt}(p) \\ e(\beta) + n + 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{2.65}$$

Since by f's definition $\forall m < \omega$: $\overline{\overline{f^{-1}``\{m\}}} < \omega$ by lemma 2.37 there exists a $q_f \in \mathbb{P}_\alpha$ such that $f_q = f$ and

$$\forall F \in [\mathrm{supt}(q)]^{<\omega}, m < \omega, r \in \mathbb{P}_\alpha (q \leqslant^{F,m} r \leftrightarrow q_f \leqslant^{F,m} r). \tag{2.66}$$

We have $f_{q_f} = f \supset f_p$ and $f^{-1}``(n+1) = f_p^{-1}``(n+1)$, both by definition of f. Because of (2.66) we also get

$$q_f \leqslant^{f^{-1}``(n+1),n} p \Leftrightarrow q \leqslant^{f^{-1}``(n+1),n} p \Leftrightarrow q \leqslant^{f_p^{-1}``(n+1),n} p. \tag{2.67}$$

But the inequality on the right hand side is true so the same holds for the inequality on the left hand side. So $q_f \leqslant_\alpha^n p$. With $F = \emptyset$ or $m = 0$ (2.66) implies that since $q \leqslant_\alpha^{F,m} q$, $q_f \leqslant_\alpha q$. So we can conclude that

$$q_f \Vdash_{\mathbb{P}_\alpha} \text{“}\dot{\alpha} \in \check{C}\text{”}. \tag{2.68}$$

This finishes the proof.

⊣

We also need the fact that countable support iterations of forcings from $\mathcal{AA}^*$ are in $\mathcal{AA}^*$. This seems to be easier to believe than to prove, nevertheless a proof is given below. For the proof we use the fact that the regular open algebra of a poset $\mathbb{P}$ is unique up to isomorphism—lemma 2.23 provides this and that “it” can be characterized as “the” complete Boolean algebra such that $\mathbb{P}$ can be mapped densely into it.

2.39. Corollary. If $\mathbb{P}_\alpha$ is an iterated forcing construction of length α of forcing notions from $\mathcal{AA}^*$ with countable support, then $\mathbb{P}_\alpha \in \mathcal{AA}^*$.

Proof. By induction. We are going to define a sequence of complete Boolean algebras $(\mathbb{B}_\gamma|\gamma < \alpha)$ together with an iterated forcing construction $\Big(\big((\mathbb{Q}_\gamma, \leqslant_{\mathbb{Q}_\gamma}, \mathbb{1}_{\mathbb{Q}_\gamma}), (\psi_\gamma, \preccurlyeq_{\varrho_\gamma}, \varepsilon_{\varrho_\gamma})\big)|\gamma < \alpha\Big)$ by induction on γ. Our inductive hypothesis for γ is that $\forall \xi < \gamma$: $\mathbb{1}_\xi \Vdash_{\mathbb{Q}_\xi}$ “ψ_ξ satisfies Axiom A.”(, hence $\mathbb{Q}_\gamma \in \mathcal{AA}$ by theorem 2.38) and that $\mathbb{P}_\gamma$ and $\mathbb{Q}_\gamma$ can both be mapped densely into $\mathbb{B}_\gamma$.

- Let $\mathbb{Q}_0 := \mathbb{P}_0$.
- $\alpha = \gamma + 1$ for some $\gamma < \Omega$. By the inductive hypothesis we have $\mathbb{Q}_\gamma \in \mathcal{AA}$ and we know that there is a complete Boolean algebra $\mathbb{B}_\gamma$ and dense embeddings $\delta_{\mathbb{P}_\gamma} : \mathbb{P}_\gamma \longrightarrow \mathbb{B}_\gamma$, $\delta_{\mathbb{Q}_\gamma} : \mathbb{Q}_\gamma \longrightarrow \mathbb{B}_\gamma$. These mappings allow us to conceive of $\mathbb{B}_\gamma$ as a superset of (the set of equivalence classes of) $\mathbb{P}_\gamma$, $\mathbb{Q}_\gamma$ and thus of $\mathbb{P}_\gamma$–names and $\mathbb{Q}_\gamma$–names as certain kinds of $\mathbb{B}_\gamma$–names. In this sense because of $\mathbb{P}_\gamma$'s and $\mathbb{Q}_\gamma$'s density in $\mathbb{B}_\gamma$ for every $\mathbb{B}_\gamma$–name there exists a $\mathbb{P}_\gamma$–name and a $\mathbb{Q}_\gamma$–name which both name the same object in every $\mathbb{B}_\gamma$–generic extension. Now $\mathbb{1}_\gamma \Vdash_{\mathbb{P}_\gamma}$ “$\pi_\gamma \in \mathcal{AA}^*$” holds. We fix witnesses for this, i.e. $\mathbb{B}_\gamma$–names $\psi_\gamma, \beta_\gamma, \delta_\pi, \delta_\psi$—where ψ_γ is chosen as a $\mathbb{Q}_\gamma$–name such that

$$\mathbb{1}_{\mathbb{B}_\gamma} \Vdash_{\mathbb{B}_\gamma} \text{“}\psi_\gamma \in \mathcal{AA}, \beta_\gamma \text{ is a complete Boolean algebra and } \delta_{\pi_\gamma} : \pi_\gamma \longrightarrow \beta_\gamma, \delta_{\psi_\gamma} : \psi_\gamma \longrightarrow \beta_\gamma \text{ are dense embeddings.”}. \tag{2.69}$$

Then $\mathbb{Q}_\gamma \star \psi_\gamma \in \mathcal{A\!A}$ by theorem 2.38 and so it suffices to show that $\mathbb{Q}_\gamma \star \psi_\gamma$ can be densely mapped into $\mathbb{B}_{\gamma+1} := \mathrm{ro}(\mathbb{P}_\gamma \star \pi_\gamma)$. Recall that $\mathbb{B}_{\gamma+1}$ consists of all regular cuts in $\mathbb{P}_\gamma \star \pi_\gamma$. Define:

$$\begin{aligned} \delta_{\mathbb{Q}_{\gamma+1}} : \mathbb{Q}_\gamma \star \psi_\gamma &\longrightarrow \mathbb{B}_{\gamma+1} \\ (q,\tau) &\longmapsto \big\{(p,\sigma) \big| (p,\sigma) \in \mathbb{P}_\gamma \star \pi_\gamma \wedge \delta_{\mathbb{P}_\gamma}(p) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{Q}_\gamma}(q) \\ &\qquad \wedge \delta_{\mathbb{P}_\gamma}(p) \Vdash_{\mathbb{B}_\gamma} \text{“}\delta_{\pi_\gamma}(\sigma) \preccurlyeq_{\beta_\gamma} \delta_{\psi_\gamma}(\tau)\text{”}\big\}. \end{aligned} \tag{2.70}$$

Of course the elements of $\mathrm{ran}(\delta_{\mathbb{Q}_{\gamma+1}})$ are cuts.

- They are also regular—Proof:
  Let $(q,\tau) \in \mathbb{Q}_\gamma \star \psi_\gamma$ and $(p,\sigma) \in \mathbb{P}_\gamma \star \pi_\gamma \setminus \delta\big((q,\tau)\big)$. We distinguish two cases:
  * $\delta_{\mathbb{P}_\gamma}(p) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{Q}_\gamma}(q)$. By definition of $\delta_{\mathbb{Q}_{\gamma+1}}$, $\delta_{\mathbb{P}_\gamma}(p) \not\Vdash_{\mathbb{B}_\gamma}$ “$\delta_{\pi_\gamma}(\sigma) \preccurlyeq_{\beta_\gamma} \delta_{\psi_\gamma}(\tau)$”. So one can choose an $r \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{P}_\gamma}(p)$ such that $r \Vdash_{\mathbb{B}_\gamma}$ “$\delta_{\pi_\gamma}(\sigma) \not\preccurlyeq_{\beta_\gamma} \delta_{\psi_\gamma}(\tau)$”. By the maximal principle, since Boolean algebras are separative and because of δ_{π_γ}'s density we can then choose a $\mathbb{P}_\gamma$–name η for an element of π_γ's interpretation such that $r \Vdash_{\mathbb{B}_\gamma}$ “$\eta \preccurlyeq_{\pi_\gamma} \sigma \wedge \delta_{\pi_\gamma}(\eta) \perp_{\beta_\gamma} \delta_{\psi_\gamma}(\tau)$”. Take an $s \in \mathbb{P}_\gamma$ such that $\delta_{\mathbb{P}_\gamma}(s) \leqslant_{\mathbb{B}_\gamma} r$. We have $(s,\eta) \leqslant_{\mathbb{P}_\gamma \star \pi_\gamma} (p,\sigma)$ and $C_{(s,\eta)} \cap \delta_{\mathbb{Q}_{\gamma+1}}\big((q,\tau)\big) = \emptyset$.
  * Not so. Since $\mathbb{B}_\gamma$ is separative one can then choose a $b \in \mathbb{B}_\gamma$ such that $b \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{P}_\gamma}(p)$ and $b \perp_{\mathbb{B}_\gamma} \delta_{\mathbb{Q}_\gamma}(q)$. By density of $\delta_{\mathbb{P}_\gamma}$ one can then find an $r \in \mathbb{P}_\gamma$ with $\delta_{\mathbb{P}_\gamma}(r) \leqslant_{\mathbb{B}_\gamma} b$. So $C_{(r,\varepsilon_\gamma)} \cap \delta_{\mathbb{Q}_{\gamma+1}}\big((q,\tau)\big) = \emptyset$.

  ⊣(The cuts are regular.)
- The mapping defined in (2.70) is a dense embedding—Proof:
  * Clearly for $q,s \in \mathbb{Q}_{\gamma+1}$ $s \leqslant_{\mathbb{Q}} q$ implies $\delta_{\mathbb{Q}_{\gamma+1}}(s) \subset \delta_{\mathbb{Q}_{\gamma+1}}(q)$.
  * Let $(q,\tau) \perp_{\mathbb{Q}_{\gamma+1}} (r,\eta)$ and suppose that $(p,\sigma) \in \delta_{\mathbb{Q}_{\gamma+1}}\big((q,\tau)\big) \cap \delta_{\mathbb{Q}_{\gamma+1}}\big((r,\eta)\big)$. Choose an $s \in \mathbb{Q}_\gamma$ such that $\delta_{\mathbb{Q}_\gamma}(s) \leqslant_{\mathbb{B}} \delta_{\mathbb{P}_\gamma}(p)$. Then by $\delta_{\mathbb{Q}_\gamma}(s) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{P}_\gamma}(p) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{Q}_\gamma}(q)$ we have $\delta_{\mathbb{Q}_\gamma}(q) \|_{\mathbb{B}_\gamma} \delta_{\mathbb{Q}_\gamma}(s)$ and thus $q \|_{\mathbb{Q}_\gamma} s$. Choose a $t \leqslant_{\mathbb{Q}} q,s$. By the same argument $t \|_{\mathbb{Q}_\gamma} r$ for which we choose a witness $u \leqslant_{\mathbb{Q}_\gamma} r,t$. Now $\delta_{\mathbb{Q}_\gamma}(u) \Vdash_{\mathbb{B}_\gamma}$ “$\delta_{\psi_\gamma}(\tau) \|_{\beta_\gamma} \delta_{\psi_\gamma}(\eta)$”. So by contraposition of the second condition defining dense embeddings, $u \Vdash_{\mathbb{Q}_\gamma}$ “$\tau \|_{\mathbb{Q}_\gamma} \eta$”. By the maximal principle choose a $\mathbb{Q}_\gamma$–name witnessing this, i.e. a name χ such that $u \Vdash_{\mathbb{Q}_\gamma}$ “$\chi \preccurlyeq_{\psi_\gamma} \tau, \eta$”. But then clearly $(u,\chi) \leqslant_{\mathbb{Q}_{\gamma+1}} (q,\tau), (r,\eta)$.
  * Let C be any regular nonempty cut in $\mathbb{P}_\gamma \star \pi_\gamma$. Let $(p,\sigma) \in C$ witness C's nonemptyness. Since $\delta_{\mathbb{Q}}$ is dense there exists a $q \in \mathbb{Q}$ such that

$\delta_{\mathbb{Q}}(q) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{P}_\gamma}(p)$. Since $\mathbb{1}_{\mathbb{B}_\gamma} \Vdash_{\mathbb{B}_\gamma}$ "$\delta_\psi : \psi \longrightarrow \beta$ is a dense embedding." by the maximal principle there exists a name τ—which can be chosen as a $\mathbb{Q}$–name—such that $\mathbb{1}_{\mathbb{B}_\gamma} \Vdash_{\mathbb{B}_\gamma}$ "$\delta_\psi(\tau) \leqslant_\beta \delta_{\pi_\gamma}(\sigma)$". The definition of $\delta_{\mathbb{Q}\star\psi}$ in (2.70) yields $C_{(q,\tau)} \subset C$ so we are finished. $\dashv$ ((2.70) is dense.)

- $\alpha \in \mathrm{Lim}$. The inductive hypothesis implies that $\forall\xi < \gamma : \mathbb{1}_{\mathbb{Q}_\xi} \Vdash_{\mathbb{Q}_\xi}$ "ψ_ξ satisfies Axiom A.". Theorem 2.38 tells us that then $\mathbb{Q}_\gamma$ also satisfies Axiom A. We will show that $\mathbb{Q}_\gamma$ can be mapped densely into $\mathbb{B}_\gamma := \mathrm{ro}(\mathbb{P}_\gamma)$. The following function does just this:

$$\begin{aligned} \delta_{\mathbb{Q}_\gamma} : \mathbb{Q}_\gamma &\longrightarrow \mathbb{B}_\gamma \\ q &\longmapsto \big\{p \in \mathbb{P}_\gamma \big| \forall\xi < \gamma \colon \delta_{\mathbb{P}_\xi}(p \restriction \xi) \leqslant_{\mathbb{B}_\xi} \delta_{\mathbb{Q}_\xi}(q \restriction \xi)\big\}. \end{aligned} \tag{2.71}$$

As above now it can be easily seen that the elements of $\mathrm{ran}(\delta_{\mathbb{Q}_\gamma})$ are cuts.

  – They are regular—Proof: Choose $q \in \mathbb{Q}_\gamma$ and $p \in \mathbb{P}_\gamma$ such that $p \notin \delta_{\mathbb{Q}_\gamma}(q)$. Then by definition of $\delta_{\mathbb{Q}_\gamma}$ there exists a $\xi < \gamma$ such that $\delta_{\mathbb{P}_{\xi+1}}\big(p \restriction (\xi+1)\big) \not\leqslant_{\mathbb{B}_{\xi+1}} \delta_{\mathbb{Q}_{\xi+1}}\big(q \restriction (\xi+1)\big)$. Choose the least ξ with this property, then $\delta_{\mathbb{P}_\xi}(p \restriction \xi) \not\Vdash_{\mathbb{B}_\xi}$ "$\delta_{\pi_\xi}\big(p(\xi)\big) \preccurlyeq_{\beta_\xi} \delta_{\psi_\xi}\big(q(\xi)\big)$". Then there is a $b \in \mathbb{B}_\xi$ such that $b \leqslant_{\mathbb{B}_\xi} \delta_{\mathbb{P}_\xi}(p \restriction \xi)$ and $b \Vdash_{\mathbb{B}_\xi}$ "$\delta_{\pi_\xi}\big(p(\xi)\big) \not\preccurlyeq_{\beta_\xi} \delta_{\psi_\xi}\big(q(\xi)\big)$". By separativity of β_ξ's interpretation, under usage of the maximal principle and by density of δ_{π_ξ}'s interpretation there exists a $\mathbb{P}_\xi$–name η such that $b \Vdash_{\mathbb{B}_\xi}$ "$\eta \preccurlyeq_{\pi_\xi} p(\xi) \wedge \delta_{\pi_\xi}(\eta) \perp_{\beta_\xi} \delta_{\psi_\xi}\big(q(\xi)\big)$". Choose an $r \in \mathbb{P}_\xi$ such that $\delta_{\mathbb{P}_\xi}(r) \leqslant_{\mathbb{B}_\xi} b$ and define

$$\begin{aligned} s &\in \mathbb{P}_\gamma \\ \nu &\mapsto \begin{cases} r(\nu) & \text{iff } \nu < \xi, \\ \eta & \text{iff } \nu = \xi, \\ p(\nu) & \text{iff } \nu \in \gamma \setminus \xi. \end{cases} \end{aligned} \tag{2.72}$$

  Then $s \leqslant_{\mathbb{P}_\gamma} p$ and $C_s \cap \delta_\gamma(q) = \emptyset$.

  – $\delta_{\mathbb{Q}_\gamma}$ is a dense embedding—Proof:
    * Clearly if $q, s \in \mathbb{Q}_\gamma$ and $s \leqslant_{\mathbb{Q}} q$ then $\delta_{\mathbb{Q}_\gamma}(s) \subset \delta_{\mathbb{Q}_\gamma}(q)$.
    * Suppose $q, r \in \mathbb{Q}_\gamma$ and $p \in \delta_{\mathbb{Q}_\gamma}(q) \cap \delta_{\mathbb{Q}_\gamma}(r)$. We are inductively going to construct an $s \leqslant_{\mathbb{Q}_\gamma} q, r$. Our inductive hypothesis for $\xi \leqslant \gamma$ is $s \restriction \xi \leqslant_{\mathbb{Q}_\xi} q \restriction \xi, r \restriction \xi \wedge \delta_{\mathbb{Q}_\xi}(s \restriction \xi) \leqslant_{\mathbb{B}_\xi} \delta_{\mathbb{P}_\xi}(p \restriction \xi)$.
      · Suppose $\xi < \gamma$ and $s \restriction \xi$ has been constructed. Clearly $\delta_{\mathbb{P}_\xi}(p \restriction \xi) \Vdash_{\mathbb{B}_\xi}$

"$\delta_{\pi_\xi}\big(p(\xi)\big) \preccurlyeq_{\beta_\xi} \delta_{\psi_\xi}\big(q(\xi)\big), \delta_{\psi_\xi}\big(r(\xi)\big)$". Now we choose a name η for an element of ψ_ξ's interpretation such that $\mathbb{1}_{\mathbb{B}_\xi} \Vdash_{\mathbb{B}_\xi}$ "$\delta_{\psi_\xi}(\eta) \preccurlyeq_{\beta_\xi} \delta_{\pi_\xi}\big(p(\xi)\big)$". Together with the second condition in the definition of dense embeddings these forcing relations imply $\delta_{\mathbb{P}_\xi}(p \restriction \xi) \Vdash_{\mathbb{B}_\xi}$ "$\eta \|_{\psi_\xi} q(\xi)$". Let χ be a name such that $\delta_{\mathbb{P}_\xi}(p\restriction\xi) \Vdash_{\mathbb{B}_\xi}$ "$\chi \preccurlyeq_{\psi_\xi} \eta, q(\xi)$". By the kind of argument just applied $\delta_{\mathbb{P}_\xi}(p\restriction\xi) \Vdash_{\mathbb{B}_\xi}$ "$\chi\|_{\psi_\xi} r(\xi)$" so let finally be $s(\xi)$ a $\mathbb{Q}_\xi$–name such that $\delta_{\mathbb{P}_\xi}(p\restriction\xi) \Vdash_{\mathbb{B}_\xi}$ "$s(\xi) \preccurlyeq_{\psi_\xi} \chi, r(\xi)$". Together with the inductive hypothesis for ξ this proves the inductive hypothesis for $\xi + 1$. The limit step is trivial if one supposes w.l.o.g. that the Boolean algebras form an ascending chain.

∗ Let C be any regular nonempty cut in $\mathbb{P}_\gamma$ and let $p \in C$ witness its nonemptyness. One can define a $q \in \mathbb{Q}_\gamma$ with $\delta_{\mathbb{Q}_\gamma}(q) \subset C$ inductively. The inductive hypothesis for $\xi < \gamma$ is $\delta_{\mathbb{Q}_\xi}(q\restriction\xi) \leqslant_{\mathbb{B}_\xi} \delta_{\mathbb{P}_\xi}(p\restriction\xi)$.

  · Suppose $q \restriction \xi$ has been defined. $\mathbb{1}_{\mathbb{B}_\xi} \Vdash_{\mathbb{B}_\xi}$ "$\delta_{\psi_\xi} : \psi_\xi \longrightarrow \beta_\xi$ is a dense embedding." hence by the maximal principle and the fact that $\delta_{\mathbb{Q}_\xi} : \mathbb{Q}_\xi \longrightarrow \mathbb{B}_\xi$ is a dense embedding one can choose a $\mathbb{Q}_\xi$–name η for an element of ψ_ξ's interpretation such that $\mathbb{1}_{\mathbb{B}_\xi} \Vdash_{\mathbb{B}_\xi}$ "$\delta_{\psi_\xi}(\eta) \preccurlyeq_{\beta_\xi} \delta_{\pi_\xi}\big(p(\xi)\big)$". Let $q(\xi) := \eta$.

This inductive construction provides a $q \in \mathbb{Q}_\gamma$ such that $\delta_{\mathbb{Q}_\gamma}(q) \leqslant_{\mathbb{B}_\gamma} \delta_{\mathbb{P}_\gamma}(p)$. So by definition of $\delta_{\mathbb{Q}_\gamma}$ we have $\delta_{\mathbb{Q}_\gamma}(q) \subset C$.

$\dashv$

3 An equivalent formulation and BAAFA

In chapter 2 we introduced Axiom A* and elucidated the relationship between partial orders and complete Boolean algebras. The reason for this is that Axiom A was designed for arbitrary posets while Bounded Forcing Axioms are commonly stated by reference to Boolean algebras. For the latter fact there are two reasons. On the one hand the Bounded Forcing Axiom for a specific poset might in fact be trivially true if all its maximal antichains are large. On the other hand in the construction of a model for BPFA in the following chapter one needs a Boolean algebra in order to construct a sub–forcing–notion of limited size which contains all antichains in question but nevertheless always contains a witness for the compatibility of two conditions in the larger notion of forcing. Even if one has a dense embedding from some arbitrary poset in this Boolean algebra and each antichain from the family considered lies in its image it is in general unclear how to construct a filter for the family of preimages of the antichains. So in order to define the Bounded Forcing Axioms properly let $\mathfrak{B}$ denote the class of Boolean algebras.

3.1. Definition. If $\kappa, \lambda \in \mathrm{Card}$ and $\mathcal{C}$ is a class of forcing notions, the forcing axiom for $\mathcal{C}$ and κ, bounded by λ—BFA(A, κ, λ) says that whenever $\mathbb{P}$ is a forcing notion in $\mathcal{C}$ and $\mathcal{A}$ is a family of less than κ maximal antichains each of which has size less than λ, there is a filter $G \subset \mathbb{P}$ such that $\forall A \in \mathcal{A} :\ A \cap G \supsetneq \emptyset$. In the following we are going to list some common forcing axioms.

- Martin's Axiom—MA is BFA(c.c.c. $\cap \mathfrak{B}, 2^{\aleph_0}, \Omega$).
- The proper forcing axiom—PFA is BFA($\mathcal{P}_{rop} \cap \mathfrak{B}, \aleph_2, \Omega$).
- Martin's Maximum—MM is BFA($\mathcal{C} \cap \mathfrak{B}, \aleph_2, \Omega$), where $\mathcal{C}$ is the class of all forcing notions that do not destroy the stationarity of any set $S \subset \aleph_1$.
- The bounded proper forcing axiom—BPFA is BFA($\mathcal{P}_{rop} \cap \mathfrak{B}, \aleph_2, \aleph_2$).

- Bounded Martin's maximum—BMM is $\mathrm{BFA}(\mathcal{C} \cap \mathfrak{B}, \aleph_2, \aleph_2)$, where $\mathcal{C}$ is as in Martin's maximum.

3.2. Definition. If n is a natural number, $\mathcal{C}_0$ a class of forcing notions and $\mathcal{C}_1$ any class then $\Sigma_n(\mathcal{C}_0, \mathcal{C}_1)$-absoluteness is the following assertion:
Whenever $\mathbb{P} \in \mathcal{C}_0, a \in \mathcal{C}_1$ and φ is a Σ_n-formula in the language of set theory then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(a)$" iff $\varphi(a)$.

3.3. Lemma. For any $\kappa \in \mathrm{Card} \setminus \omega$ there is a $\Delta_1\big(\{\kappa\}\big)$-definable surjection $\varphi : \mathfrak{P}(\kappa) \twoheadrightarrow H_{\kappa^+}$.

Proof. There is—see the proof of theorem I.10.12 in [Ku]—a $\Delta_1\big(\{\kappa\}\big)$-definable well-order of $\kappa \times \kappa$ of length κ. So one can define a bijection $\psi : \kappa \times \kappa \longleftrightarrow \kappa$ in a $\Delta_1\big(\{\kappa\}\big)$-fashion. This function can be used to code relations on κ as subsets of κ. Furthermore if R is a binary and well-founded relation on κ let $\pi_R : (\kappa, R) \succ (\pi_R``\kappa, \in)$ denote the function collapsing κ onto a transitive set. This function is $\Delta_1\big(\{\kappa, R\}\big)$. We define

$$\begin{aligned} &\varphi : \mathfrak{P}(\kappa) \twoheadrightarrow H_{\kappa^+} \\ &A \longmapsto \begin{cases} \bigcup \big\{x \in \pi_R``\kappa \big| \forall y \in \pi_R``\kappa \colon x \notin y\big\} \text{ iff} & R := \psi^{-1}``A \text{ and} \\ & (\kappa, R) \text{ is well-founded} \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned} \tag{3.1}$$

We essentially have to verify two claims:

- φ is surjective. In order to show this let $a \in H_{\kappa^+}$ be given and set $X := \mathrm{trcl}\big(\{a\}\big)$. We have that $\overline{\overline{X}} = \overline{\overline{\mathrm{trcl}(a)}} + 1 \leqslant \kappa + 1 = \kappa$. Fix any surjection $f : \kappa \twoheadrightarrow X$ and define $\varepsilon := \big\{(\alpha, \beta) \big| f(\alpha) \in f(\beta)\big\}$. By definition ε is a well-founded relation on κ. Moreover it is extensional on any $S \subset \kappa$ such that $f \restriction S$ is one to one. Let π_ε denote the collapsing function of the transitive collapse of (κ, ε), then $\pi_\varepsilon : (\kappa, \varepsilon) \succ (X, \in)$ and $\pi_\varepsilon : (S, \varepsilon) \simeq (X, \in)$ for any $S \subset \kappa$ such that $f \restriction S$ is one to one and S is maximal with respect to this property. We have that

$$\big\{x \in X \big| \forall y \in X \colon x \notin y\big\} = \{a\} \tag{3.2}$$

  Thus by setting $A := \psi``\varepsilon$ we get $\varphi(A) = a$.

- φ is Δ_1. We know that ψ is $\Delta_1\big(\{\kappa\}\big)$ and that the collapsing function is Δ_1 in the parameters κ, R and thus in the parameters κ, A. The only quantifier occuring in

φ's definition is bounded. So it remains to be shown that being well-founded is Δ_1.

- "(κ, R) is well-founded." is Π_1 since it can be written as follows:

$$\forall f : \omega \longrightarrow \kappa \, \exists n < \omega \neg \big(f(n+1) \; R \; f(n)\big) \tag{3.3}$$

- On the other hand it is Σ_1 because the following formulation is possible:

$$\exists f \Big(\mathrm{dom}(f) = \kappa \wedge \mathrm{ran}(f) \subset \Omega \wedge \forall \alpha, \beta < \Omega(\alpha \; R \; \beta \rightarrow f(\alpha) < f(\beta))\Big) \tag{3.4}$$

$\dashv$

3.4. Remark. In many arguments to come, given a family $\mathcal{A}$ of maximal antichains in a Boolean algebra $\mathbb{B}$ we will consider the subalgebra finitely generated by $\mathcal{A}$, that is

$$\mathbb{S} := \left\{ \bigwedge_{B \in C} \bigvee_{b \in B} b \middle| C \in \Big[\big[\{ b \in \mathbb{B} | \exists A \in \mathcal{A} : b \in A \vee \neg b \in A \} \big]^{<\omega} \Big]^{<\omega} \right\} \tag{3.5}$$

We also will sometimes use the fact that given a regular cardinal κ such that $\mathcal{A}$ is a family of less than κ subsets—normally maximal antichains—of a Boolean algebra $\mathbb{B}^*$ each of which has size smaller than κ the Boolean subalgebra $\mathbb{S}^*$ finitely generated by $\mathcal{A}$ has size smaller than κ. Because of this fact it is simple to construct a Boolean algebra $\mathbb{B}$ isomorphic to $\mathbb{B}^*$ by an isomorphism $\psi : \mathbb{B}^* \simeq \mathbb{B}$ such that $\psi``\mathbb{S}^* \in H_\kappa$. In general this finitely generated subalgebra will not be complete. For the remaining part of this thesis we employ the following convention: We are repeatedly going to talk about a notion of forcing extending another one—that is $\mathbb{P} \subset \mathbb{Q}$. where $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ and $\mathbb{Q} = (Q, \leqslant_{\mathbb{Q}})$ may be arbitrary posets, separative posets, Boolean algebras, complete Boolean algebras or other notions of forcing. Important in this respect is only the fact that a partial order is given or canonically definable on the forcing notions and that not only the subset relation holds with respect to P and Q but also $p \leqslant_{\mathbb{P}} q$ if and only if $p \leqslant_{\mathbb{Q}} q$ for all $p, q \in P$. One can for example up to isomorphism conceive of $\mathbb{P}$ and $\mathbb{Q}$ this way if $\mathbb{P}$ is antisymmetric and separative and there exists a dense embedding $\delta : \mathbb{P} \hookrightarrow \mathbb{Q}$ which then has to be one–to–one. But of course in the general case $\mathbb{P} \subset \mathbb{Q}$ by no means implies that $\mathbb{P}$ has to be dense in $\mathbb{Q}$.

The following theorem was proved by Joan Bagaria during the summer of 1995, presented in Oberwolfach in January 1996 and appeared in print in [Bag 1].

3.5. THEOREM. (Joan Bagaria, 2000) Let $\kappa \in \mathrm{Card}$, $\mathrm{cf}(\kappa) > \omega$ and $\mathbb{B}$ be a Boolean algebra. The following expressions are equivalent.

(1) $\Sigma_1\big(\{\mathbb{B}\}, \mathfrak{P}(\kappa)\big)$-absoluteness,

(2) $\Sigma_1(\{\mathbb{B}\}, H_{\kappa^+})$-absoluteness,

(3) $\mathrm{BFA}(\{\mathbb{B}\}, \kappa^+, \kappa^+)$.

Proof.

- (1) $\Rightarrow$ (2): Suppose (1) holds true, $\varphi(\bullet)$ is a Σ_1-formula in the language of set theory, $a \in H_{\kappa^+}$ and $\mathbb{B}$ is a complete Boolean algebra such that $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}}$ "$\varphi(a)$". Let $f : \mathfrak{P}(\kappa) \twoheadrightarrow H_{\kappa^+}$ be the $\Delta_1(\kappa)$-definable surjection from lemma 3.3 and let $b \subset \kappa$ be such that $f(b) = a$. Then $\psi :\equiv \varphi \circ f$ is a Σ_1-formula in the language of set theory and $\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}}$ "$\psi(b)$". By (1) $\psi(b)$ is really true. But then $\varphi(a)$ is true too. $\dashv \big((1) \Rightarrow (2)\big)$

- (2) $\Rightarrow$ (3): Suppose (2) holds and $\mathcal{A}_{\mathbb{B}} = \big\{A_\alpha \big| \alpha < \kappa\big\}$ is a family of at most κ maximal antichains of cardinality at most κ. By remark 3.4 let $\mathbb{D}$ be isomorphic to $\mathbb{B}$ by an isomorphism $\psi : \mathbb{B} \simeq \mathbb{D}$ such that the subalgebra finitely generated by the family $\mathcal{A}_{\mathbb{D}} := \big\{\psi``A_\alpha \big| \alpha < \kappa\big\}$ is in H_κ. Then in particular $\mathcal{A}_{\mathbb{D}}$ is in H_κ. We have that

$$\mathbb{1}_{\mathbb{B}} \Vdash_{\mathbb{B}} \text{“}\check{\psi}``\Gamma \text{ is } \mathcal{A}_{\mathbb{D}}\text{–generic.”}. \tag{3.6}$$

By (2) we can infer

$$\exists H \subset \mathbb{D} : H \text{ is } \mathcal{A}_{\mathbb{D}}\text{–generic.} \tag{3.7}$$

But then for such an H it follows that

$$\psi^{-1}``H \text{ is } \mathcal{A}_{\mathbb{B}}\text{–generic.} \tag{3.8}$$

$\dashv \big((2) \Rightarrow (3)\big)$

- (3) $\Rightarrow$ (1): This is the laborious part. Suppose that BFA $\big(\{\mathbb{B}\},\kappa^+,\kappa^+\big)$ holds true, $a\subset\kappa$, φ is a Σ_0–formula in the language of set theory and

$$\mathbb{1}_{\mathbb{B}}\Vdash_{\mathbb{B}}\text{“}\exists x\varphi(x,a)\text{”}. \tag{3.9}$$

Let G be $\mathbb{B}$–generic. At first we argue in $V[G]$. We know that κ is still a cardinal there because BFA $\big(\{\mathbb{B}\},\kappa^+,\kappa^+\big)$ implies MA_κ. Since $\exists x\varphi(x,a)$ is a Σ_1– and so in particular a Π_2–statement lemma 1.2 implies that $H_{\kappa^+}\models$ “$\exists x\varphi(x,a)$”. Let $M\prec H_{\kappa^+}$ be such that $\kappa\cup\{a\}\subset M$ and $\overline{\overline{M}}=\kappa$. Now let $f:\kappa\longleftrightarrow M$ be an enumeration such that $f(1)=a$ and $f(2\cdot\alpha)=\alpha$ for all $\alpha<\kappa$. Note that f can be chosen to be one–to–one since $\big\{\{\alpha\}\big|\alpha<\kappa\big\}\subset M$ because M is an elementary submodel of H_{κ^+}.

Since κ is still a cardinal and $\in$ is well–founded, for every $\alpha<\kappa$ there exists a function $g_\alpha:\alpha\longrightarrow\kappa$ such that $g_\alpha(\beta)<g_\alpha(\gamma)$ whenever $f(\beta)\in f(\gamma)$ for all $\beta,\gamma<\alpha$—though of course in general not vice versa.

Now back to V. We are going to produce a transitive model containing a which satisfies $\exists x\varphi(x,a)$. Because our generic filter G was arbitrarily chosen there is a $\mathbb{B}$–name Ξ for M and $\mathbb{B}$–names $\dot f,\dot g_0,\dot g_1,\ldots$ for $f,g_0,g_1,\ldots$ such that

$$\begin{gathered}[\![\Xi\models\text{“}\exists x\varphi(x,a)\text{”}]\!],[\![\dot f(1)=\check a]\!],[\![\forall\alpha<\kappa:\dot f(2\cdot\alpha)=\alpha]\!]\text{ and} \\ [\![\forall\beta,\gamma<\alpha\big(\dot f(\beta)\in\dot f(\gamma)\rightarrow\dot g_\alpha(\beta)<\dot g_\alpha(\gamma)\big)]\!]\text{ for }\alpha<\kappa\text{ are all equal to }\mathbb{1}_{\mathbb{B}}.\end{gathered} \tag{3.10}$$

For every formula ψ and every $(\beta_0,\ldots\beta_n)\in{}^{<\omega}\kappa$ consider the maximal antichains

$$\begin{aligned}A^0_{\psi,(\beta_0,\ldots,\beta_n)}:=\Big(&\big\{[\![\alpha\text{ is minimal such that}\Xi\models\text{“}\psi\big(\dot f(\alpha),\dot f(\check\beta_0),\ldots,\dot f(\check\beta_n)\big)\text{”}]\!]\big|\alpha<\kappa\big\} \\ &\cup\big\{[\![\Xi\models\text{“}\nexists x\psi\big(x,\dot f(\check\beta_0),\ldots,\dot f(\check\beta_n)\big)\text{”}]\!]\big\}\Big)\setminus\{\mathbb{0}_{\mathbb{B}}\},\end{aligned} \tag{3.11}$$

together with the maximal antichains:

$$A^1_{\alpha,\beta}:=\big\{[\![\dot g_\alpha(\check\beta)=\check\gamma]\!]\big|\gamma<\kappa\big\}\setminus\{\mathbb{0}_{\mathbb{B}}\},\beta<\alpha<\kappa. \tag{3.12}$$

By BFA $\big(\{\mathbb{B}\},\kappa^+,\kappa^+\big)$ there exists a filter H meeting all these antichains. Now we define a binary relation R on κ as follows:

$$R := \big\{(\alpha,\beta)\big|\alpha,\beta<\kappa \wedge [\![\dot f(\check\alpha)\in\dot f(\check\beta)]\!]\in H\big\}. \tag{3.13}$$

Notice the following:

- R is extensional—Proof:

  Let $\alpha<\beta<\kappa$. $[\![\dot f(\check\alpha)\neq\dot f(\check\beta)]\!]=\mathbb{1}_{\mathbb{B}}$. So by the definition of the antichains in (3.11) there exists a $\gamma<\kappa$ such that $[\![\gamma$ is minimal such that $\dot f(\gamma)\in\dot f(\check\alpha)\bigtriangleup\dot f(\check\beta)]\!]\in H$. But then $\gamma\in\big\{\eta<\kappa\big|\eta\ R\ \alpha\big\}\bigtriangleup\big\{\eta<\kappa\big|\eta\ R\ \beta\big\}$. $\dashv$ (R is extensional)

- R is well–founded—Proof:

  Suppose towards a contradiction that this is wrong and let $(\beta_n|n<\omega)$ be a sequence of ordinals from κ such that $\beta_m\ R\ \beta_n$ whenever $n<m<\omega$. Define $\gamma := (\sup_{n<\omega}\beta_n)+1$. Since $\mathrm{cf}(\kappa)>\omega$ it follows that $\gamma<\kappa$. Then we can consider $\dot g_\gamma$. We have that $[\![\dot f(\check\beta_m)\in\dot f(\check\beta_n)]\!]\in H$ for all $n<m<\omega$. As a consequence $[\![\dot g_\gamma(\check\beta_m)<\dot g_\gamma(\check\beta_n)]\!]\in H$ for all $n<m<\omega$. But H intersects each A^1_{γ,β_n}. So this gives us a sequence of ordinals $(\gamma_n|n<\omega)$ such that $[\![\dot g_\gamma(\check\beta_n)=\gamma_n]\!]\in H$ for all $n<\omega$. By choice of $\dot g_\gamma$ this sequence must be decreasing. This is absurd. $\dashv$ (R is well–founded.)

Now we consider the Mostowski–collapse $(N,\in)$ of the structure (κ,R), let $\pi:(\kappa,R)\simeq(N,\in)$ denote the collapsing morphism.

Claim: $\pi(1)=a$.

Proof of Claim:

Let us define $X := \big\{\alpha\big|\alpha=1\vee\exists\beta<\kappa:\alpha=2\cdot\beta\big\}$. Note that X is closed under R, i.e. if $\beta\in X$ and $\alpha\ R\ \beta$ then $\alpha\in X$. Define

$$\varrho : X\simeq\kappa\cup\{a\} \tag{3.14}$$
$$\alpha\mapsto\begin{cases} a & \text{iff } \alpha=1\\ \beta & \text{iff } 2\cdot\beta=\alpha.\end{cases}$$

Since $\mathbb{1}_{\mathbb{B}}\in H$ and because of (3.10) we have $\alpha\ R\ \beta$ if and only if $\varrho(\alpha)\in\varrho(\beta)$.

So ϱ is an isomorphism from (X,R) onto the transitive structure $(\{a\}\cup\kappa,\in)$. By uniqueness of the collapsing function and since X is closed under R it follows that $\pi\restriction X=\varrho$. $\dashv$(Claim)

By induction on the complexity of a formula ψ in the language of set theory one can see that $(\kappa, R) \models$ "$\psi(\beta_0, \ldots, \beta_n)$"—with the symbol $\in$ interpreted as the relation R—if $[\![\Xi \models \text{"}\psi\big(\dot{f}(\check{\beta}_0), \ldots, \dot{f}(\check{\beta}_n)\big)\text{"}]\!] \in H$. So in the case of φ this means that again with this interpretation $(\kappa, R) \models$ "$\varphi(\alpha, 1)$" for some $\alpha < \kappa$. But then $(N, \in) \models$ "$\varphi\big(\pi(\alpha), a\big)$" and since φ is a Σ_0–formula and N is transitive really $\varphi\big(\pi(\alpha), a\big)$ and in particular $\exists x \varphi(x, a)$. $\dashv$ $\big((3) \Rightarrow (1)\big)$

$\dashv$

3.6. Corollary. Let $\mathcal{C}$ be any class of forcing notions and $\kappa \in$ Card. Then the following are equivalent:

(1) $\Sigma_1\big(\mathcal{C}, \mathfrak{P}(\kappa)\big)$-absoluteness,

(2) $\Sigma_1(\mathcal{C}, H_{\kappa^+})$-absoluteness,

(3) $\Sigma_1\big(\mathfrak{rh}(\mathcal{C}), \mathfrak{P}(\kappa)\big)$-absoluteness,

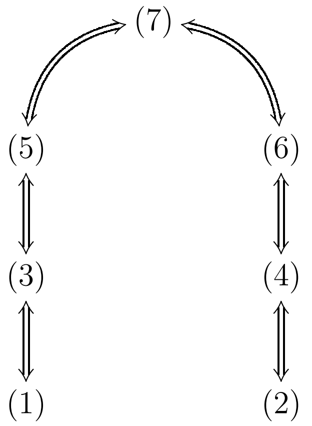
(4) $\Sigma_1\big(\mathfrak{rh}(\mathcal{C}), H_{\kappa^+}\big)$-absoluteness,

(5) $\Sigma_1\big(\mathfrak{rh}(\mathcal{C}) \cap \mathfrak{B}, \mathfrak{P}(\kappa)\big)$-absoluteness,

(6) $\Sigma_1\big(\mathfrak{rh}(\mathcal{C}) \cap \mathfrak{B}, H_{\kappa^+}\big)$-absoluteness,

(7) BFA $\big(\mathfrak{rh}(\mathcal{C}) \cap \mathfrak{B}, \kappa^+, \kappa^+\big)$.

Proof.



The equivalences (5) $\Leftrightarrow$ (7) and (6) $\Leftrightarrow$ (7) follow from theorem 3.5. The other equivalences follow from the definition of the reasonable hull and the fact that for any forcing notion $\mathbb{P}$ forcing with ro($\mathbb{P}$) always yields the same generic extension as forcing with $\mathbb{P}$.

$\dashv$

Now we are going to introduce a new bounded forcing axiom—BAAFA—the bounded Axiom A forcing axiom. The reason for us doing so is that it is a natural as well as proper weakening of BPFA which nevertheless has the same consistency strength. We will **not** define BAAFA $:\Longleftrightarrow$ BFA$(\mathcal{AA} \cap \mathfrak{B}, \aleph_2, \aleph_2)$ though. The reason is simply that it seems to be unknown whether $\mathcal{AA}$ is reasonable. So in order to be able to treat BAAFA just as the other Forcing Axioms we make the following definition:

3.7. Definition.

BAAFA $:\Longleftrightarrow$ BFA$(\mathcal{AA}^* \cap \mathfrak{B}, \aleph_2, \aleph_2)$ is the *Bounded Axiom A forcing Axiom.*

4 The consistency of BPFA

This chapter aims at the definition of an iterated forcing construction by which one attains a generic extension in which BPFA holds true. For this one needs a reflecting cardinal. The iteration will consist of proper notions of forcing and will have this cardinal as length. In order to describe the iteration and to prove that it provides what was demanded we need some technical knowledge.

Two technical lemmata are immediately following in order to eventually prove lemma 4.3. They are concerned with technical aspects of forcing and do not touch the central line of argument which starts with 4.3.

4.1. Lemma. Let $\kappa \in \mathrm{Card}, \mathbb{P} \in H_{\kappa^+}$ and $\lambda \in \mathrm{Reg}$ such that $\lambda > 2^\kappa$. Then for every $\mathbb{P}$–name σ there is a $\mathbb{P}$–name $\tau \in H_\lambda$ such that $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ "$\sigma \in \dot{H}_\lambda \Rightarrow \sigma = \tau$" where $\dot{H}_\lambda$ is a $\mathbb{P}$-name for the set of all sets with transitive closure smaller than λ in the generic extension.

Proof. By induction on the rank of σ. Suppose the claim has been proved for all $\mathbb{P}$–names σ with $\mathrm{rk}(\sigma) < \alpha$. Now let σ be a $\mathbb{P}$–name of rank α and $A \subset \mathbb{P}$ an antichain maximal in the set $\{p \in \mathbb{P} | p \Vdash_\mathbb{P} \text{"}\sigma \in \dot{H}_\lambda\text{"}\}$. We are now going to construct a name τ_p for every $p \in A$. Let $e : \mu \hookrightarrow H_\lambda$ be an enumeration of all $\mathbb{P}$–names from H_λ for the appropriate $\mu \in \mathrm{Card}$. Define τ_p inductively as follows:

- $\tau_0 := \emptyset$.
- If τ_β has been defined set $\tau_{\beta+1} := \tau_\beta \cup \big(\{e(\beta)\} \times A_\beta\big)$ where A_β is an antichain maximal in the set $\{q \leqslant_\mathbb{P} p | q \Vdash_\mathbb{P} \text{"}e(\beta) \in \sigma\text{"} \wedge \forall \gamma < \beta : q \Vdash_\mathbb{P} \text{"}e(\gamma) \neq e(\beta)\text{"}\}$.
- If $\gamma \in \mathrm{Lim}$ and τ_β has been defined for all ordinals $\beta < \gamma$ then set $\tau_\gamma := \bigcup_{\beta<\gamma} \tau_\beta$.

In fact only less than λ of these antichains can be nonempty—$\overline{\overline{\{\beta < \mu | A_\beta \supsetneq \emptyset\}}} < \lambda$—
Proof:
Suppose otherwise. Then there is a $Q \in [\mu]^\lambda$ such that $\forall \beta \in Q : A_\beta \supsetneq \emptyset$. Since there are at most 2^κ subsets of $\mathbb{P}$ there are at most 2^κ nonempty antichains in $\mathbb{P}$. So by the

pigeonhole principle there is an $S \in [Q]^\lambda$ such that $\forall \beta, \gamma \in S : A_\beta = A_\gamma$. Take a $q \in A_{\min(S)}$. Then for any two $\beta, \gamma \in S : q \Vdash_{\mathbb{P}}$ "$e(\beta) \in \sigma \wedge e(\gamma) \in \sigma \wedge e(\beta) \neq e(\gamma)$". Define a name $\zeta := \{e(\beta) | \beta \in S\} \times \{\mathbb{1}_{\mathbb{P}}\}$. Then $q \Vdash_{\mathbb{P}}$ "$\overline{\overline{\zeta}} = \lambda \wedge \zeta \subset \sigma \in H_\lambda$". $\lightning$ $\dashv$
Set $\tau_p := \bigcup_{\beta<\mu} \tau_\beta$. So we now have constructed a name τ for a subset of σ. Since λ is regular, $\tau \in H_\lambda$. We still have to show that $p \Vdash_{\mathbb{P}}$ "$\sigma = \tau$". To this end let $q \leqslant_{\mathbb{P}} p$ and ν a $\mathbb{P}$–name such that $q \Vdash_{\mathbb{P}}$ "$\nu \in \sigma$". We will show that $\{r \in \mathbb{P} | r \Vdash_{\mathbb{P}}$ "$\nu \in \tau$"$\}$ is dense below q. So let $r \leqslant_{\mathbb{P}} q$ be arbitrary and let $\eta \in \mathrm{dom}(\sigma), s \leqslant r$ be such that $s \Vdash_{\mathbb{P}}$ "$\nu = \eta$". $\mathrm{rk}(\eta) < \alpha$ so by the inductive hypothesis there is a $\mathbb{P}$–name $\vartheta \in H_\lambda$ such that $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\vartheta = \eta$". At some point in the construction ϑ is considered. We distinguish two cases:

- There is a $t \in A_{e^{-1}(\vartheta)}$ such that $s \|_{\mathbb{P}} t$ which is witnessed by $u \leqslant_{\mathbb{P}} s, t$. Then $u \Vdash_{\mathbb{P}}$ "$\nu = \eta = \vartheta \wedge \vartheta \in \tau$" so $u \Vdash_{\mathbb{P}}$ "$\nu \in \tau$" and $u \leqslant_{\mathbb{P}} s \leqslant_{\mathbb{P}} r$.

- $s \perp_{\mathbb{P}} t$ for all $t \in A_{e^{-1}(\vartheta)}$. Define $X := \{\beta < e^{-1}(\vartheta) | s \nVdash_{\mathbb{P}}$ "$e(\beta) \neq \vartheta$"$\}$. Because of $s \Vdash_{\mathbb{P}}$ "$\vartheta \in \sigma$" and the maximality condition for $A_{e^{-1}(\vartheta)}$ we have that $X \supsetneq \emptyset$. Let $\beta := \min X$, $\xi := e(\beta)$ and $t \leqslant_{\mathbb{P}} s$ be such that $t \Vdash_{\mathbb{P}}$ "$\xi = \vartheta$". Then A_β is predense below t so there are $u \in \mathbb{P}$, $v \in A_\beta$ such that $u \leqslant_{\mathbb{P}} t, v$. But then $u \Vdash_{\mathbb{P}}$ "$\nu = \eta = \vartheta = \xi \wedge \xi \in \tau$". So $u \Vdash_{\mathbb{P}}$ "$\nu \in \tau$" and $u \leqslant_{\mathbb{P}} t \leqslant_{\mathbb{P}} s \leqslant_{\mathbb{P}} r$.

So we succeeded in finding for each $p \in A$ a name $\tau_p \in H_\lambda$ such that $p \Vdash_{\mathbb{P}}$ "$\sigma = \tau_p$". Set $\tau := \bigcup_{p \in A} \tau_p$. Since each τ_p was in H_λ and $\overline{\overline{A}} \leqslant \overline{\overline{\mathbb{P}}} \leqslant \kappa < 2^\kappa < \lambda$, $\tau \in H_\lambda$.

Now let $q \in \mathbb{P}$ be such that $q \Vdash_{\mathbb{P}}$ "$\sigma \in \dot{H}_\lambda$". We will show that $\{r \leqslant_{\mathbb{P}} q | r \Vdash_{\mathbb{P}}$ "$\sigma = \tau$"$\}$ is dense below q. To this end let $r \leqslant_{\mathbb{P}} q$ be arbitrary and take a $p \in A$ such that $p \|_{\mathbb{P}} r$. Let $s \leqslant_{\mathbb{P}} p, r$ be a witness to this fact. Then clearly $s \Vdash_{\mathbb{P}}$ "$\sigma = \tau_p$". But also $s \Vdash_{\mathbb{P}}$ "$\tau_p = \tau$" since $\forall t \in A \setminus \{p\}, u \in \mathrm{ran}(\tau_t) : u \in \mathbb{P}_t$, A is an antichain and $s \leqslant_{\mathbb{P}} p$. $\dashv$

4.2. Lemma. Let $\kappa \in \mathrm{Card}$, $\mathbb{P} \in H_{\kappa^+}$ a forcing notion and $\lambda \in \mathrm{Reg}$ such that $\lambda > 2^\kappa$. Then for every formula φ and every $a \in H_\lambda$:

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\dot{(H)}_\lambda \models \text{“}\varphi(a)\text{””} \iff H(\lambda) \models \text{“}\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{“}\varphi(a)\text{””}$$

Proof. By induction on the complexity of the formula φ. We define $\mathrm{rk}(\exists x \psi(x)) := \mathrm{rk}(\psi(\bullet)) + 1$, $\mathrm{rk}(\psi \to \tau) := \mathrm{rk}(\psi) + \mathrm{rk}(\tau) + 1$ and $\mathrm{rk}(\bot) := 0$. Then for all formulae φ with $\mathrm{rk}(\varphi) = 0$ $H_\lambda \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi \leftrightarrow \bot$"" and $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\varphi \leftrightarrow \bot$""

- $\varphi \equiv \bot$. This is trivial.

- $\varphi \equiv \psi \rightarrow \tau$ for some formulae ψ, τ.
  Suppose $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\psi \rightarrow \tau$"". We distinguish two cases.
  - $\mathrm{rk}(\tau) > 0$. Let $A \subset \mathbb{P}_p$ be an antichain maximal below p deciding $H_\lambda \models$ "ψ" and $H_\lambda \models$ "τ". $\lambda > 2^\kappa \geqslant \kappa^+ > \overline{\overline{\mathrm{trcl}(A)}}$ hence $A \in H_\lambda$, $H_\lambda \models$ "A is an antichain maximal in below p.". $q \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\neg\psi$"" or $q \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "τ"" for every $q \in A$. By the inductive hypothesis $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "$\neg\psi$"" or $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "τ"" for every $q \in A$. But then $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\psi \rightarrow \tau$"".
  - $\mathrm{rk}(\tau) = 0$ In this case $\tau \equiv \bot$. We prove the statement by contraposition. Suppose $H_\lambda \not\models$ '$p \Vdash_{\mathbb{P}}$ "$\psi \rightarrow \bot$". Then $H_\lambda \models$ "$p \not\Vdash_{\mathbb{P}}$ "$\psi \rightarrow \bot$"". So $H_\lambda \models$ "$\exists q \leqslant_{\mathbb{P}} p : q \Vdash_{\mathbb{P}}$ "ψ"" hence there is a $q \leqslant_{\mathbb{P}} p$ with $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "ψ"". By the inductive hypothesis $q \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "ψ"" so $p \not\Vdash_{\mathbb{P}}$ "$H_\lambda \not\models$ 'ψ" hence $p \not\Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\psi \rightarrow \bot$"".

  Assume that $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\psi \rightarrow \tau$"". Again we distinguish two cases.
  - $\mathrm{rk}(\psi) > 0$. Suppose $p \not\Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\psi \rightarrow \tau$"". Then there is a $q \leqslant_{\mathbb{P}} p$ such that $q \Vdash_{\mathbb{P}}$ "$H_\lambda \not\models$ '$\psi \rightarrow \tau$". But then $q \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "ψ"" and $q \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\neg\tau$"". By the inductive hypothesis $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "ψ"" so $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "τ"" by our assumption. But—again by the inductive hypothesis $H_\lambda \models$ "$q \Vdash_{\mathbb{P}}$ "$\neg\tau$"". $\lightning$
  - $\mathrm{rk}(\psi) = 0$. Then $\psi \equiv \bot$ and φ is equivalent to $\top$ but then this is clearly trivial.

- $\varphi \equiv \exists x \psi(x)$.
  - Suppose $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\exists x \psi(x)$"". Then $p \Vdash_{\mathbb{P}}$ "$\exists x \in H_\lambda : H_\lambda \models$ "$\psi(x)$"". So by the maximal principle there is a $\mathbb{P}$–name σ such that $p \Vdash_{\mathbb{P}}$ "$\sigma \in H_\lambda \wedge H_\lambda \models$ "$\psi(\sigma)$"". By lemma 4.1 we can assume w.l.o.g. that $\sigma \in H_\lambda$. So $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\psi(\sigma)$"" by the inductive hypothesis hence $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\exists x \psi(x)$"" in particular.
  - Assume $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\exists x \psi(x)$"". Then $H_\lambda \models$ "$\exists x : p \Vdash_{\mathbb{P}}$ "$\psi(x)$"" so there is a $\mathbb{P}$–name $\sigma \in H_\lambda$ such that $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\psi(\sigma)$"". By the inductive hypothesis $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\psi(\sigma)$"". So in particular $p \Vdash_{\mathbb{P}}$ "$\exists x \in H_\lambda : H_\lambda \models$ "$\psi(x)$"" but then $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\exists x \psi(x)$"".

At the end we have to prove $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\sigma \in \tau$"" $\Leftrightarrow H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$"" for every $p \in \mathbb{P}$ and $\mathbb{P}$–names $\sigma, \tau \in H_\lambda$. Clearly in this case $p \Vdash_{\mathbb{P}}$ "$H_\lambda \models$ "$\sigma \in \tau$"" $\Leftrightarrow p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$". So it suffices to prove $p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$" $\Leftrightarrow H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$"". This will be done by an induction on $\mathrm{rk}(\tau)$ in which we distinguish two cases:

- $\mathrm{rk}(\tau) = 0$. Then τ is a name for the empty set hence $p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$", $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$"" are both clearly false.

- $\mathrm{rk}(\tau) > 0$. Suppose $p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$". By definition of the forcing relation this means that $D_{\sigma,\tau} := \{q \mid \exists(\zeta, r) \in \tau(q \leqslant_{\mathbb{P}} r \wedge q \Vdash_{\mathbb{P}} \text{"}\sigma = \zeta\text{"})\}$ is dense below p. But then by the inductive hypothesis $D_{\sigma,\tau}^{H_\lambda} = D_{\sigma,\tau}$ and $H_\lambda \models$ "$D_{\sigma,\tau}$ is dense below p.", so by the inductive hypothesis $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$"".

  On the other side assume that $H_\lambda \models$ "$p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$"". Then there is $D_{\sigma,\tau}^{H_\lambda} \in H_\lambda$ and $H_\lambda \models$ "$D_{\sigma,\tau}$ is dense below p.". Again $D_{\sigma,\tau}^{H_\lambda} = D_{\sigma,\tau}$ and $D_{\sigma,\tau}$ is really dense below p. With the inductive hypothesis it follows that $p \Vdash_{\mathbb{P}}$ "$\sigma \in \tau$".

Because H_λ satisfies extensionality we do not need to deal with " = ". ⊣

4.3. Lemma. Let κ be a reflecting cardinal and $\mathbb{P} \in H_\kappa$ a forcing notion. Then $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "κ is reflecting.".

Proof. Let κ and $\mathbb{P}$ be as above. Since reflecting cardinals are in particular inaccessible and hence limit cardinals by the lemmata 1.7 and 1.8 we can fix a $\lambda \in \mathrm{Card} \cap \kappa$ such that $\mathbb{P} \in H_\lambda$. Now let $a \in H_\kappa$, $\mu \in \mathrm{Card} \setminus \kappa$ and suppose that

$$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{"}H_\mu \models \text{"}\varphi(a)\text{""}. \tag{4.1}$$

Lemma 4.2 implies $H_\mu \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(a)$"". If ν is large enough then by lemma 1.1 $H_\nu \models$ "$H_\mu \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(a)$""" and in particular

$$H_\nu \models \text{"}\exists\vartheta \in \mathrm{Card} \setminus \lambda : H_\vartheta \models \text{"}\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}} \text{"}\varphi(a)\text{""""} \tag{4.2}$$

Let ϑ be a witness to this. Lemma 1.2 implies that $\vartheta \geqslant \lambda$ is in fact a cardinal and in conjunction with lemma 1.1 it also yields $H_\vartheta \models$ "$\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$\varphi(a)$"". Since $\vartheta \geqslant \lambda$ and $\mathbb{P} \in H_\lambda$, forcing with $\mathbb{P}$ does not collapse ϑ. Finally by lemma 4.2 we attain $\mathbb{1}_{\mathbb{P}} \Vdash_{\mathbb{P}}$ "$H_\vartheta \models$ "$\varphi(a)$"".

⊣

At this point we cite a generalized Δ–system–lemma.

4.4. Lemma. ([Ku], II.1.6. Theorem) Let $\lambda \in \mathrm{Card} \setminus \omega$ and $\kappa \in \mathrm{Reg} \setminus \lambda^+$ such that $\forall \nu < \kappa : \overline{\overline{[\nu]^{<\lambda}}} < \kappa$. If $\mathcal{B}$ is a family of size at least κ such that $\forall x \in \mathcal{B} : \overline{\overline{x}} < \lambda$ then there is a $\mathcal{C} \in [\mathcal{B}]^\kappa$ that forms a Δ–system.

We will need this in order to see that a certain forcing notion fulfills a particular chain condition.

4.5. Fact. If $\alpha \in \mathrm{Lim}$, $\mathbb{P}_\alpha$ is an iterated forcing construction of length α with finite or countable support, G_α is $\mathbb{P}_\alpha$–generic, $\big(\beta < \mathrm{cf}(\alpha)\big)^{V[G_\alpha]}$ and $S \in \mathfrak{P}(\beta) \cap V[G_\alpha]$ then already $S \in V[G_\gamma]$ for some $\gamma < \alpha$.

Proof. We can suppose w.l.o.g. that $\mathrm{cf}(\alpha)$ is uncountable since otherwise β is a natural number which instantaneously renders the statement above true. Let σ be a $\mathbb{P}_\alpha$–name for S. For every $\delta \in S$ let $p_\delta \in G_\alpha$ be such that $p_\delta \Vdash_{\mathbb{P}_\alpha}$ "$\check{\delta} \in \sigma$". Let $\gamma := \sup_{\delta \in S} \Big(\sup\big(\mathrm{supt}(p_\delta)\big)\Big)$. Since the support of every condition is countable and the cofinality of α is uncountable and greater than β, $\gamma < \alpha$. Then $S = \big\{\delta < \beta \big| \exists p \in G_\gamma : p \Vdash_{\mathbb{P}_\alpha}$ "$\check{\delta} \in \sigma$"$\big\}$. The exact value of γ of course depends on S but that is not the point. $\Vdash_{\mathbb{P}_\alpha}$ is definable in V so S is already definable from G_γ. $\dashv$

4.6. Theorem. (Saharon Shelah, 1995) Let κ be a reflecting cardinal. Then there is a κ–c.c. proper notion of forcing $\mathbb{P}$ such that whenever G is $\mathbb{P}$–generic over V, $V[G] \models$ "$\mathrm{ZFC} + \mathrm{BPFA} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 = \kappa$".

The following proof is an analogue of the classical construction of a generic extension for ZFC + ¬ CH + MA by iterated forcing as written down for example in [Ku], chapter 8, §6 which has been adapted for the proof of the statement above. The theorem was first proved with a slightly different notation in [G–S].

Proof. Let $f : \kappa \longleftrightarrow \kappa^2$ be a bijection with the property that $\forall \gamma < \kappa : f(\gamma) < \kappa \cdot (\gamma + 1)$—in fact a surjection with this feature would suffice. We are going to define an iterated forcing construction with countable support $\Big(\big((\mathbb{P}_\gamma, \leqslant_\gamma, \mathbb{1}_\gamma), (\pi_\gamma, \preccurlyeq_\gamma, \varepsilon_\gamma)\big) \big| \gamma < \kappa\Big)$ as follows: Let $\gamma < \kappa$ and

$$e_\gamma : \kappa \longleftrightarrow \Big\{(\sigma, \alpha) \Big| \alpha, \sigma \in H_\kappa \text{ are } \mathbb{P}_\gamma\text{–names and } \mathbb{1}_{\mathbb{P}_\gamma} \Vdash_{\mathbb{P}_\gamma} \text{“}\sigma \in H_{\check{\kappa}} \text{ is a separative partial order and } \alpha \text{ is a family of maximal antichains in } \sigma.\text{”}\Big\} \quad (4.3)$$

be an enumeration. Whenever $\eta \leqslant \gamma$ one canonically can embed $\mathbb{P}_\eta$ into $\mathbb{P}_\gamma$. Hence we always can conceive of $\mathbb{P}_\eta$–names as $\mathbb{P}_\gamma$–names in such a situation. So for the names σ, α from (4.3) let $\beta^\gamma_{(\alpha,\sigma)}$ be a $\mathbb{P}_\gamma$–name from H_κ such that[1]

$$\mathbb{1}_{\mathbb{P}_\gamma} \Vdash_{\mathbb{P}_\gamma} \text{“}\big(\exists \beta \in H_{\check{\kappa}} : \beta \supset \sigma \wedge \beta \text{ is a proper and complete Boolean algebra such that } \forall A \in \alpha : A \text{ remains a maximal antichain in } \beta\big) \rightarrow \beta^\gamma_{(\alpha,\sigma)} \text{ is such a } \beta \text{, otherwise } \beta^\gamma_{(\alpha,\sigma)} \text{ is the trivial Boolean algebra.”.} \tag{4.4}$$

Now we can use f for bookkeeping. Set $\pi_\gamma := \beta^\gamma_{e_\eta(\xi)}$ iff $f(\gamma) = \kappa \cdot \eta + \xi$ for $\gamma < \kappa$.

Note that since the π_γ are all chosen from H_κ all initial segments $\mathbb{P}_\gamma$—where $\gamma < \kappa$—of our iterated forcing construction $\mathbb{P}_\kappa$ are in H_κ too.

- $\mathbb{P}_\kappa$ has the κ–c.c.—Proof:
  Suppose towards a contradiction that $\{p_\alpha | \alpha < \kappa\}$ was an antichain in $\mathbb{P}_\kappa$. Consider $\mathcal{B} := \{\text{supt}(p_\alpha) | \alpha < \kappa\}$. We distinguish two cases:
  - If $\overline{\overline{\mathcal{B}}} = \kappa$ apply lemma 4.4 for $\lambda := \aleph_1$. Then there is a Δ–system $\mathcal{C} \subset \mathcal{B}$ with root r of size κ. Then define $\mathcal{D} := \{p_\alpha | \alpha < \kappa \wedge \text{supt}(p_\alpha) \in \mathcal{C}\}$.
  - If $\overline{\overline{\mathcal{B}}} < \kappa$ then by the pigeonhole principle there exists an $r \in [\kappa]^{<\omega_1}$ and an $s \in [\kappa]^\kappa$ such that $\forall \alpha \in s : \text{supt}(p_\alpha) = r$. Define $\mathcal{D} := \{p_\alpha | \alpha \in s\}$.

  Set $\gamma := \sup(r) + 1$. In both cases $\{p \restriction \gamma | p \in \mathcal{D}\}$ is an antichain of size κ in $\mathbb{P}_\gamma$. But $\mathbb{P}_\gamma \in H_\kappa$ so in particular $\overline{\overline{\mathbb{P}_\gamma}} < \kappa$. ↯ ⊣ ($\mathbb{P}_\kappa$ has the κ–c.c..)

  This immediately implies that κ remains a cardinal in the generic extension.

- $\mathbb{P}_\kappa$ is proper.
  This is simply a consequence of theorem 2.16 since $\mathbb{1}_\gamma \Vdash_{\mathbb{P}_\gamma}$ “π_γ is proper.” for all $\gamma < \kappa$. ⊣ ($\mathbb{P}_\kappa$ is proper.)

- Now we show that $V[G_\kappa] \models$ “$\kappa \leqslant \aleph_2$”—Proof:
  The iterated forcing construction as a whole is proper so in particular $\aleph_1$ is preserved.(As a consequence we dispense with the superscripts when talking about $\aleph_1$.) But all cardinals between $\aleph_1$ and κ are collapsed. In order to see this we

[1]The word “otherwise” in (4.4) refers to the part of (4.4) in brackets.

consider the following notion of forcing:

$$\begin{aligned} \mathbb{Q} &:= (Q, \leqslant_{\mathbb{Q}}) \text{ where} \\ Q &:= \{f : C \hookrightarrow \aleph_1 \big| C \in [\aleph_1]^{<\omega_1}\} \text{ and} \\ p \leqslant_{\mathbb{Q}} q &:\Longleftrightarrow p \supset q. \end{aligned} \tag{4.5}$$

Now let $\lambda \in (\mathrm{Card} \cap \kappa \setminus \aleph_2)^V$ and $\mathcal{A} := \{A_\eta \big| \eta < \lambda\}$ a family of maximal antichains where $A_\eta := \{\{(\xi, \eta)\} \big| \xi < \aleph_1\}$ for every $\eta < \lambda$. Furthermore let σ, α be $\mathbb{P}_0$–names from H_κ for $\mathbb{Q}^V, \mathcal{A}$ respectively. Eventually σ, α are considered—say in step $\gamma < \kappa$ in the iteration. Note that $\mathrm{id} : \mathbb{Q}^V \hookrightarrow \mathbb{Q}^{V[G_\gamma]}$ is a dense embedding. The first two conditions in the definition of dense embedings are easily fulfilled. Lemma 2.15 implies that $\mathbb{Q}$ is dense in $\mathbb{Q}^{V[G_\gamma]}$. $\mathbb{B}^* := \big(\mathrm{ro}(\mathbb{Q})\big)^{V[G_\gamma]}$ is a complete Boolean algebra which is proper in $V[G_\gamma]$ since it is defined there from a countably closed and hence proper forcing notion. Since $\mathbb{Q}^{V[G_\gamma]}$ is separative the canonical dense embedding from $\mathbb{Q}^{V[G_\gamma]}$ into $\mathbb{B}^*$ is one–to–one. So there exists a one–to–one dense embedding from $\mathbb{Q}^V$ into $\mathbb{B}^*$. By an isomorphic correction as in remark 3.4 we attain a complete and proper Boolean algebra $\mathbb{B}$ such that $\mathbb{Q}$ is dense in $\mathbb{B}$. So every maximal antichain from $\mathbb{Q}^V$ stays maximal in $\mathbb{B}$, in particular all from $\mathcal{A}$.

So we showed that in $V[G_\gamma]$ there exists a complete and proper Boolean algebra extending $\mathbb{Q}^V$ such that all $A \in \mathcal{A}$ stay maximal. Now by definition of our forcing iteration $\pi_\gamma^{G_\gamma}$ is such a thing. But then $F_\gamma^{\gamma+1} \in V[G_{\gamma+1}]$ is a filter meeting every A_η and $\bigcup\big(\mathbb{Q}^{V[G_\gamma]} \cap F_\gamma^{\gamma+1}\big) : \aleph_1 \longleftrightarrow \lambda$. So $V[G_{\gamma+1}] \models$ "$\lambda < \aleph_2$" and hence in particular

$$V[G_\kappa] \models \text{“}\lambda < \aleph_2\text{”}. \tag{4.6}$$

Since λ was arbitrarily chosen from the cardinals below κ that finishes this part.

$\dashv$ ($V[G_\kappa] \models$ "$\kappa \leqslant \aleph_2$")

- Next we want to see that $V[G_\kappa] \models$ "$2^{\aleph_1} \leqslant \kappa$"—Proof:
From fact 4.5 we know that every subset of $\aleph_1$ in $V[G_\kappa]$ is already in some $V[G_\gamma]$ where $\gamma < \kappa$. Every set in $\mathfrak{P}(\aleph_1) \cap V[G_\gamma]$ is represented by a nice $\mathbb{P}_\gamma$–name. $\mathbb{P}_\gamma \in H_\kappa$, let $\lambda_\gamma := \overline{\overline{\mathbb{P}_\gamma}} + \aleph_1$. There are at most $(2^{\lambda_\gamma})^{\aleph_1} = 2^{\lambda_\gamma} < \kappa$ such names. So there are at most $\Sigma_{\gamma<\kappa} 2^{\lambda_\gamma} \leqslant \kappa \cdot \kappa = \kappa$ subsets of $\aleph_1$ in $V[G_\kappa]$. Note that at this point we needed κ's regularity.

$\dashv$ ($V[G_\kappa] \models$ "$2^{\aleph_1} \leqslant \kappa$")

Since $2^{\aleph_1} > \aleph_1$ this also shows for the second time that κ is not collapsed and so remains a cardinal in the generic extension. Moreover it is now clear that $V[G_\kappa] \models$ "$\kappa = \aleph_2$". Note that if we succeed in showing $V[G_\kappa] \models$ "BPFA" because of BPFA $\Rightarrow$ $\mathrm{MA}_{\aleph_1} \Rightarrow 2^{\aleph_0} \geqslant \aleph_2$ we will be finished.

- Finally $V[G_\kappa] \models$ "BPFA"—Proof:
  At first we work in $V[G_\kappa]$. Let $\mathbb{B}^*$ be a proper Boolean algebra and $\mathcal{A}_{\mathbb{B}^*} := \{A_\eta \mid \eta < \aleph_1\}$ a given family of maximal antichains in $\mathbb{B}^*$ all of which have size at most $\aleph_1$. By an isomorphic correction choose a $\mathbb{B}$ and an isomorphism $\psi : \mathbb{B}^* \simeq \mathbb{B}$ such that the subalgebra $\mathbb{S}$ finitely generated by $\mathcal{A} = \{\psi``A_\eta \mid \eta < \aleph_1\}$ is in $H_{\aleph_2}$. Clearly $\mathbb{B}$ is also proper and $\mathcal{A}$ is again a family of maximal antichains. By lemma 3.3 $\mathcal{A}$ and $\mathbb{S}$ are coded by subsets of $\aleph_1$ which by fact 4.5 already appear at an intermediate stage in the iteration. So choose a $\gamma < \kappa$ such that $\mathbb{S}, \mathcal{A} \in V[G_\gamma]$.

  Claim: For all $\eta \in \kappa \setminus \gamma$ in $V[G_\eta]$ there exists a complete and proper Boolean superalgebra $\mathbb{B}'$ of $\mathbb{S}$ such that all antichains from $\mathcal{A}$ remain maximal in $\mathbb{B}'$. In fact if $p \in \frac{\mathbb{P}_\kappa}{G_\eta}$, β is a $\frac{\mathbb{P}_\kappa}{G_\eta}$–name such that

$$p \Vdash_{\frac{\mathbb{P}_\kappa}{G_\eta}} \text{"}\beta \text{ is a proper Boolean superalgebra of } \check{\mathbb{S}} \text{ such that all antichains from } \check{\mathcal{A}} \text{ remain maximal in } \beta.\text{"}, \tag{4.7}$$

  $Q := \{q \in \frac{\mathbb{P}_\kappa}{G_\eta} \mid q \leqslant_{\frac{\mathbb{P}_\kappa}{G_\eta}} p\}$, $\mathbb{Q} := (Q, \leqslant_{\frac{\mathbb{P}_\kappa}{G_\eta}} \upharpoonright Q)$ and β^+ is a $\frac{\mathbb{P}_\kappa}{G_\eta}$–name for $(\beta^{G_\kappa})^+$ then $\mathbb{B}' := \mathrm{ro}(\mathbb{Q} \star \beta^+)$ is such an algebra up to isomorphism.

  Note that the existence of such p, β is witnessed by $\mathbb{B}$'s existence in $V[G_\kappa]$.

  Proof of Claim: Let $\eta \in \kappa \setminus \gamma$ be arbitrarily chosen and let $p, \beta, Q, \mathbb{Q}$ and $\mathbb{B}'$ be as above. Let $\delta : \mathbb{Q} \star \beta^+ \longrightarrow \mathbb{B}'^+$ be the canonical dense embedding. At the beginning we prove that $\mathbb{B}'$ can be seen as an extension of $\mathbb{S}$ by the following embedding:

$$\chi : \mathbb{S} \longrightarrow \mathbb{B}' \qquad s \longmapsto \begin{cases} \delta\big((p, \check{s})\big) & \text{iff } s \neq \mathbb{0}_{\mathbb{S}} \\ \mathbb{0}_{\mathbb{B}'} & \text{otherwise.} \end{cases} \tag{4.8}$$

  - It is immediate that $\mathbb{0}_{\mathbb{S}} = \mathbb{0}_{\mathbb{B}'}$ and $\mathbb{1}_{\mathbb{S}} = \mathbb{1}_{\mathbb{B}'}$.
  - Also $\chi(\neg s) = \neg\chi(s)$ for all $s \in \mathbb{S}$. In order to show this let $s \in \mathbb{S} \setminus \{\mathbb{0}_{\mathbb{S}}, \mathbb{1}_{\mathbb{S}}\}$ be arbitrarily chosen. Suppose towards a contradiction that $\chi(\neg s) \neq \neg\chi(s)$. Since the partial order on a Boolean algebra is in particular antisymmetric

we can infer that $\chi(\neg s) \nleqslant_{\mathbb{B}'} \neg\chi(s)$ or $\neg\chi(s) \nleqslant_{\mathbb{B}'} \chi(\neg s)$. So we distinguish these two cases:

* $\chi(\neg s) \nleqslant_{\mathbb{B}'} \neg\chi(s)$. Since $\mathbb{B}'$ is a Boolean algebra this means that $\chi(\neg s) \|_{\mathbb{B}'^+} \chi(s)$. By density of δ's image choose a (q,τ) such that $\delta\big((q,\tau)\big) \leqslant_{\mathbb{B}'} \chi(\neg s), \chi(s)$. Then $(p,\check{s}) \|_{\mathbb{Q}\star\beta^+} (q,\tau)$—let $(r,\nu) \leqslant_{\mathbb{Q}\star\beta^+} (q,\tau),(p,\check{s})$ be a witness for this. Furthermore let $(t,\vartheta) \leqslant_{\mathbb{Q}\star\beta^+} (r,\nu),(p,\check{\neg s})$ witness $(r,\nu)\|_{\mathbb{Q}\star\beta^+}(p,\check{\neg s})$. So $t \Vdash_{\mathbb{Q}}$ "$\vartheta \in \beta^+ \wedge \vartheta \preccurlyeq_\beta \check{s}, \check{\neg s}$" which is clearly nonsense.
* $\neg\chi(s) \nleqslant_{\mathbb{B}'} \chi(\neg s)$. Since $\mathbb{B}'$ is a Boolean algebra the canonical partial order on it is separative. Using this fact and the density of δ's image one can find a $(q,\tau) \in \mathbb{Q}\star\beta^+$ such that $\delta\big((q,\tau)\big) \leqslant_{\mathbb{B}'} \neg\chi(s)$ but $\delta\big((q,\tau)\big) \perp_{\mathbb{B}'^+} \chi(\neg s)$. So $\delta\big((q,\tau)\big) \perp_{\mathbb{B}'^+} \chi(s)$ and hence $(q,\tau) \perp_{\mathbb{Q}\star\beta^+} (p,\check{s})$. But also $(q,\tau)\perp_{\mathbb{Q}\star\beta^+}(p,\check{\neg s})$. Since $q \leqslant_{\mathbb{Q}} p$ this implies $q \Vdash_{\mathbb{Q}}$ "$\tau \in \beta^+ \wedge \tau \perp_{\beta^+} \check{s}, \check{\neg s}$"—↯ $\quad \dashv \big(\chi(\neg s) = \neg\chi(s)\big)$

– Finally $\chi(s \wedge t) = \chi(s) \wedge \chi(t)$ for all $s,t \in \mathbb{S}$. Again by separativity it suffices to distinguish two cases in search for a contradiction:

* $\chi(s \wedge t) \nleqslant_{\mathbb{B}'} \chi(s) \wedge \chi(t)$. By separativity and density choose a $(q,\tau) \in \mathbb{Q}\star\beta^+$ such that $\delta\big((q,\tau)\big) \leqslant_{\mathbb{B}'} \chi(s\wedge t)$ but $\delta\big((q,\tau)\big) \perp_{\mathbb{B}'^+} \big(\chi(s)\wedge\chi(t)\big)$. Since $(q,\tau)\|_{\mathbb{Q}\star\beta^+}\big(p,(s\wedge t)\check{}\big)$ we can choose a $(r,\nu) \leqslant_{\mathbb{Q}\star\beta^+} (q,\tau),\big(p,(s\wedge t)\check{}\big)$. We distinguish two subcases:
  · $\delta\big((r,\nu)\big)\perp_{\mathbb{B}'^+}\chi(s)$. Then $(r,\nu)\perp_{\mathbb{Q}\star\beta^+}(p,\check{s})$ but also $(r,\nu) \leqslant_{\mathbb{Q}\star\beta^+} (p,(s\wedge t)\check{})$. So $r \Vdash_{\mathbb{Q}}$ "$\nu \in \beta^+ \wedge \nu \preccurlyeq_\beta (s\wedge t)\check{} \wedge \nu\perp_{\beta^+} s$"—↯
  · $\delta\big((r,\nu)\big)\|_{\mathbb{B}'^+}\chi(s)$. Then we can choose a $(u,\vartheta) \leqslant_{\mathbb{Q}\star\beta^+} (r,\nu)$ such that $\delta\big((u,\vartheta)\big) \leqslant_{\mathbb{B}'} \chi(s)$. Since $(u,\vartheta) \leqslant_{\mathbb{Q}\star\beta^+} (q,\tau)$ it follows that $\delta\big((u,\vartheta)\big)\perp_{\mathbb{B}'^+}\chi(t)$. So $(u,\vartheta) \leqslant_{\mathbb{Q}\star\beta^+} \big(p,(s\wedge t)\check{}\big)$ but $(u,\vartheta)\perp_{\mathbb{Q}\star\beta^+}(p,\check{t})$. Hence $u \Vdash_{\mathbb{Q}}$ "$\vartheta \in \beta^+ \wedge \vartheta \preccurlyeq_\beta (s\wedge t)\check{} \wedge \vartheta\perp_{\beta^+}\check{t}$"—↯
* $\chi(s)\wedge\chi(t) \nleqslant_{\mathbb{B}'} \chi(s\wedge t)$. By separativity and density choose a condition (q,τ) from $\mathbb{Q}\star\beta^+$ such that $\delta\big((q,\tau)\big) \leqslant_{\mathbb{B}'} \chi(s)\wedge\chi(t)$ but $\delta\big((q,\tau)\big)\perp_{\mathbb{B}'^+}\chi(s\wedge t)$. Since obviously $(q,\tau)\|_{\mathbb{Q}\star\beta^+}(p,\check{s})$ we can choose a condition $(r,\nu) \leqslant_{\mathbb{Q}\star\beta^+} (q,\tau),(p,\check{s})$ and subsequently a $(u,\vartheta) \leqslant_{\mathbb{Q}\star\beta^+} (r,\nu),(p,\check{t})$ by the same form of argument. $(u,\vartheta)\perp_{\mathbb{Q}\star\beta^+}\big(p,(s\wedge t)\check{}\big)$ so $u \Vdash_{\mathbb{Q}}$ "$\vartheta \in \beta^+ \wedge \vartheta \preccurlyeq_\beta \check{s},\check{t} \wedge \vartheta\perp_{\beta^+}\check{s}\wedge\check{t}$"—↯ $\quad \dashv \big(\chi(s\wedge t) = \chi(s)\wedge\chi(t)\big)$

– This actually implies that χ is one–to–one. For if $s,t \in \mathbb{S}$ and $s \neq t$ by antisymmetry we can suppose w.l.o.g. that $s \nleqslant_{\mathbb{S}} t$. Then by separativity there exists a $u \in \mathbb{S}^+$ with $u \leqslant_{\mathbb{S}} s$—that is $u \wedge_{\mathbb{S}} s = u$ such that $u\perp_{\mathbb{S}^+}t$—that it $u \wedge_{\mathbb{S}} t = \mathbb{0}_{\mathbb{S}}$. But then $\chi(u) \wedge_{\mathbb{B}'} \chi(s) = \chi(u)$ and $\chi(u) \wedge_{\mathbb{B}'} \chi(t) = \mathbb{0}_{\mathbb{B}'}$. If

$\chi(s) = \chi(t)$ we would have $\chi(u) = \mathbb{0}_{\mathbb{B}'}$ and hence $u = \mathbb{0}_{\mathbb{B}'}$—↯

⊣ (χ is one–to–one.)

– Suppose towards a contradiction that $A \in \mathcal{A}$ and that $b' \in \mathbb{B}'$ is incompatible with all elements from A. Since $\delta``(\mathbb{Q}\star\beta)$ is dense in $\mathbb{B}'$ there is a $(q,\tau) \in \mathbb{Q}\star\beta$ such that $\delta\big((q,\tau)\big) \leqslant b'$. We have

$$\forall a \in A : \delta\big((q,\tau)\big) \perp_{\mathbb{B}'} \chi(a). \tag{4.9}$$

Since δ is a dense embedding it follows immediately that

$$\forall a \in A : (q,\tau) \perp_{\mathbb{Q}\star\beta} (p, \check{a}). \tag{4.10}$$

But then

$$\forall a \in A : q \Vdash_{\mathbb{Q}} \text{“}\tau \perp_\beta \check{a}\text{”}. \tag{4.11}$$

Proof of (4.11):

Suppose otherwise. From $p = \mathbb{1}_{\mathbb{Q}}$ we get $q \leqslant_{\mathbb{Q}} p$. So there would be an $a \in A$ and an $r \leqslant_{\mathbb{Q}} q$ such that $r \Vdash_{\mathbb{Q}}$ “$\tau \|_\beta \check{a}$”—that is $r \Vdash_{\mathbb{Q}}$ “$\exists \xi \preccurlyeq_\beta \tau, \check{a}$”. By the maximal principle there exists a $\mathbb{Q}$–name ξ such that $r \Vdash_{\mathbb{Q}}$ “$\xi \preccurlyeq_\beta \tau, \check{a}$”. But then $(r,\xi) \leqslant_{\mathbb{Q}\star\beta} (q,\tau), (p,\check{a})$ contradicting (4.10). ⊣(4.11)

But (4.11) says that $q \Vdash_{\mathbb{Q}}$ “$\check{A}$ is no longer maximal in β.” contradicting (4.7).

⊣(All antichains from $\mathcal{A}$ remain maximal in $\mathbb{B}'$.)

⊣(Claim)

As $\mathbb{S}$ is a Boolean algebra one in particular can conceive of it as a separative poset. Now let σ be a $\mathbb{P}_\gamma$–name for $\mathbb{S}$ and α be a $\mathbb{P}_\gamma$–name for $\mathcal{A}$. Then by definition of our bookkeeping function f at some later point in the iteration the pair of them is considered—in fact at $\eta := f^{-1}\Big(\kappa \cdot \gamma + e_\gamma^{-1}\big((\sigma,\alpha)\big)\Big)$. We have

$$V[G_\eta] \models \text{“}\exists \mathbb{B}'\big(\mathbb{B}' \text{ is a complete and proper Boolean superalgebra of } \mathbb{S} \text{ and } \forall A \in \mathcal{A} : A \text{ is a maximal antichain in } \mathbb{B}'.\big)\text{”}. \tag{4.12}$$

Lemma 2.10 implies that the assertion believed by $V[G_\eta]$ is $\Sigma_2\big(\{\mathbb{S},\mathcal{A}\}\big)$. Since $\mathbb{P}_\eta \in H_\kappa$ lemma 4.3 implies that κ is still reflecting and thus by lemma 1.8 Σ_2–

correct in $V[G_\eta]$. But then of course

$$H_\kappa[G_\eta] \models \text{“}\exists \mathbb{B}' \big(\mathbb{B}' \text{ is a complete and proper Boolean superalgebra of } \mathbb{S} \text{ and } \forall A \in \mathcal{A} : A \text{ is a maximal antichain in } \mathbb{B}'.\text{”}\big). \tag{4.13}$$

Let $\mathbb{E} \in H_\kappa[G_\eta]$ be a witness to this, then because of κ's inaccessiblity and lemma 1.2:

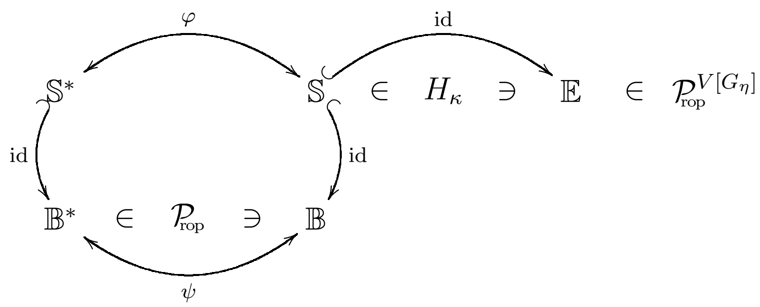
$$V[G_\eta] \models \text{“}\mathbb{E} \in H_\kappa, \mathbb{E} \text{ is a complete and proper Boolean superalgebra of } \mathbb{S} \text{ and } \forall A \in \mathcal{A} : A \text{ is a maximal antichain in } \mathbb{E}.\text{”}. \tag{4.14}$$

Suppose now that σ is a $\mathbb{P}_\eta$–name for $\mathbb{S}$ and α is a $\mathbb{P}_\eta$–name for $\mathcal{A}$. Then by definition of our iterated forcing construction π_η is chosen in such a way that

$$\mathbb{1}_\eta \Vdash_{\mathbb{P}_\eta} \text{“}\pi_\eta \in \dot{H}_\kappa, \pi_\eta \text{ is a complete and proper Boolean superalgebra of } \sigma \text{ and every antichain from } \alpha \text{ stays maximal in } \pi_\eta.\text{”}. \tag{4.15}$$

Hence in our iterated forcing construction $F_\eta^{\eta+1} \in V[G_{\eta+1}] \subset V[G_\kappa]$ is a filter $\pi_\eta^{G_{\eta+1}}$–generic over $V[G_\eta]$. That means it intersects every maximal antichain in $\pi_\eta^{G_{\eta+1}}$ especially all those from $\mathcal{A}$.

But now $H := \psi^{-1}\text{“}(F_\eta^{\eta+1} \cap \mathbb{S})$ is an $\mathcal{A}_{\mathbb{B}^*}$–generic filter. The following diagram illustrates the situation:



$\dashv$ ($V[G_\kappa] \models$ “BPFA”)

$\dashv$

5 The consistency strength of BAAFA and BPFA

It turns out that the existence of a reflecting cardinal is indeed equiconsistent with BAAFA as well as BPFA. In order to show this we need some more definitions. $\mathcal{AA}^* \subset \mathcal{P}_{\text{rop}}$ implies that BPFA $\Rightarrow$ BAAFA so it suffices to show that BAAFA has the consistency strength of a reflecting cardinal.

5.1. Lemma. Let T be a tree. The following two assertions are equivalent:

(1) $\exists f : T \longrightarrow \omega \, \forall s, t \in T\big(s <_T t \rightarrow f(s) \neq f(t)\big)$.

(2) There is a sequence $(A_n | n < \omega)$ of antichains such that $T = \bigcup_{n<\omega} A_n$.

Here of course $<_T := \leqslant_T \setminus (= \restriction T)$.

Proof. (2) $\Rightarrow$ (1). Define f by setting $t \mapsto \min \big\{n \big| n < \omega \wedge t \in A_n\big\}$ for $t \in T$.
(1) $\Rightarrow$ (2). Set $A_n := f^{-1}``\{n\}$. $\dashv$

5.2. Definition. A tree T is called *special* by definition if and only if the assertions above hold. The function f in clause (1) is called the *specializing function.*

5.3. Definition. Jensen's *global square-principle*, denoted $\Box$ asserts the existence of a sequence $(C_\alpha | \alpha \in \text{Lim} \setminus \text{Reg})$ the size of a proper class with the following properties for any $\alpha \in \text{Lim} \setminus \text{Reg}$:

- C_α is club in α,
- $\text{otyp}(C_\alpha) < \alpha$,
- $\forall \beta \in \lim(C_\alpha) : \beta \notin \text{Reg} \wedge C_\beta = C_\alpha \cap \beta$.

The following theorem asserts the existence of a well-known state of affairs.

5.4. Theorem. $L \models$ "$\Box$".

This is proved for example in [De 2], VI.6. A careful analysis of this proof yields the following...

5.5. Fact. There is a $\Sigma_1\big(\{\aleph_1\}\big)$–definable witness for the truth of $\Box$ in L. To put it more formally:

There is a $\Sigma_1\big(\{\aleph_1\}\big)$–formula φ such that for every L–singular limit ordinal α there exists exactly one C_α such that $\varphi(\alpha, C_\alpha)$.

We now sketch this analysis:

Proof. Devlin's proof starts at the bottom of page 286. On page 288 he defines $Q := \big\{\alpha \big| \Phi``(\alpha \times \alpha) \subset \alpha\big\}$ where Φ is Gödel's pairing function—which is Δ_1–definable. So Q is Δ_1–definable too. He proceeds by distinguishing five cases.

(1) $\alpha < \aleph_1$.

In the first case C_α shall be any ω–sequence cofinal in α. So we can take the $<_L$–least one—which is Σ_1–definable.

(2) $\alpha > \aleph_1$ and $\alpha \notin Q$.

(3) $\alpha > \aleph_1$, $\alpha \in Q$ and $\sup(Q \cap \alpha) < \alpha$.

In cases two and three the sets C_α are defined via ordinal arithmetic. These definitions are all absolute and the reason for this is that they are Δ_1 since in the end they are defined by transfinite recursion where every single step is Δ_1.

For the two remaining cases Devlin defines:

$$\beta := \text{"the least } \beta \text{ such that } \alpha \text{ is singular over } J_\beta\text{."} \tag{5.1}$$

$$\text{and } n := \text{"the least } n \text{ such that } \alpha \text{ is } \Sigma_n\text{–singular over } J_\beta. \tag{5.2}$$

(4) $\alpha > \aleph_1$, $\alpha \in \lim(Q)$, $n = 1$ and β is a successor ordinal.

Case four is similar to case one. The cofinality of α is ω so one can take any ω–sequence cofinal in α. Again one can simply take the Σ_1–definable $<_L$–least one.

(5) $\alpha > \aleph_1$, $\alpha \in \lim(Q)$ and ($n > 1$ or β is a limit ordinal).

In the last case C_α is eventually defined on page 294:

$$C_\alpha := \{\alpha_{t(\nu)} | \nu < \tilde{\theta}\} \tag{5.3}$$

This uses the function t defined on page 293 by transfinite recursion from Gödel's pairing function, the parameter κ defined on page 291, the sequences (α_ν) and (X_ν), α itself, the function $\alpha \mapsto J_\alpha$, the $n+1^{st}$ standard–parameter over β and the canonical Σ_1 Skolem function $h_{\varrho_\beta^{n-2}, A_\beta^{n-2}}$. Indeed this whole construction—the heart of which is the recursive definition of the functions k and m and the sequences (X_ν) and (α_ν) on page 291—amounts to a Σ_1–definition. The important points in this respect are the following:

- All fine–structural elements of the construction—the canonical Σ_1–Skolem functions, the standard codes and the standard parameters are Σ_1–definable.
- The function $\beta \mapsto J_\beta$ is Σ_1–definable—see for example [De 2], corollary VI.2.6.
- Whenever the formulation "Let ... be the least ordinal such that ..." appears, notice that $\beta \mapsto J_\beta$, h, h_τ, etc. are Σ_1–definable **functions**, not Σ_1–definable yet otherwise arbitrary relations. So "$\bullet \in J_\beta$" as well as "$\bullet \notin J_\beta$" are both $\Sigma_1(\{\beta\})$–definable predicates and hence denying the property "$\bullet \in J_\gamma$" for all ordinals γ smaller than β does not "contaminate" the definition with a Π_1–statement. Similar considerations obtain for h, h_τ, etc.
- A set recursively defined by Σ_1–statements is at whole Σ_1–definable—see for example [Je 2], lemma 13.12.

Finally we have to check that the cases can be distinguished in a Σ_1–fashion. "$\bullet \in Q$" is a Δ_1–predicate, a supremum of a set of ordinals can simply be calculated by taking the union, so $\sup(Q \cap \alpha) < \alpha$ and $\alpha \in \lim(Q)$ are both $\Delta_1(\{\alpha\})$.

Now only the definition of β in (5.1) and the definition of n in (5.2) remain to be checked. The first can be formulated thus:

$$\begin{aligned} &\exists A \big(A \subset \alpha \wedge \bigcup A = \alpha \wedge \text{otyp}(A) < \alpha \wedge A \in J_\beta\big) \wedge \\ &\qquad \forall \gamma < \beta, A \in J_\gamma \big((A \subset \alpha \wedge \bigcup A = \alpha) \rightarrow \text{otyp}(A) = \alpha\big). \end{aligned} \tag{5.4}$$

For the latter we use the fact that the property "m is the Gödel-number of a Σ_n–formula." is $\Delta_1(\{m, n\})$. If we denote this property by FC then we can express (5.2) as follows:

$$\exists A \in J_\beta, m < \omega \big(A \subset \alpha \wedge \bigcup A = \alpha \wedge \operatorname{otyp}(A) < \alpha \\ \wedge \operatorname{Sat}(J_\beta, A, m) \wedge \operatorname{FC}(m, n) \big) \wedge \forall k < n, m < \omega \nexists A \in J_\beta : \\ \operatorname{FC}(m, k) \wedge \operatorname{Sat}(J_\beta, A, m) \wedge A \subset \alpha \wedge \bigcup A = \alpha \wedge \operatorname{otyp}(A) < \alpha. \tag{5.5}$$

⊣

5.6. Remark. If $(C_\alpha | \alpha \in \mathrm{Lim} \setminus \mathrm{Reg})$ is a witness to the truth of $\square$ there is a tree $\mathcal{T}$, which corresponds to this sequence in a canonical way. Define $\mathcal{T} := (\mathrm{Lim} \setminus \mathrm{Reg}, \preccurlyeq)$ and set $\beta \preccurlyeq \gamma :\Longleftrightarrow \beta$ is a limit point of C_γ. $\preccurlyeq$ is reflexive, transitive and antisymmetrical. It also inherits the property of being well-ordered from the ordinals. So $\preccurlyeq$ is a strict partial well-order. Finally

$$\forall \beta, \gamma \in \mathrm{Lim} \setminus \mathrm{Reg} \big(\exists \eta \in \mathrm{Lim} \setminus \mathrm{Reg} (\beta \preccurlyeq \eta \wedge \gamma \preccurlyeq \eta) \rightarrow (\beta \preccurlyeq \gamma \vee \gamma \preccurlyeq \beta) \big), \tag{5.6}$$

so T is a tree.

Proof. We will only prove the last assertion.

Let β, γ, η be singular limit ordinals such that $\beta, \gamma \preccurlyeq \eta$. Suppose w.l.o.g. that $\beta < \gamma$. γ is a limit point of C_η so $C_\gamma = C_\eta \cap \gamma$. Since β is a limit point of C_η and $\beta < \gamma$, β is a limit point of C_γ. So $\beta \preccurlyeq \eta$. The other postulates regarding $\mathcal{T}$ are comparably simple to prove. ⊣

For the rest of this chapter we fix a witness for the the truth of $\square$ in L, i.e. we fix a class–sized sequence $(C_\alpha | \alpha \in \mathrm{Lim} \setminus \mathrm{Reg})$ and we suppose it to be Σ_1–definable. We are going to analyse initial segments of its corresponding tree. The tree will be denoted as $\mathcal{T}_\square = (\mathrm{Lim} \setminus \mathrm{Reg}, \preccurlyeq_\square)$. When writing $\mathcal{T}_\square \restriction S$ for $S \subset \mathrm{Lim} \setminus \mathrm{Reg}$ we mean $\big(S, \preccurlyeq_\square \cap (S \times S)\big)$.

5.7. Lemma. If $0^\#$ does not exist, then $\forall \alpha \geqslant \aleph_2 : \operatorname{cf}^V \big(\alpha^{+L}\big) \geqslant \overline{\overline{\alpha}}^V$.

Proof. Suppose towards a contradiction that the lemma is false, i.e. that $\alpha \geqslant \aleph_2$, $0^\#$ does not exist but $\operatorname{cf}^V \big(\alpha^{+L}\big) < \overline{\overline{\alpha}}^V$. Let $C \subset \alpha^{+L}$ be a cofinal subset of α^{+L} such that $\operatorname{otyp}(C) < \overline{\overline{\alpha}}$. Since $0^\#$ does not exist Jensen's Covering lemma holds true. $\aleph_1 \cup C$

is uncountable and so there is an $S \in L$ such that $S \supset \aleph_1 \cup C$ yet $\overline{\overline{S}} = \overline{\overline{\aleph_1 \cup C}}$. Let $X := S \cap \alpha^{+L}$, then X is cofinal in α^{+L} and $X \in L$. Since L satisfies full Choice and hence in particular all its successor cardinals are regular we infer that $\mathrm{otyp}(X) = \alpha^{+L}$. At the same time $\overline{\overline{X}} \leqslant \overline{\overline{S}} = \overline{\overline{\aleph_1 \cup C}} \leqslant \aleph_1 + \overline{\overline{C}} < \overline{\overline{\alpha}}$ and hence $\mathrm{otyp}(X) < \alpha$. Contradiction!

$\dashv$

5.8. LEMMA. Suppose ϱ is regular in L, $C^* \in L$ a club set in ϱ such that $C^* \cap \mathrm{Reg}^L = \emptyset$ and $C \subset C^*$ a club set in ϱ such that $\mathrm{otyp}(C) = \omega_1$. Then all branches in $\mathcal{T}_\square \restriction C$ are countable.

Proof. Throughout this proof when we use the phrase "cofinal branch" we are **not** referring to this branch being cofinal in the tree but to the fact that the set of the elements of this branch is cofinal in ϱ. So suppose towards a contradiction that $b \subset C$ is an uncountable branch. Since $\mathrm{otyp}(b) = \omega_1$ it is cofinal.

$$\text{There is exactly one cofinal branch in } \mathcal{T}_\square \restriction C^*. \tag{5.7}$$

Proof of (5.7):

- There is at least one: $b^* := \left\{\gamma \in C^* \middle| \exists \eta \in b \setminus \gamma : \gamma \preccurlyeq_\square \eta\right\}$ is a cofinal branch in $\mathcal{T}_\square \restriction C^*$.

  For if $\gamma \in b^*$ and w.l.o.g. $\beta \in \gamma \cap b^*$ there are witnesses $\eta, \nu \in b$ such that $\beta \preccurlyeq_\square \eta$ and $\gamma \preccurlyeq_\square \nu$. Since b is a branch there is a $\zeta \in b$ such that $\zeta \succcurlyeq_\square \eta, \nu$. Hence $\beta, \gamma \preccurlyeq_\square \zeta$ which implies $\beta \preccurlyeq_\square \gamma$. So b^* is a branch.

  b^* is also cofinal in ϱ since b is cofinal in ϱ and clearly $b^* \supset b$.

- There is at most one: Suppose that b^* and c^* are both cofinal branches in $\mathcal{T}_\square \restriction C^*$. Suppose $\beta_0 \in b^* \bigtriangleup c^*$. W.l.o.g. assume that $\beta_0 \in b^* \setminus c^*$. Define two sequences $(\beta_n | n < \omega)$, $(\gamma_n | n < \omega)$ as follows:

$$\gamma_n := \min(c^* \setminus \beta_n) \tag{5.8}$$
$$\beta_{n+1} := \min(b^* \setminus \gamma_n) \tag{5.9}$$

  Since C^* is a club $\zeta := \sup_{n<\omega} \beta_n = \sup_{n<\omega} \gamma_n$ is an element of C^*. Choose an $\eta \in c^* \setminus \zeta$. Then since γ_n is a limit point of C_η and $C_{\gamma_n} = C_\eta \cap \gamma_n$ for all $n < \omega$

we have that ζ is a limit point of C_η and $C_\zeta = C_\eta \cap \zeta$. Furthermore choose a $\nu \in b^* \setminus \zeta$. Like before—since β_n is a limit point of C_ν and $C_{\beta_n} = C_\nu \cap \beta_n$ for all $n < \omega$ it follows that ζ is a limit point of C_ν and $C_\zeta = C_\nu \cap \zeta$. So $\zeta \in b^* \cap c^*$. By $\zeta \in b^*$ we get $\beta_0 \preccurlyeq_\Box \zeta$ which in conjunction with $\zeta \in c^*$ implies $\beta_0 \in c^*$. Contradiction! ⊣(5.7)

Since there is exactly one cofinal branch in $\mathcal{T}_\Box \upharpoonright C^*$ it can be defined from C^*. If b^* is this branch then obviously

$$b^* = \{\gamma \in C^* \big| \forall \eta < \varrho \exists \zeta \in C^* \setminus \eta : \gamma \leqslant_{\mathcal{T}_{w\Box^*_\chi}} \zeta\}. \tag{5.10}$$

Since $C^* \in L$ this means that in fact $b^* \in L$. Moreover $\bigcup_{\gamma \in b^*} C_\gamma$ is in L. ϱ is regular in L hence $\operatorname{otyp}(\bigcup_{\gamma \in b^*} C_\gamma) = \varrho$. Since the C_γ are end-extensions of each other and b^* is unbounded in ϱ this implies that

$$\forall \eta < \varrho \exists \gamma \in b^* \setminus \eta : \operatorname{otyp}(C_\gamma) \geqslant \eta. \tag{5.11}$$

Now we are finally approaching contradiction. We can construct a sequence of ordinals $(\gamma_n | n < \omega)$ inductively as follows:

$$\begin{aligned} \gamma_0 &:= 0 \\ \gamma_{n+1} &:= \min\{\eta \in b^* \big| \operatorname{otyp}(C_\eta) \geqslant \gamma_n\}. \end{aligned} \tag{5.12}$$

Since $\operatorname{otyp}(C_\gamma) < \gamma$ for all $\gamma \in \mathrm{Lim} \setminus \mathrm{Reg}$ this sequence has to be properly ascending. Let $\zeta := \sup_{n<\omega} \gamma_n$. Since C^* is club ζ is an element of C^*. Obviously $\operatorname{cf}(\zeta) = \omega$ so C_ζ exists. Since b^* is cofinal we can choose an $\eta \in b^* \setminus \zeta$. Because γ_n is a limit point of C_η and $C_{\gamma_n} = C_\eta \cap \gamma_n$ for all $n < \omega$ it follows that ζ is a limit point of C_η and $C_\zeta = C_\eta \cap \zeta$. Let $\xi := \operatorname{otyp}(C_\zeta)$. $\xi < \zeta$ by the properties of the square-sequence. By definition of ζ there exists an $n < \omega$ such that $\operatorname{otyp}(C_{\gamma_n}) \geqslant \xi$. But $\gamma_n \preccurlyeq_\Box \zeta$—again by definition of ζ. Contradiction! ⊣

5.9. Definition. For trees $\mathcal{T} = (T, \leqslant_\mathcal{T})$ let $\mathbb{Q}_\mathcal{T} = (Q, \leqslant_\mathbb{Q})$ be the following notion of forcing:

$$
\begin{aligned}
Q := \big\{f \big| \quad & \operatorname{dom}(f) \in [T]^{<\omega} \wedge \operatorname{ran}(f) \subset \omega \\
& \wedge \forall n < \omega, s, t \in f^{-1}“(n)(s \not\leqslant_{\mathcal{T}} t \wedge t \not\leqslant_{\mathcal{T}} s)\big\} \\
p \leqslant_{\mathbb{Q}} q :\Longleftrightarrow & p \supset q
\end{aligned}
\tag{5.13}
$$

Subsequent occurences of $\mathbb{Q}_{\mathcal{T}}$ refer to this notion of forcing. We want to see that $\mathbb{Q}_{\mathcal{T}}$ satisfies the c.c.c. for a certain kind of trees, for this we recall the following...

5.10. Definition. Let $X \supsetneq \emptyset$. An Ultrafilter U over X is called *uniform* by definition iff $U \subset [X]^{\overline{\overline{X}}}$.

5.11. Lemma. Whenever $\mathcal{T}$ is a tree whose chains are all countable, $\mathbb{Q}_{\mathcal{T}}$ satisfies the countable chain condition.

Proof. Suppose $P \in [Q]^{\geqslant\omega_1}$. We have to show that P is no antichain so suppose it was one. $U := \big\{\operatorname{dom}(q)\big|q \in P\big\}$ is an uncountable family of finite sets so the Δ-system-lemma can be applied. Let $D \in [U]^{\geqslant\omega_1}$ be a Δ-system with root r and $P' := \big\{q \in P\big| \operatorname{dom}(q) \in D\big\}$. Since $\overline{\overline{\big\{q \in Q\big| \operatorname{dom}(q) = r\big\}}} < \aleph_1$ there is a $q \in Q$ with $\operatorname{dom}(q) = r$ such that $P_q := \big\{p \in P'\big|p \restriction r = q\big\}$ is uncountable. Fix such a q, let $n := \max\big\{l < \omega\big|\overline{\overline{\{p \in P_q|\overline{\overline{\operatorname{dom}(p \setminus q)}} < l\}}} < \aleph_1\big\}$ and $P^* := \big\{p \in P_q\big|\overline{\overline{\operatorname{dom}(p \setminus q)}} = n\big\}$. Clearly $\overline{\overline{P^*}} \geqslant \aleph_1$ and $P^* \subset P_q \subset P' \subset P$.
Now we have that

$$
\forall p, r \in P^* \exists x \in \operatorname{dom}(p \setminus q), y \in \operatorname{dom}(r \setminus q)(x \leqslant_{\mathcal{T}} y \vee y \leqslant_{\mathcal{T}} x). \tag{5.14}
$$

Otherwise for a counterexample $\{p, r\}$ we would have $p \cup r \leqslant_{\mathbb{Q}} p, r$ contradicting the fact that P^* is an antichain. For each $p \in P^*$ let $e_p : n \longleftrightarrow \operatorname{dom}(p \setminus q)$ be an enumeration. Define $O_l^x := \big\{p\big|p \in P^* \wedge \big(e_p(l) \leqslant_{\mathcal{T}} x \vee x \leqslant_{\mathcal{T}} e_p(l)\big)\big\}$ for every $x \in \bigcup_{p \in P^*} \operatorname{dom}(p \setminus q)$ and every $l < n$. Now because of (2) we can write O as a finite union for all $p \in P^*$:

$$
P^* = \bigcup_{x \in \operatorname{dom}(p \setminus q)} \bigcup_{l<n} O_l^x. \tag{5.15}
$$

Choose a uniform ultrafilter U over P^*. Then one can pick $x_p \in \operatorname{dom}(p \setminus q), l_p < n$ such that $O_{l_p}^{x_p} \in U$ for every $p \in P^*$. We have that

$$
\forall l < n\big(b_l := \big\{x_p\big|p \in P^* \wedge l_p = l\big\} \text{ is a chain in } \mathcal{T}.\big) \tag{5.16}
$$

Proof of (5.16):
Take $l < n$ and $y, z \in b_l$. Then $y = x_p, z = x_r$ and $l_p = l = l_r$ for $p, r \in P^*$. Because of $O_l^y, O_l^z \in U$ we have $O_l^y \cap O_l^z \in U$. U is uniform hence $\overline{\overline{O_l^y \cap O_l^z}} \geqslant \aleph_1$. We ensured that $\forall p, r \in P^* \big(\operatorname{dom}(p \setminus q) \cap \operatorname{dom}(r \setminus q) = \emptyset \vee p = q\big)$ so as a consequence $Z := \{e_p(l) | p \in O_l^y \cap O_l^z\}$ is uncountable. $Z \subset c^y \cup c^z \cup (T^y \cap T^z)$ but $\overline{\overline{c^y}}, \overline{\overline{c^z}} < \aleph_1$. So $\aleph_1 \leqslant \overline{\overline{Z \setminus (c^y \cup c^z)}} \leqslant \overline{\overline{T^y \cap T^z}}$ and $T^y \cap T^z \supsetneq \emptyset$ in particular. But then $y \leqslant_{\mathcal{T}} z \vee z \leqslant_{\mathcal{T}} y$. ⊣(5.16)
Let $m := \max\{l \leqslant n | \overline{\overline{\{p \in P^* | l_p < l\}}} < \aleph_1\}$. Then $m < n$ and b_m is an uncountable chain in $\mathcal{T}$. ↯ ⊣

5.12. Lemma. Suppose $\mathcal{T}$ is a tree with $\aleph_1$ elements whose branches are all countable and let G be $\mathbb{Q}_{\mathcal{T}}$-generic. Then $V[G] \models$ "$\mathcal{T}$ is special.".

Proof. Let $\mathcal{T}$ be as above and G be $\mathbb{Q}_{\mathcal{T}}$-generic. Then $\bigcup G$ is a specializing function for $\mathcal{T}$. As a union of functions it clearly is a function itself. Moreover, $\operatorname{dom}(\bigcup G) = T$ since $\mathcal{D}_s := \{q | q \in Q \wedge s \in \operatorname{dom}(q)\}$ is dense for every $s \in T$. On the other hand $\mathcal{D}_n := \{q | q \in Q \wedge q^{-1}``(n) \supsetneq \emptyset\}$ is dense for each $n < \omega$ so $\operatorname{ran}(f) = \omega$. Suppose counterfactually that there are $s, t \in T$ such that $(\bigcup G)(s) = (\bigcup G)(t)$, yet $s \leqslant_{\mathcal{T}} t$. First take a $p \in \mathcal{D}_s \cap G$, then a $q \in \mathcal{D}_t \cap G$ such that $q \leqslant_{\mathbb{Q}} p$. It follows that $q(s) = q(t)$. ↯ ⊣

Now we will prove the following theorem:

5.13. Theorem. (Saharon Shelah, 1995) If BAAFA holds, then $\aleph_2$ is reflecting in L.

In [G–S] Shelah proved that BPFA $\Rightarrow$ "$\aleph_2$ is reflecting in L". Later on Stevo Todorčević gave a simplified proof for this. In fact he already proved the theorem above. Before we will enter the proof some comments on the idea are given. If one looks at the formulation of Bounded Forcing Axioms in terms of forcing absoluteness it can be considered a straightforward idea to collapse κ below $\aleph_2$ with a proper notion of forcing—"$\exists \alpha < \aleph_2 : L_\alpha \models$ "$\varphi(a)$"" is a Σ_1–assertion because one typically formulates this assertion by restricting the quantifiers in φ to L_α and the function $\alpha \mapsto L_\alpha$ is Δ_1 by [Je 2], lemma 13.14. Unfortunately this direct approach does not work since we also require this α to be a cardinal (in L). Since being a cardinal is a Π_1-property we would end up with a Σ_2-assertion which does not lead us anywhere. The solution is to find a Σ_1-formalizable strengthening of the quality of being a cardinal in L.

Proof.(Stevo Todorčević) We are distinguishing two cases.

Case 1: $0^\#$ exists.

Then we are finished quickly. $\aleph_2$ is regular so all the more it is regular in L. But it is also Σ_2–correct in L by lemma 1.15. Finally lemma 1.8 implies that it has to be reflecting in L.

Case 2: $0^\#$ does not exist.

First recall that $H^L_\kappa = L_\kappa$ for all $\kappa \in \mathrm{Card}^L$. Suppose φ is a first order formula in the language of set theory, $a \in L_{\aleph_2}$ and $\kappa \in \big\{(\lambda^+)^L \big| \lambda \in \mathrm{Card}^L\big\} \setminus \aleph_2$ such that

$$L_\kappa \models \text{“}\varphi(a)\text{”}. \tag{5.17}$$

Let $\mathbb{P} := \mathrm{Fn}(\aleph_1, \kappa, \aleph_1)$ be the usual forcing which adds a surjection from $\aleph_1$ to κ with countable conditions. We have that $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ “$\overline{\overline{\kappa}} = \aleph_1$” so $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ “$\mathrm{cf}(\kappa) \leqslant \omega_1$”. But then $\mathbb{1}_\mathbb{P} \Vdash_\mathbb{P}$ “$\mathrm{cf}(\kappa) = \omega_1$” since Lemma 5.7 yields that $\mathrm{cf}^V(\kappa) \geqslant \omega_2$ and because of $\mathbb{P}$'s being σ–closed, forcing with $\mathbb{P}$ adds no new countable sets. So we arrive at

$$V[G] \models \text{“}\exists \kappa \in \mathrm{Reg}^L \cap \aleph_2 \big(\mathrm{cf}(\kappa) = \omega_1 \wedge L_\kappa \models \text{“}\varphi(a)\text{”}\big)\text{”}. \tag{5.18}$$

κ is a successor cardinal in L so in particular κ is not Mahlo in L. Let $C^* \in \mathfrak{P}(\kappa) \cap L$ be a club witnessing this, i.e. $C^* \cap \mathrm{Reg}^L = \emptyset$. Let $(\gamma_\alpha | \alpha < \aleph_1)$ be a properly ascending sequence of ordinals from $V[G]$ which is cofinal in κ. Then by setting $E := \big\{\min(C \setminus \gamma_\alpha) \big| \alpha < \aleph_1\big\}$ and $C := E \cup \lim(E)$ one can see that there is a club subset C of κ of order type ω_1 containing solely limit ordinals singular in L.

Lemma 5.8 says that

$$V[G] \models \text{“All branches of } \mathcal{T}_\Box \restriction C \text{ are countable.”}. \tag{5.19}$$

Now by lemma 5.12 we are able to add a specializing function with $\mathbb{Q}_{\mathcal{T}_\Box \restriction C}$. $\mathbb{Q}_{\mathcal{T}_\Box \restriction C}$ satisfies the countable chain condition by lemma 5.11. So let H be $\mathbb{Q}_{\mathcal{T}_\Box \restriction C}$-generic over $V[G]$. $\bigcup H$ is a specializing function for $\mathcal{T}_\Box \restriction C$. Consider the following statement:

$$\begin{aligned} \exists \kappa < \aleph_2, C, f : C \to \omega \big(a \in L_\kappa \wedge L_\kappa \models \text{“}\varphi(a)\text{”} \wedge C \cap \mathrm{Reg}^L = \emptyset \\ \wedge\, \mathrm{otyp}(C) = \omega_1 \wedge C \text{ is club in } \kappa \wedge f \text{ is a specializing function for } \mathcal{T}_\Box \restriction C\big). \end{aligned} \tag{5.20}$$

This is indeed a $\Sigma_1\big(\{\aleph_1\}\big)$-assertion. In order to see this notice that

- $\kappa < \aleph_2 \equiv \exists f : \aleph_1 \twoheadrightarrow \kappa$—this is $\Sigma_1\big(\{\aleph_1\}\big)$.

- C is unbounded in $\kappa \equiv \forall\beta < \kappa\, \exists\gamma < \kappa : \gamma \in C \setminus \beta$—this is $\Sigma_0\big(\{\kappa\}\big)$.

- C is closed in $\kappa \equiv \forall\beta < \kappa\big((C$ is unbounded in $\beta \wedge \beta \in \mathrm{Lim}) \rightarrow \beta \in C\big)$—again this is $\Sigma_0\big(\{\kappa\}\big)$.

- $\mathrm{otyp}(C) < \alpha$ is $\Delta_1\big(\{C, \alpha\}\big)$: It can be characterized as $\exists\beta < \alpha, f : C \longrightarrow \beta$: "f is order–preserving." but also as $\nexists f : \alpha \longrightarrow C$: "f is order–preserving."

  As a consequence $\mathrm{otyp}(C) = \alpha$ is $\Delta_1(\{C, \alpha\})$ too since it in turn can be characterized as $\mathrm{otyp}(C) < \alpha + 1 \wedge \mathrm{otyp}(C) \not< \alpha$.

- $C \cap \mathrm{Reg}^L = \emptyset \equiv \forall\alpha \in C\, \exists\beta, A\big(A \in L_\beta \wedge A \subset \alpha \wedge \mathrm{otyp}(A) < \alpha \wedge A$ is unbounded in $\alpha\big)$—this is $\Sigma_1\big(\{C\}\big)$ since the function $\beta \mapsto L_\beta$ is Σ_1 by [Je 2], lemma 13.14

- α is a limit point of $A \equiv \forall\beta < \alpha\, \exists\gamma \in A \cap \alpha \setminus (\beta + 1)$—this is $\Sigma_0(\{\alpha, A\})$.

- f is a specializing function for $\mathcal{T}_\Box \restriction C \equiv \mathrm{ran}(f) = \omega \wedge \mathrm{dom}(f) = C \wedge \forall\alpha, \beta \in C\big(\alpha$ is a limit point of $C_\beta \rightarrow f(\alpha) \neq f(\beta)\big)$, this is $\Sigma_1\big(\{\aleph_1\}\big)$ since C_β is $\Sigma_1\big(\{\aleph_1\}\big)$–definable from β by fact 5.5.

So we arrived at

$$V[G][H] \models \text{“(5.20)”}. \tag{5.21}$$

But (5.20) is a Σ_1–statement in the single parameter $\aleph_1 \in H_{\aleph_2}$ and BAAFA holds. $\mathbb{P}$ is countably closed and $\mathbb{Q}_{\mathcal{T}_\Box \restriction C}$ satisfies the countable chain condition. By the lemmata 2.20, 2.19 and 2.30 it follows that $\mathbb{P} \star \mathbb{Q}_{\mathcal{T}_\Box \restriction C} \in \mathcal{AA} \subset \mathcal{AA}^*$. So by corollary 3.6 (5.20) already holds in V. Finally we have to prove that the α mentioned in (5.20) is an L–cardinal. In fact it is even regular. This is not excessively surprising since we know by lemma 1.5 that in this context one can replace the requirement of the existence of an $\alpha \in \mathrm{Card}^L \cap \aleph_2$ with $L_\alpha \models$ "$\varphi(a)$" by the call for an $\alpha < \aleph_2$ which is a successor cardinal—and hence regular—in L with this property without limiting the scope of our notion of "reflecting cardinal".

So the κ from (5.20) is regular in L—Proof:
Suppose it was not. Then C_κ from the sequence corresponding to $\mathcal{T}_\Box$ exists. First notice that

$$\mathrm{cf}(\kappa) = \omega_1. \tag{5.22}$$

This is because C is club in κ and $\operatorname{otyp}(C) = \omega_1$. By definition $\operatorname{cf}(\kappa) \leqslant \omega_1$. If $\operatorname{cf}(\kappa) = \omega$ there would be a properly ascending sequence $(\gamma_n | < \omega)$ cofinal in κ. Then $C = \bigcup_{n<\omega}(C \cap \gamma_{n+1} \setminus \gamma_n)$. By the pigeonhole principle then there has to be an $n < \omega$ such that $\overline{\overline{C \cap \gamma_{n+1} \setminus \gamma_n}} = \aleph_1$. Since $\operatorname{otyp}(C) = \omega_1$ it follows that $C \setminus \gamma_{n+1} = \emptyset$. C is unbounded in κ so $\gamma_{n+1} \geqslant \kappa$.↯ ⊣(5.22)
Since $\kappa < \aleph_2$ we immediately have that

$$\overline{\overline{C_\kappa}} = \aleph_1. \tag{5.23}$$

Let now $\lim(C_\kappa)$ denote the set of limit points of C_κ. Then we have

$$\overline{\overline{C \cap \lim(C_\kappa)}} = \aleph_1. \tag{5.24}$$

(5.23) easily implies $\overline{\overline{C \cap \lim(C_\kappa)}} \leqslant \aleph_1$. For the other direction suppose $\operatorname{otyp}\big(C \cap \lim(C_\kappa)\big) < \omega_1$. Since $\operatorname{cf}(\kappa) = \omega_1$ it follows that $\beta_0 := \sup\big(C \cap \lim(C_\kappa)\big) < \kappa$. Now define inductively for $n < \omega$:

$$\gamma_n := \min(C_\kappa \setminus \beta_n), \tag{5.25}$$

$$\beta_{n+1} := \min\big(C \setminus (\gamma_n + 1)\big). \tag{5.26}$$

Let $\gamma_\omega := \sup_{n<\omega} \gamma_n$. $\gamma_\omega \in C \cap \lim(C_\kappa) \setminus (\beta_0 + 1)$.↯ ⊣ (5.24)
But it also is a fact that

$$C \cap \lim(C_\kappa) \text{ is a chain in } \mathcal{T}_{\kappa+\omega} \restriction C. \tag{5.27}$$

In order to see this let $\beta, \gamma \in C \cap \lim(C_\kappa)$ be different. W.l.o.g. one can assume that $\beta < \gamma$. Since both β and γ are limit points of C_κ it follows that $C_\beta = C_\kappa \cap \beta$ and $C_\gamma = C_\kappa \cap \gamma$. But then β is also a limit point of C_γ and $C_\beta = C_\gamma \cap \beta$—in other words, $\beta \preccurlyeq_{\mathcal{T}_\square \restriction C} \gamma$. ⊣ (5.27)
But now (5.27) immediately yields a contradiction since $f : C \longrightarrow \omega$ was supposed to be a specializing function yet by (5.24) and the pigeonhole principle there have to be different $\beta, \gamma \in C \cap \lim(C_\kappa)$ with $f(\beta) = f(\gamma)$. ⊣ ($\kappa \in \operatorname{Reg}^L$)
⊣

6 BAAFA **does not imply** BPFA

When one considers the fact that by the use of a reflecting cardinal BPFA can be forced while at the same time BAAFA suffices in order to have $\aleph_2$ reflecting in L, the question whether BAAFA $\Rightarrow$[1] BPFA is not a remote one. To put the question more formally: Is there a model of set theory which satisfies BAAFA but fails to satisfy BPFA? The question arises in particular because we have just seen that BAAFA and BPFA have the same consistency strength. If that would not be the case one simply could start with a large cardinal insufficient for forcing BPFA and force BAAFA—this is for example the situation with BMM and BPFA(or BMM and BSPFA)—see [Sch 2]. But if a ZFC–statement p implies a ZFC–statement q and both statements are consistent modulo the same large cardinal in general it is unclear whether $p \wedge \neg q$ is consistent and if yes modulo which large cardinal. As a trivial example p implies p but there is no model of $p \wedge \neg p$. Clearly the minimal consistency strength of $p \wedge \neg q$ is the consistency strength of $p, \neg q$ respectively.

After all—in this situation a reflecting cardinal suffices to arrive at a model of BAAFA $\wedge \neg$ BPFA. In order to see this recall the forcing which adds a set club below $\aleph_1$ with finite conditions. An important fact in this respect is that it is absolute between transitive models with the same $\aleph_1$.

6.1. Lemma. Let $\mathbb{P}$ be the forcing adding a set club below $\aleph_1$ with finite conditions—see example 2.25. "$p \in \mathbb{P}$" is a $\Delta_1(\{\aleph_1\})$-relation.

Proof. $\mathbb{P}$ has been defined as follows:

$$\begin{aligned}\mathbb{P} := \big\{p \big| p \text{ is a function with } \operatorname{dom}(p) \in [\aleph_1]^{<\omega} \text{ and } \operatorname{ran}(p) \subset \aleph_1 \\ \text{such that there is a normal function with} \\ \text{domain } \aleph_1 \text{ and range cofinal in } \aleph_1 \text{ extending } p.\big\}.\end{aligned} \tag{6.1}$$

- In order to see that "$p \in \mathbb{P}$" is a Σ_1–relation it suffices to analyse the original definition from example 2.25 which we restated in (6.1). So note that...

[1] For once in mathematics "$\Rightarrow$" here does not stand for the material implication.

- "$\mathrm{dom}(p) \in [\aleph_1]^{<\omega}$" if and only if "$\exists n < \omega, f : n \twoheadrightarrow \mathrm{dom}(p)$" and that f's being onto can be expressed by a general quantifier bounded by $\mathrm{dom}(p)$.
- "There is a normal function with domain $\aleph_1$ and range cofinal in $\aleph_1$ extending p." can be expressed as follows:

$$\exists f \supset p\Big(f \in \mathrm{Func} \wedge \forall \alpha, \beta \in \mathrm{dom}(f)\big(\alpha < \beta \to f(\alpha) < f(\beta)\big) \\ \wedge \forall \alpha \in \mathrm{Lim} \cap \mathrm{dom}(f), \beta < f(\alpha) \exists \gamma < \alpha : f(\gamma) > \beta\Big). \tag{6.2}$$

- It seems to be somewhat more involved to find a Π_1–formulation for "$p \in \mathbb{P}$". The Π_1–definition provided below relies on the fact that whether a normal function extending a given p exists is mainly a question of ordinal distance. First note that "$\mathrm{dom}(p) \in [\aleph_1]^{<\omega}$" if and only if "$\nexists f : \omega \hookrightarrow \mathrm{dom}(p)$" which is a Π_1–assertion since f's being one–to–one is expressed by a general quantifier. So let p be a function such that $\mathrm{dom}(p) \in [\aleph_1]^{<\omega}$ and $\mathrm{ran}(p) \subset \aleph_1$.

  Claim: There is a normal function extending p if and only if there is one extending $\big\{\big(\alpha, p(\alpha)\big), \big(\beta, p(\beta)\big)\big\}$ for every $\alpha, \beta \in \mathrm{dom}(p)$ such that $\alpha < \beta$ and $\nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma < \beta$.

  Proof:

  - Necessity is trivial for if there is a normal function extending p then clearly this normal function extends every subset of p too.
  - In order to prove sufficiency let $n := \overline{\overline{\mathrm{dom}(p)}}$ and $e : n \longleftrightarrow \mathrm{dom}(p)$ be an order–preserving enumeration. For $m < n$ let $f_m \supset \big\{\big(e(m), (p \circ e)(m)\big), \big(e(m+1), (p \circ e)(m+1)\big)\big\}$ be a normal function. Then clearly the following is a normal function extending p:

$$f : \aleph_1 \longrightarrow \aleph_1 \tag{6.3}$$
$$\alpha \longmapsto \begin{cases} f_0(\alpha) & \text{iff } \alpha \leqslant e(1) \\ \vdots & \\ f_m(\alpha) & \text{iff } e(m) < \alpha \leqslant e(m+1) \\ \vdots & \\ f_{n-1}(\alpha) & \text{iff } e(n-1) < \alpha. \end{cases}$$

    Each f_m is continuous. Whenever one considers an infinitely ascending sequence of countable ordinals the pigeonhole principle implies that infinitely—and hence cofinally—many of them fall into an intervall on which f is defined to be identical with some f_m. The limit of these ordinals then also lies in this very intervall. So f is continuous too. Moreover suppose that $\alpha < \beta$. Then there is an $m < n$ such that $\alpha \leqslant e(m+1) < \beta$. Hence $f(\alpha) = f_m(\alpha) \leqslant (f_m \circ e)(m+1) = (f_{m+1} \circ e)(m+1) < f_{m+1}(\beta) = f(\beta)$.

Now the following formula provides a Π_1–formulation for "$p \in \mathbb{P}$":

$$\forall \alpha, \beta \in \mathrm{dom}(p)\big((\alpha < \beta \wedge \nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma \wedge \gamma < \beta) \rightarrow \big(p(\alpha) < p(\beta) \wedge \beta \leqslant p(\beta) \wedge \varphi(\alpha, \beta) \wedge (\beta \in \mathrm{Lim} \rightarrow \psi(\alpha, \beta))\big)\big). \tag{6.4}$$

We are going to show that there exists a normal function with domain $\aleph_1$ and range cofinal in $\aleph_1$ extending p if and only if (6.4) holds for suitable Π_1–formulae φ, ψ. To this end suppose that $p(\alpha) < p(\beta)$ and $\beta \leqslant p(\beta)$ hold true for all $\alpha, \beta \in \mathrm{dom}(p)$ such that $\nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma < \beta$—otherwise the sought–after normal function could not exist for trivial reasons. We define φ and ψ as follows:

$$\varphi \equiv \forall \gamma < \beta \nexists g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha : \text{"}g \text{ is order–preserving."}, \tag{6.5}$$

$$\psi \equiv \forall \gamma < p(\beta) \exists \eta < \beta \forall \zeta < \beta \nexists g : p(\beta) \setminus \gamma \rightarrow \zeta \setminus \eta : \text{"}g \text{ is order–preserving."}. \tag{6.6}$$

One cannot simply dispense with the involved extra–treatment of limit ordinals. Whereas φ suffices as a characterization if β is a successor ordinal it does not if it is a limit ordinal—clearly $p(\beta)$ then has to be a limit ordinal too. But even this does not suffice—there is for example no normal function extending $\{(\omega^2, \omega^2 + \omega)\}$.

– Suppose there are $\alpha, \beta \in \mathrm{dom}(p)$ with $\nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma < \beta$ such that (6.5) fails although there is a normal function extending p. Let us call this normal function f. By the failure of (6.5) let $\gamma < \beta$ and $g : p(\beta) \setminus p(\alpha) \longrightarrow \gamma \setminus \alpha$ be order–preserving. Then $g \circ (f \restriction \beta) : \beta \setminus \alpha \longrightarrow \gamma \setminus \alpha$ is order–preserving and so one is able to define an order–preserving function

$$h : \beta \longrightarrow \gamma \qquad \eta \longmapsto \begin{cases} \eta & \text{iff } \eta < \alpha \\ \big(g \circ (f \restriction \beta)\big)(\eta) & \text{otherwise.} \end{cases} \tag{6.7}$$

yet $\gamma < \beta$—↯

- Suppose furthermore that there are $\alpha \in \mathrm{dom}(p)$, $\beta \in \mathrm{Lim} \cap \mathrm{dom}(p)$ with $\nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma < \beta$ and that there is a normal function f extending $\{(\alpha, p(\alpha)), (\beta, p(\beta))\}$ although $\psi(\alpha, \beta)$ fails. By the failure of $\psi(\alpha, \beta)$ choose a $\gamma < p(\beta)$ such that

$$\forall \eta < \beta \exists \zeta < \beta, g : p(\beta) \setminus \gamma \longrightarrow \zeta \setminus \eta : \text{“}g \text{ is order–preserving.”}. \tag{6.8}$$

  Since as a normal function f is continuous in β one can define $\eta := \min \{\xi | \xi < \Omega \wedge f(\xi) \geqslant \gamma\}$. By (6.8) let $\zeta < \beta$ and g be an order–preserving function from $p(\beta) \setminus \gamma$ into $\zeta \setminus \eta$. Then $g \circ (f \restriction (\beta \setminus \eta))$ is an order–preserving function from $\beta \setminus \eta$ into $\zeta \setminus \eta$—↯ (—one attains contradiction just as in (6.7))

- Now suppose that (6.5) holds for all $\alpha, \beta \in \mathrm{dom}(p)$ such that $\nexists \gamma \in \mathrm{dom}(p) : \alpha < \gamma < \beta$ yet there is no normal function extending p. The claim implies that there is a least $\alpha \in \mathrm{dom}(p)$ such that with $\beta := \min(\mathrm{dom}(p) \setminus (\alpha + 1))$ there is no normal function extending $\{(\alpha, p(\alpha)), (\beta, p(\beta))\}$. We distinguish two cases:

  * β is a successor ordinal. (6.5) expresses that $\mathrm{otyp}(p(\beta) \setminus p(\alpha)) > \mathrm{otyp}(\gamma \setminus \alpha)$ for every $\gamma < \beta$—in other words: $\mathrm{otyp}(p(\beta) \setminus p(\alpha)) \geqslant \mathrm{otyp}(\beta \setminus \alpha)$. But then there has to be an order–preserving function from $\beta \setminus \alpha$ into $p(\beta) \setminus p(\alpha)$. In particular

$$\begin{aligned} g : \beta \setminus \alpha &\longrightarrow p(\beta) \setminus p(\alpha) \\ \eta &\longmapsto p(\alpha) + (\eta - \alpha) \end{aligned} \tag{6.9}$$

    is such a function. Note that it is in particular continuous at limit ordinals. The choice of α implies that there is a normal function f^* extending $p \restriction \beta$. But then

$$\begin{aligned} f : \aleph_1 &\longrightarrow \aleph_1 \\ \eta &\longmapsto \begin{cases} f^*(\eta) & \text{iff } \eta < \alpha \\ g(\eta) & \text{iff } \eta \in \beta \setminus \alpha \\ p(\beta) + (\eta - \beta) & \text{otherwise} \end{cases} \end{aligned} \tag{6.10}$$

    is a normal function extending $p \restriction (\beta + 1)$ contradicting the choice of α.

* β is a limit ordinal. First note that $\psi(\alpha, \beta)$ implies in particular that $p(\beta)$ is a limit ordinal. Now we choose a properly ascending sequence $(\chi_n | n < \omega)$ of countable ordinals cofinal in $p(\beta)$. By the choice of α there exists a normal function f^* extending $p \restriction \beta$. For $\xi < p(\beta)$ let $n(\xi)$ denote the minimal n such that $\chi_n \geqslant \xi$. Define inductively[2]

$$f : \aleph_1 \longrightarrow \aleph_1 \tag{6.11}$$
$$\xi \longmapsto \begin{cases} f^*(\xi) & \text{iff } \xi \leqslant \alpha \\ \sup_{\vartheta < \xi} f(\vartheta) & \text{iff } \xi \in \mathrm{Lim} \cap \beta \setminus (\alpha + 1) \\ \chi_{(n \circ f)(\vartheta)} & \text{iff } q \\ f(\vartheta) + 1 & \text{iff } r \\ p(\beta) + (\xi - \beta) & \text{otherwise, i.e. iff } \xi \geqslant \beta. \end{cases}$$

Here q and r stand for the following cases:

$$\begin{aligned} q : \xi &= \vartheta + 1 \text{ for some } \vartheta \in \beta \setminus \alpha \text{ and} \\ &\mathrm{otyp}(\beta \setminus \vartheta) \leqslant \mathrm{otyp}\big(p(\beta) \setminus \chi_{(n \circ f)(\vartheta)}\big). \\ r : \xi &= \vartheta + 1 \text{ for some } \vartheta \in \beta \setminus \alpha \text{ and} \\ &\mathrm{otyp}(\beta \setminus \vartheta) > \mathrm{otyp}\big(p(\beta) \setminus \chi_{(n \circ f)(\vartheta)}\big). \end{aligned}$$

Now f really is a normal function with domain $\aleph_1$ and range cofinal in $\aleph_1$ extending $p \restriction (\beta + 1)$. This is immediately clear by the definition on $\alpha + 1$ and on $\aleph_1 \setminus \beta$. For the part of f's domain lying in between note the following:

- The second case in (6.11) ensures that f is continuous at limit ordinals.
- Furthermore take note of the fact that $\varphi(\alpha, \beta)$ yields $\mathrm{otyp}(\beta \setminus \alpha) \leqslant \mathrm{otyp}\big(p(\beta) \setminus p(\alpha)\big)$. The discrimination between cases q and r in (6.11) inductively ensures that $\forall \xi \in \beta \setminus \alpha : \mathrm{otyp}(\beta \setminus \xi) \leqslant \mathrm{otyp}\big(p(\beta) \setminus f(\xi)\big)$ and hence in particular $f``(\beta \setminus \alpha) \subset p(\beta) \setminus p(\alpha)$.
- Suppose $f``\beta$ would not be cofinal in $p(\beta)$. Let $n < \omega$ be minimal

[2] For the clause in the middle note that we can conceive of n as a function $n : p(\beta) \longrightarrow \omega$.

such that $\chi_n \geqslant \sup(f``\beta)$. Then for χ_n in the role of γ in ψ we attain

$$\exists\eta < \beta\forall\zeta < \beta\nexists g : p(\beta) \setminus \chi_n \to \zeta \setminus \eta : \text{“}g \text{ is order–preserving.”}. \tag{6.12}$$

Choose such an $\eta < \beta$. Then

$$\forall\zeta < \beta : \operatorname{otyp}\big(p(\beta) \setminus \chi_n\big) > \operatorname{otyp}(\zeta \setminus \eta). \tag{6.13}$$

But this means that $\operatorname{otyp}\big(p(\beta) \setminus \chi_n\big) \geqslant \operatorname{otyp}(\beta \setminus \eta)$. If one now considers a successor ordinal $\xi \in \beta \setminus \eta$ which is large enough such that there is no $m < n$ fulfilling $\chi_m \geqslant f(\xi)$ one sees that by case q in (6.11) one must have $f(\xi) \geqslant \chi_n$ and hence $f(\xi + 1) > \chi_n$—$\lightning$

But f should not exist in light of our choice of α—$\lightning$

$\dashv$

6.2. Corollary. Let still $\mathbb{P}$ be the forcing adding a club below $\aleph_1$ with finite conditions. $\mathbb{P}$ is identical in every inner model and in every forcing extension which share their $\aleph_1$.

Proof. Let M, N be transitive models of ZFC, $\aleph_1^M = \aleph_1^N$ and $M \subset N$. M could be an inner model of N or N a forcing extension of M. Since “$p \in \mathbb{P}$” is Δ_1 by lemma 6.1 we have that “$X \subset \mathbb{P}$” is $\Delta_1(\{\aleph_1, X\})$ and hence absolute between M and N. Suppose towards a contradiction that $\mathbb{P}^M \neq \mathbb{P}^N$. Then obviously $\mathbb{P}^M \subsetneq \mathbb{P}^N$. Choose $p \in \mathbb{P}^N \setminus \mathbb{P}^M$. $p \in [\aleph_1]^{<\omega}$ so obviously $p \in M$. Since $p \in \mathbb{P}$ is $\Delta_1(\{p, \aleph_1\})$ it follows that $p \in \mathbb{P}^M$. Contradiction! $\dashv$

The idea behind the construction of a model of BAAFA $\wedge\neg$ BPFA is to force BAAFA with some forcing satisfying Axiom A* and then to argue that BPFA cannot hold in the generic extension. For this we need the following lemma:

6.3. Lemma. To satisfy Axiom A is a Σ_2-property.

Proof. Let $\mathbb{P} = (P, \leqslant_{\mathbb{P}})$ be a poset. First recognize the following facts:

- That a set D is dense below some condition $p \in \mathbb{P}$ is $\Sigma_0(\{D, \mathbb{P}\})$—as the following formula shows:

$$\forall q \in P(q \leqslant_{\mathbb{P}} p \to \exists r \in D : r \leqslant_{\mathbb{P}} q). \tag{6.14}$$

- That a set C is countable is $\Sigma_1(\{C\})$—the following formula says that C is countable:

$$\exists f\big(f \text{ is a function with} \operatorname{dom}(f) = \omega \wedge \forall c \in C \exists n < \omega \colon f(n) = c\big). \tag{6.15}$$

The following formula describes what it means for $\mathbb{P}$ to satisfy Axiom A—remember lemma 2.18:

$$\begin{aligned} &\exists X, (\leqslant_n | n < \omega) \\ &\Big(\forall S \subset P \colon S \in X \wedge P \times P \supset \leqslant_{\mathbb{P}} = \leqslant^0 \wedge \forall n < \omega : \leqslant^{n+1} \subset \leqslant^n \wedge \\ &\forall (p_n | n < \omega)\big(\forall n < \omega(p_n \in P \wedge p_{n+1} \leqslant^n p_n) \\ &\to \exists p_\omega \in P(\forall n < \omega : p_\omega \leqslant^n p_n)\big) \wedge \\ &\forall p \in P, n < \omega, D \in X\big(D \text{ is dense below } p \\ &\to \exists q \in P, C \in X(q \leqslant^n p, C \subset D \text{ is countable and predense below } q)\big)\Big). \end{aligned} \tag{6.16}$$

Here in the first line there is an unbounded existential quantifier while the two lines following start with an unbounded general quantifier. All other quantifiers appearing in this formula are bounded. Being countable is Σ_1 but the corresponding assertion in the formula above is not preceded by any unbounded general quantifier. Hence it is $\Sigma_2(\{\mathbb{P}\})$. This is also the reason for the reference to P's powerset. One would simplify the notation at the beginning at the price of ending up with an unbounded general quantifier preceding the assertion of the countability of C by dispensing with it. $\dashv$

In fact this result is not quite enough. One also has to contemplate the following fact—which is not very deep.

6.4. Lemma. Being a complete Boolean algebra is a Π_1–assertion.

Proof. Let $\mathbb{B} = (B, \mathbb{0}_{\mathbb{B}}, \mathbb{1}_{\mathbb{B}}, \neg_{\mathbb{B}}, \wedge_{\mathbb{B}}, \vee_{\mathbb{B}})$ be a Boolean algebra. That $\mathbb{B}$ is a Boolean algebra is $\Sigma_0\big(\{\mathbb{B}\}\big)$ whenever the ordered pair is reasonably coded. It is complete iff the infinitary product exists for every subset of its domain B. The following formula says that this state of affairs obtains:

$$\forall X\big(X \subset B \rightarrow \exists b \in B \forall x \in X : b \wedge_{\mathbb{B}} x = b\big) \tag{6.17}$$

So being a complete Boolean algebra is a Π_1–property. ⊣

6.5. Corollary. To satisfy Axiom A* is a Σ_2–property.

Proof. Let $\mathbb{P}$ be a notion of forcing. $\mathbb{P}$ satisfies Axiom A* iff the following holds:

$$\exists \mathbb{B}, \mathbb{Q}, \delta_{\mathbb{P}}, \delta_{\mathbb{Q}}\big(\mathbb{B} \text{ is a complete Boolean algebra, } \mathbb{Q} \text{ is a notion of forcing satisfying Axiom A and } \delta_{\mathbb{P}} : \mathbb{P} \longrightarrow \mathbb{B}, \delta_{\mathbb{Q}} : \mathbb{Q} \longrightarrow \mathbb{B} \text{ are dense embeddings.}\big) \tag{6.18}$$

Now since being a complete Boolean algebra is Π_1 by lemma 6.4, being a notion of forcing satisfying Axiom A is Σ_2 by lemma 6.3 and the assertions about $\delta_{\mathbb{P}}$ and $\delta_{\mathbb{Q}}$ in (6.18) are all Σ_0 the proof is finished. ⊣

6.6. Lemma. Let $(\alpha_n | n < \omega)$ be a sequence of countable indecomposable ordinals and $(\beta_n | n < \omega)$ a sequence of ordinals such that $\forall n < \omega : \beta_n < \alpha_{n+1}$. Consider the forcing $\mathbb{Q}$ which adds a club with finite conditions below $\aleph_1$. The following sets are dense in $\mathbb{Q}$:

$$D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}} := \big\{q \big| q \in \mathbb{Q} \wedge \exists n < \omega, \gamma \in \aleph_1 \setminus \beta_n : (\alpha_n, \gamma) \in q\big\}. \tag{6.19}$$

Proof. Choose any $p \in \mathbb{Q}$. Let $\alpha_\omega := \sup_{n<\omega} \alpha_n$, $\eta := \max\big(\mathrm{dom}(p) \cap \alpha_\omega\big)$ and $n := \min\big\{m \big| m < \omega \wedge \eta < \alpha_m\big\}$. Let f witness that $p \in \mathbb{Q}$ by being a normal function extending p. We distinguish two cases:

- $f(\alpha_n) \geqslant \beta_n$. Then instantaneously all is well. Set $q := p \cup \big\{\big(\alpha_n, f(\alpha_n)\big)\big\}$. f witnesses that $q \in \mathbb{Q}$.
- $f(\alpha_n) < \beta_n$. Then set $q := p \cup \{(\alpha_n, \beta_n + \alpha_n)\}$. One can define a normal function

g extending q as follows:

$$g : \aleph_1 \longrightarrow \aleph_1, \tag{6.20}$$
$$\delta \longmapsto \begin{cases} f(\delta) & \text{iff } \delta \in \big(\eta+1\big) \cup \big(\aleph_1 \setminus (\alpha_{n+1}+1)\big) \\ \beta_n + \delta & \text{iff } \delta \in (\alpha_{n+1}+1) \setminus (\eta+1). \end{cases}$$

g clearly is a normal function on $\eta+1$, $(\alpha_{n+1}+1) \setminus (\eta+1)$ and $\aleph_1 \setminus (\alpha_{n+1}+1)$ because f and $\delta \mapsto \beta_n + \delta$ are normal functions. Moreover we have

$$g(\eta) = f(\eta) < f(\alpha_n) < \beta_n < \beta_n + \eta + 1 = g(\eta+1) \tag{6.21}$$

$$\text{and } g(\alpha_{n+1}) = \beta_n + \alpha_{n+1} = \alpha_{n+1} \tag{6.22}$$
$$< \alpha_{n+1} + 1 \leqslant f(\alpha_{n+1}+1) = g(\alpha_{n+1}+1).$$

In any of both cases $q \in \mathbb{Q} \cap D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$ is an extension of p. Since p was arbitrarily chosen this shows the density of $D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$. ⊣

We now present a variation of our forcing construction from chapter 4:

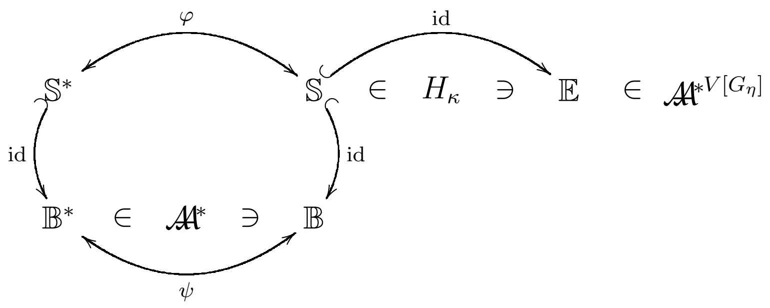
6.7. Theorem. (Thilo Weinert, 2007) Let κ be a reflecting cardinal. Then there is a forcing $\mathbb{P} \in \mathcal{AA}^*$ that satisfies the κ–c.c. such that whenever G is $\mathbb{P}$-generic

$$V[G] \models \text{“ZFC} + \text{BAAFA} + \neg\,\text{BPFA} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2\text{”}. \tag{6.23}$$

Proof. The first part of the proof is almost identical to the proof of theorem 4.6. Simply substitute every occurrence of “proper forcing notion” in the proof of theorem 4.6 by “forcing notion from $\mathcal{AA}^*$”. Then corollary 2.39 takes the role of theorem 2.16 and corollary 6.5 takes the role of lemma 2.10. This shows that

$$V[G] \models \text{“ZFC} + \text{BAAFA} + 2^{\aleph_0} = 2^{\aleph_1} = \aleph_2\text{”}. \tag{6.24}$$

We provide the following diagram in order to call into mind the idea of the proof:



We will now show $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa}$ "$\neg$ BPFA". Suppose towards a contradiction that this is wrong, i.e. suppose there was a $p \in \mathbb{P}_\kappa$ such that $p \Vdash_{\mathbb{P}_\kappa}$ "BPFA". Consider now the forcing from example 2.25 that adds a club with finite conditions below $\aleph_1$. In this proof let us call it $\mathbb{Q}$.

$$\mathcal{D} := \{D \big| D \subset \mathbb{Q} \wedge D \text{ is dense.}\} \tag{6.25}$$

We are indeed analysing this very set defined in our ground model V and **not** the set defined by the same conditions in the generic extension. $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa}$ "$\forall D \in \check{\mathcal{D}} : \overline{\overline{D}} < \aleph_2$" since every $D \in \mathcal{D}$ is a subset of $\mathbb{Q}$ and hence $(\overline{\overline{D}} < \aleph_2)^V$ for all $D \in \mathcal{D}$ already. Also $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa}$ "$\overline{\overline{\check{\mathcal{D}}}} < \aleph_2$" since $\overline{\overline{\mathbb{Q}}} = \aleph_1$ and if λ is the least inaccessible $\overline{\overline{\mathcal{D}}} \leqslant 2^{\aleph_1} < \lambda < \kappa$. Note that one can argue as in the proof of theorem 4.6 in order to see that $\mathbb{P}_\kappa$ adds a surjection from $\aleph_1$ to each $\alpha < \kappa$. Now as usual let $G \ni p$ be a $\mathbb{P}_\kappa$–generic filter. Corollary 6.2 yields that $\mathbb{Q}$ still can be defined as "the forcing notion adding a club with finite conditions below $\aleph_1$" in $V[G]$. In conjuction with lemma 2.29 this in particular yields that $\mathbb{Q}$ remains proper in $V[G]$. Now let $\mathbb{B}$ be the regular open algebra of $\mathbb{Q}$ calculated in $V[G]$ and let $\delta : \mathbb{Q} \longrightarrow \mathbb{B}$ be the corresponding dense embedding. Let furthermore $\mathcal{D}_\mathbb{B} := \{\delta``D \big| D \in \mathcal{D}\}$. That a set D is dense in a poset $\mathbb{Q}$ is $\Sigma_0\big(\{D, \mathbb{Q}\}\big)$ hence $\mathbb{1}_{\mathbb{P}_\kappa} \Vdash_{\mathbb{P}_\kappa}$ "$\forall D \in \check{\mathcal{D}}$: "D is dense in $\mathbb{Q}$."". But then $V[G] \models$ "$\overline{\overline{\mathcal{D}_\mathbb{B}}} < \aleph_2 \wedge \forall D \in \mathcal{D}_\mathbb{B}(\overline{\overline{D}} < \aleph_2$ and D is dense in $\mathbb{B}$.)". Moreover $\mathbb{B}$ is proper. So after all it must be the case that

$$p \Vdash_{\mathbb{P}_\kappa} \text{"}\exists H : H \text{ is a } \check{\mathcal{D}}_\mathbb{B}\text{-generic filter over } \mathbb{B}.\text{"}. \tag{6.26}$$

Now we are going to show that with the help of such a filter H one can define a certain normal function. In order to do this note that the sets

$$D^*_\alpha := \{q \in \mathbb{Q} \big| \alpha \in \operatorname{dom}(q)\}. \tag{6.27}$$

are dense in $\mathbb{Q}$. Now in the generic extension we define our normal function as follows:

$$\begin{aligned} f : \aleph_1 &\longrightarrow \aleph_1 \\ \alpha &\longmapsto \text{the unique } \beta < \aleph_1 \text{ such that } \exists q \in \mathbb{Q}\big(\alpha \in \operatorname{dom}(q) \wedge q(\alpha) = \beta \wedge \delta(q) \in H\big) \end{aligned} \tag{6.28}$$

Note the following:

- This is indeed well–defined. On the one hand there always is such a β because $\delta``D^*_\alpha \in \mathcal{D}_{\mathbb{B}}$. On the other hand: If $\beta, \gamma < \aleph_1$ are given such that $\exists q, r \in \mathbb{Q}\big(\alpha \in \operatorname{dom}(q) \cap \operatorname{dom}(r) \wedge q(\alpha) = \beta \wedge r(\alpha) = \gamma \wedge \delta(q), \delta(r) \in H\big)$ then $\delta(q), \delta(r)$ are compatible in $\mathbb{B}$ and hence q, r are compatible in $\mathbb{Q}$. But then $q(\alpha) = r(\alpha)$ which in turn implies $\beta = \gamma$.

- Let $\alpha < \beta$ and suppose towards a contradiction that $f(\alpha) \geqslant f(\beta)$. By definition of f this means that there are $q, r \in \mathbb{Q}$ such that $q(\alpha) \geqslant r(\beta)$ yet $\delta(q), \delta(r) \in H$. So $\delta(q) \|_{\mathbb{B}^+} \delta(r)$ and hence $q \|_{\mathbb{Q}} r$. This means that we can take a witness $s \leqslant_{\mathbb{Q}} q, r$ for this. But this is absurd since $s(\alpha) \geqslant s(\beta)$ although there should be a normal function extending s.

- Let α be a limit ordinal. We want to show that $f(\alpha) \leqslant \sup_{\beta<\alpha} f(\beta)$. So let $\beta \in f(\alpha)$ be arbitrarily chosen. We have to find a $\gamma < \alpha$ such that $f(\gamma) \geqslant \beta$. By definition of f let $q \in \mathbb{Q}$ be such that $\alpha \in \operatorname{dom}(q)$ and $\delta(q) \in H$. Consider the following set, defined in V!

$$D := \big\{ r \in \mathbb{Q} \big| q \perp_{\mathbb{Q}} r \vee (r \leqslant q \wedge \operatorname{ran}(r) \cap q(\alpha) \setminus \beta \supsetneq \emptyset) \big\} \tag{6.29}$$

  Note that D is dense in $\mathbb{Q}$. So we can take an $r \in D$ such that $\delta(r) \in H$. Since $\delta(q) \|_{\mathbb{B}^+} \delta(r)$ the definition of D shows $r \leqslant_{\mathbb{Q}} q$ as well as the existence of an $\eta \in \operatorname{ran}(r) \cap q(\alpha) \setminus \beta$. Set $\gamma := r^{-1}(\eta)$ then obviously $\gamma < \alpha$ and r is a witness to the fact that $f(\gamma) \geqslant \beta$.

We now start a play of the strengthened proper game from chapter 2 in $\mathbb{P}_\kappa$ below p. Since $\mathbb{P}_\kappa$ satisfies Axiom A* we know by clause (1) of lemma 2.27 that Player II must have a winning strategy in this game. We will now show that under the current presuppositions Player I has a winning strategy in this game. This will be our contradiction.

The game proceeds as follows:

- In the first move, I plays a name for $f(0)$.

- In the n^{th} move II plays a $B_n \in [\aleph_1]^{<\omega_1}$.
- In move $n+1$ I chooses an indecomposable ordinal $\alpha_{n+1} \in \aleph_1 \setminus (\sup B_n + 1)$ and plays a name for $f(\alpha_{n+1})$.

This game yields a sequence of indecomposable countable ordinals $(\alpha_n | n < \omega)$—here α_0 is just zero. By setting $\beta_n := \sup(B_n)+1$ for every $n < \omega$ one gets a sequence $(\beta_n | n < \omega)$ with the property that $\forall n < \omega : \beta_n < \alpha_{n+1}$. So we can consider $D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$ from (6.19). Lemma 6.6 tells us that $D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$ is dense. Our play of the strengthened proper game took place in V so $D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}} \in \mathcal{D}$. Hence if Λ is a name for H and $\dot{\delta}$ is a name for δ we get

$$p \Vdash_{\mathbb{P}_\kappa} \text{“}\Lambda \cap \dot{\delta}\text{“}\check{D}^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}} \supsetneq \emptyset\text{”}. \tag{6.30}$$

Now let μ be a $\mathbb{P}_\kappa$–name for f. We will show that

$$p \Vdash_{\mathbb{P}_\kappa} \text{“}\exists n < \omega : \mu(\check{\alpha_n}) \geqslant \check{\beta}_n\text{”}. \tag{6.31}$$

To this end choose $r \leqslant_{\mathbb{P}_\kappa} p$ arbitrarily. Because of (6.30) there is a $t \leqslant_{\mathbb{P}_\kappa} r$ and a $q \in D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$ such that $t \Vdash_{\mathbb{P}_\kappa}$ “$\dot{\delta}(\check{q}) \in \Lambda$”. By definition of $D^{(\alpha_n)_{n<\omega}}_{(\beta_n)_{n<\omega}}$ there are $n < \omega, \gamma \in \aleph_1 \setminus \beta_n$ such that $(\alpha_n, \gamma) \in q$. So by definition of f it follows that

$$t \Vdash_{\mathbb{P}_\kappa} \text{“}\mu(\check{\alpha_n}) = \check{\gamma}\text{”}. \tag{6.32}$$

Since r was arbitrarily chosen this shows (6.31). Since $B_n \subset \beta_n$ for every $n < \omega$ (6.31) implies that I has a winning strategy in the strengthened proper game in $\mathbb{P}_\kappa$ below p. But $\mathbb{P}_\kappa \in \mathcal{AA}^*$, so II had to have a winning strategy. ↯ $\dashv$

7 History and questions

7.1 Some history

In the introduction we gave a compact overview of Bounded Forcing Axioms. We did, however, not touch the subject of the consequences of Forcing Axioms on the class of possible cardinalities of the continuum. Martin's Axiom does not decide the size of the continuum. A forcing axiom which does—although this was not recognized at the time—first appeared fourteen years after Martin's article in [Ba 2]—the Proper forcing axiom. There it is introduced and moreover proven consistent modulo the existence of a supercompact cardinal. Then in 1988 Martin's Maximum is formulated in [F–M–S] and also proven consistent—again modulo the existence of a supercompact cardinal. It is also shown in this paper that MM $\Rightarrow \mathfrak{c} = \aleph_2$. Later on it was established—see [Be] and [V]—that indeed PFA suffices for this. Roughly at the same time Sakaé Fuchino wrote the article [Fu] in which he showed that for a large class of classes of forcing notions[1]—which in particular includes the class of proper forcing notions—BFA$(\mathcal{C}, \kappa, \Omega)$ is equivalent to the following statement:

$$\begin{gathered}\text{For any two structures } \mathcal{A}, \mathcal{B} \text{ of size at most } \kappa, \text{ if an} \\ \text{embedding from } \mathcal{A} \text{ into } \mathcal{B} \text{ can be forced to exist by a} \\ \text{forcing notion from } \mathcal{C} \text{ then some such embedding exists.}\end{gathered} \tag{7.1}$$

Fuchino also discussed whether for some classes of forcing notions "embedding" might be replaced by "isomorphism" without weakening the statement. For the class of proper forcing notions this was answered in the negative by Goldstern and Shelah in [G–S] three years later. In this undertaking the bounded proper forcing axiom was formulated and proven to be equiconsistent to the existence of a reflecting cardinal. In the year 2000 Bagaria's article [Bag 1] appeared in which Bounded Forcing Axioms were interpreted as assertions of absoluteness between the ground model and generic extensions. Two

[1] Believe it or not—we are still doing first order set theory.

years after that Stevo Todorčević showed in [To 3] that BMM $\Rightarrow \mathfrak{c} = \aleph_2$. This was further improved by Justin Tatch Moore who showed in [Mo] that indeed BPFA suffices for this, i.e. BPFA $\Rightarrow \mathfrak{c} = \aleph_2$.

7.2 Open questions

Although there has been a considerable amount of progress in the understanding of Bounded Forcing Axioms in recent years it seems not that difficult to formulate questions which cannot be answered outright. In the light of the basic elucidations regarding BAAFA in the chapters 5, 6 and the recent historical comments one may for example ask the following:

7.1. QUESTION.

- Does BAAFA decide the size of the continuum?
- If it does not: what is the consistency strength of BAAFA $+2^{\aleph_0} > \aleph_2$?

The first part of this question is somewhat reminiscent of the question asked by Bagaria in [Bag 3]: Questions 6.8.(4). There he formulated the following:

7.2. QUESTION. Let σ–closed $*$ ccc be the class of forcing notions consisting of an iteration of a σ–closed poset followed by a ccc poset. Such posets are proper[2]. Does BFA(σ–closed $*$ ccc, $\aleph_2, \aleph_2$)[3] imply $\mathfrak{c} = \aleph_2$?

The iterated forcing constructions by which we attained the generic extensions in which BPFA, BAAFA hold respectively were defined referring not as commonly done to Σ_1–formulae and parameters from $H_{\aleph_2}$ as in [C–V] or—which is a special case of this—to structures on $\aleph_1$ and endomorphisms of these structures as done in [G–S] but rather to filters and maximal antichains. In the light of what is really needed to attain the generic extensions in question one can make the following...

7.3. DEFINITION. Let $\mathcal{C}$ be a reasonable class of forcing notions. A cardinal κ is called $\mathcal{C}$–reflecting if and only if

- κ is regular.

[2] They even satisfy Axiom A.

[3] In fact something different is written there. But considering his explanations preceding this question this is obviously what he wanted to ask.

- If $\mathbb{B} \in \mathcal{C}$ is a Boolean algebra and $\mathcal{A}$ is a family of maximal antichains of $\mathbb{B}$ such that the to subalgebra $\mathbb{S}$ finitely generated by $\mathcal{A}$ is in H_κ then there exists a Boolean algebra $\mathbb{E} \in \mathcal{C} \cap H_\kappa$ such that $\mathbb{E} \supset \mathbb{S}$ and $\forall A \in \mathcal{A}$: A is a maximal antichain in $\mathbb{E}$.

In the light of this definition what one used to attain the generic extensions in which BAAFA, BPFA hold respectively was the fact that reflecting cardinals are $\mathcal{AA}^*$–reflecting and $\mathcal{P}_{\mathrm{rop}}$–reflecting respectively. Clearly for all cardinals κ:

$$\kappa \text{ is reflecting} \Longrightarrow \kappa \text{ is } \mathcal{P}_{\mathrm{rop}}\text{–reflecting} \Longrightarrow \kappa \text{ is } \mathcal{AA}^*\text{–reflecting} . \tag{7.2}$$

But since in both forcing extensions the $\mathcal{AA}^*$–reflecting($\mathcal{P}_{\mathrm{rop}}$–reflecting) cardinal is rendered $\aleph_2$ and BAAFA $\Rightarrow$ "$\aleph_2$ is reflecting in L." one also gets

$$L \models \text{"All } \mathcal{AA}^*\text{–reflecting cardinals are reflecting."}. \tag{7.3}$$

This leads to the somewhat amorphous...

7.4. Question. Can (7.3) be generalized so that one attains a purely combinatorial definition of reflecting cardinals?

Another fact worth noting isthat we had to introduce the notion of being reasonable in order to formulate a natural version of BAAFA. We could have dispensed with this if we knew that $\mathcal{AA}$ is reasonable. So easily one arrives at asking the following...

7.5. Question. Is $\mathcal{AA}$ reasonable?

In fact $\mathcal{AA}$ might not be reasonable. On page 50 BAAFA was said to be a natural weakening of BPFA. A good reason for this is that $\mathcal{AA}$ can be seen as a natural class of forcing notions. An argument for this in turn is that Axiom A is a straightforward generalization from properties of Sacks forcing, Laver forcing, Mathias forcing, Silver forcing, etc.. But first of all these properties are just combinatorial attributes of the posets and it is not clear at all why for example they should also apply to the respective regular open algebra. Note for example that whenever $\mathbb{B}$ is a σ–complete Boolean algebra, $\mathbb{B}^+$ cannot be σ–closed. So the the positive elements of the regular open algebra of a σ–closed poset do not themselves form a σ–closed poset.

So at the end of this thesis we have reached the classical state of mind: We managed to give one answer—theorem 6.7—but are indeed left behind with three open questions: 7.1, 7.4 and 7.5.

7.3 Is the continuum problem solved?

Since the time of Gödel some set theorists—mostly those who endorse realism as the appropriate philosophical attitude towards mathematics—have argued that a proof of the undecidability of a statement from the axioms of ZFC should not be viewed as the final solution. Rather one could hope for axioms which—added to ZFC—would provide a solution, they claimed. Gödel himself argued this way on behalf of the continuum hypothesis in [Gö]. Later on in the development several statements had been formulated by set theorists that actually were denoted as axioms—such as for example large cardinal axioms, the Axiom of Constructibility—also known as $V = L$ or Martin's Axiom. But the designation of these statements axioms was more due to the fact that they were easily seen to be unprovable from ZFC yet useful in set theory. Neither was there a great effort to argue for some such principle to be *true* nor was it even proposed to be employed as a generally usable presupposition in proofs. Later on Penelope Maddy endorsed a pair of heuristic principles which she called "Maximize" and "Unify"—see [Ma]—to be used to judge how suitable certain statements in the language of set theory are to be added to ZFC. The idea is on the one hand not to refrain from analysing certain mathematical structures just because one added an axiom to ZFC which implies that they do not exist—that is the content of the principle "Maximize". On the other hand one wishes to provide a unique framework for mathematics—this is what is demanded by "Unify". If one dispensed with "Maximize" one could add any statement consistent with ZFC to ZFC and thus attain a theory answering more questions than ZFC alone. If alternatively one did not employ "Unify" the best solution would be to allow a plenitude of theories which could then be analysed simultaneously. Her paradigmatic example in this respect is to judge the theory $\mathrm{ZFC} + V = L$ against $\mathrm{ZFC} +$ "$0^{\#}$ exists.". These theories contradict each other. However, it is possible to reinterpret the first theory in the second one simply by relativizing every formula to L. Furthermore Maddy proves that by the second theory there exists an isomorphism type which does not exist in any model of the first theory. So by Maddy's argumentation it would be advisable to prefer the second theory over the first. In [Bag 2] Joan Bagaria follows this line of thought. There he also adds another criterion by which—in the context of his article—one should judge several axiom candidates against each other—the criterion of fairness as he calls it. Additional axioms should neither discriminate between formulae of the same logical comlexity—as given by the Levy hierarchy—nor between sets of the same complexity, i.e.e.g. sets of the same rank or of the same hereditary cardinality. This in mind, he goes on to argue that because of the possible characterization as principles of generic absoluteness the Bounded Forcing Axioms are indeed "real axioms" and that they were at least as natural as axioms of large cardinals. In [Bag 3] Bagaria gives a comprehensive overview on statements of generic absoluteness. On the one hand he discusses generic absoluteness for formulae higher in the Levy hierarchy but only with hereditary countable parameters. These principles are consistent with the continuum hypothesis. On the other hand not so much seems to be known on generic absoluteness with respect to Σ_1–formulae with parameters from $H_{\aleph_3}$. If one considers the class of c.c.c. notions of forcing the corresponding principle will just be $\mathrm{MA}_{\aleph_2}$. One cannot however state the principle consistently even only for all σ–closed forcing notions since one can add a surjection $f : \aleph_1 \longrightarrow\!\!\!\!\!\rightarrow \aleph_2$ with countable conditions. The statement that such a surjection exists is $\Sigma_1\big(\{\aleph_1, \aleph_2\}\big)$ and this notion of forcing is countably closed. So the principle implies $\aleph_2 < \aleph_2$ and is hence inconsistent. This however does not exclude the possibility of the existence of a natural class of forcing notions properly extending the one of those satisfying the countable chain condition in respect to which generic absoluteness of Σ_1–formulae with parameters from $H_{\aleph_3}$ is consistent. All the same the class of proper notions of forcing can be considered a natural one but it nevertheless was not trivial to isolate. So the claim that the continuum problem is solved *because* BPFA implies that $2^{\aleph_0} = \aleph_2$ is probably premature, for even those who believe that the extension of ZFC is a reasonable goal and additionally follow Bagaria in his argumentation regarding the question what criteria one should adopt to judge between different—and sometimes inconsistent—candidates for axiomhood can demand that one has to argue why Bounded Forcing Axioms should be preferred over alternative principles of generic absoluteness. While in the context of various principles of generic absoluteness one certainly could find arguments in favour of the Bounded Forcing Axioms it seems hardly probable that even every platonist could be convinced this way at present. But our knowledge will—hopefully—increase so eventually such an argumentation might be successful or one might find other hints on how big the continuum really is.

Bibliography

[Ab] Uri Abraham, *Proper forcing,* ***The Handbook of Set Theory*** (Foreman, Kanamori, Magidor, editors)

[As] David Asperó, *A maximal bounded forcing axiom,* ***The Journal of Symbolic Logic*** Volume 67, Number 1, March 2002, pp. 130–142.

[Ba 1] James E. Baumgartner *Iterated forcing,* ***Surveys in set theory*** (A. R. D. Mathias, editor), London Mathematical Society Lecture Note Series, Volume 87, Cambridge University Press, Cambridge, 1983, pp. 1–59.

[Ba 2] —, *Applications of the proper forcing axiom,* ***Handbook of set-theoretic topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 913–959.

[Bag 1] Joan Bagaria, *Bounded Forcing Axioms as principles of generic absoluteness,* ***Archive for Mathematical Logic,*** Volume 39 (2000), pp. 393–401.

[Bag 2] —, *Natural Axioms of set theory and the continuum problem,* http://villaveces.info/tavconj2006/upcontent/uploads/2006/03/continuo.pdf

[Bag 3] —, *Axioms of generic absoluteness,* http://www.crm.es/Publications/03/pr563.pdf

[Be] M. Bekkali *Topics in Set Theory,* ***Lecture Notes in Mathematics 1476,*** Springer-Verlag, Berlin, 1991

[C–V] Andrés Eduardo Caicedo and Boban Veličković, *The bounded proper forcing axiom and well orderings of the reals* ***Mathematical Research Letters,*** 12, pp. 10001–10018 (2005)

[De 1] Keith J. Devlin, *The Yorkshireman's guide to proper forcing,* ***Surveys in set theory*** (A. R. D. Mathias, editor), London Mathematical Society Lecture Note Series, Volume 87, Cambridge University Press, Cambridge 1983, pp. 60–115.

[De 2] —, *Constructibility,* ***Perspectives in Mathematical Logic,*** Springer Verlag, Berlin • Heidelberg • New York • Tokyo, 1984

[F–M–S] Matthew Foreman, Menachem Magidor and Saharon Shelah, *Martin's Maximum, saturated ideals and nonregular ultrafilters. I.* ***Ann. of Math.*** (2) 127 (1988), no. 1, pp.1–47.

[Fu] Sakaé Fuchino, *On potential embedding and versions of Martin's Axiom,* ***Notre Dame Journal of Formal Logic,*** vol. 33 (1992), pp. 481–492.

[Gö] Kurt Gödel, *What is Cantor's continuum problem?,* ***American Mathematical Monthly,*** USA 54 (1947), pp. 515–525.

[G–S] Martin Goldstern and Saharon Shelah, *The bounded proper forcing axiom,* ***The Journal of Symbolic Logic,*** Volume 60, 1995, pp. 58–73.

[Je 1] Thomas Jech, *Set theory, United Kingdom Edition,* ***Academic Press,*** New York • San Francisco • London, 1978

[Je 2] —, *Set theory—The Third Millennium Edition, Revised and Expanded,* ***Springer Monographs in Mathematics,*** Berlin, 2002

[Ka] Akihiro Kanamori, *The Higher Infinite—Large Cardinals in Set Theory from Their Beginnings, Second Edition* ***Springer Monographs in Mathematics,*** Berlin • Heidelberg, 1994 and 2003

[Ko] Piotr Koszmider, *On coherent families of finite-to-one functions* ***The Journal of Symbolic Logic*** Volume 58, no. 1, March 1993

[Ku] Kenneth Kunen, *Set theory—An introduction to independence proofs,* ***Studies in logic and the foundations of mathematics*** Volume 102, 1980, North-Holland—Amsterdam • New York • Oxford

[Ma] Penelope Maddy *$V = L$ and Maximize,* ***Logic Colloquium 95,*** Lecture Notes Logic, Volume 11, Springer, 1998, pp. 143–178.

[Mo] Justin Tatch Moore, *Set mapping reflection,* ***Journal of Mathematical Logic*** 5 (2005), no. 1,pp. 87–98

[M–S] Donald A. Martin and Robert Solovay, *Internal Cohen extensions,* ***Ann. Math. Logic 2*** (1970), pp.143–178.

[Ro] J. Barkley Rosser, *Simplified Independence Proofs—Boolean valued models of set theory, United Kingdom Edition,* ***Academic Press,*** London, 1969

[Sch 1] Ralf Schindler, *Forcing axioms and projective sets of reals,* http://wwwmath1.uni-muenster.de/logik/Personen/rds/FOTFS.ps

[Sch 2] Ralf Schindler, *Bounded Martin's Maximum and strong cardinasls,* http://wwwmath1.uni-muenster.de/logik/Personen/rds/Paper2.ps

[Sh] Saharon Shelah, *Proper forcing,* ***Lecture Notes in Mathematics,*** Volume 940, Springer Verlag, Berlin, 1982.

[S–T] Robert Solovay and Stanley Tennenbaum, *Iterated Cohen extensions and Souslin's problem,* ***Annals of Mathematics 94*** 1971, pp.201–245.

[To 1] Stevo Todorčević, *A note on the proper forcing axiom,* ***Axiomatic set theory*** (J. Baumgartner et al., editors), American Mathematical Society, Providence, Rhode Island, 1984, pp. 209–218.

[To 2] —, *Trees and Linearly Ordered Sets,* ***Handbook of set-theoretic topology*** (K. Kunen and J. E. Vaughan, editors), North-Holland, Amsterdam, 1984, pp. 235–293.

[To 3] —, *Generic absoluteness and the continuum,* ***Mathematical Research Letters*** 9, pp. 1–7 (2002)

[V] Boban Veličković, *Forcing axioms and stationary sets,* ***Adv. Math.*** 94 (1992), no. 2, pp. 256–284.

A Notation

- Ω denotes the class of all ordinals,
- Lim the class of all limit ordinals,
- Card the class of all cardinals,
- Reg the class of all regular cardinals,
- Func the class of all functions,
- Γ the canonical name for a generic filter,
- $\mathrm{trcl}(X)$ the transitive closure of X,
- $\mathrm{otyp}(C)$ the order type of C,
- $\lim(C)$ the set of C's limit points,
- $\overline{\overline{X}}$ the cardinality of X,
- $f``X$ the pointwise image of X under f,
- $\mathfrak{P}(X)$ the power set of X,
- ${}^{X}Y$ the set of all functions from X into Y,
- $\mathfrak{B}$ the class of Boolean algebras,
- c.c.c. the class of forcing notions satisfying the countable chain condition,
- $\sigma\mathrm{cl}$ the class of countably closed forcing notions,
- $\mathcal{P}_{\mathrm{rop}}$ the class of proper notions of forcing,
- $\mathcal{AA}$ the class of notions of forcing satisfying Axiom A,
- $\mathfrak{rh}(\mathcal{C})$ the reasonable hull of $\mathcal{C}$ and

- $\mathscr{A}\!\!\mathscr{A}^*$ the class of notions of forcing satisfying Axiom A*.

Closed unbounded sets are often abbreviated as “clubs”. $\subset, \supset, \subsetneq, \supsetneq$ are used—not $\subseteq$ $, \supseteq, \subset, \supset$. The size of the font is a tribute to the sometimes excessive usage of indices. Indices are not always used solely if unavoidable. Sometimes they simply shall insinuate meaning—meaning they are supposed to simplify not to complicate the reading. $M \prec N$ means that M is an elementary submodel of N and $M \prec_{\Sigma_n} N$ means that M is a submodel of N and M and N believe in the same Σ_n-assertions. When speaking of a formula which may contain parameteres we often dispense with its designation as $\varphi(a_0, \ldots, a_n)$ but write just $\varphi(a)$ instead.

The term “forcing notion” is often used in this thesis. This is a somewhat vague notion, since one can force with partially ordered sets, Boolean algebras, topological spaces and other kinds of objects. At some places “Boolean algebra” is used instead of “forcing notion”. The easiest way perhaps is, always to conceive of a forcing notion as a poset and of a Boolean algebra as a poset which can be embedded bijectively into a “real” Boolean algebra(We for example do not care about the “fact” that posets are pairs while Boolean algebras are sextupels.). However—we do suppose that the class of forcing notions is Σ_0-definable—as is the class of posets. This is necessary in order to ensure the correctness of some calculations of complexity.

The terminology of iterated forcing is essentially the one of [Ku]. There nevertheless is an important detail which is different. As it is said in the text in the definition of the two–step iteration we do not require the name of a condition to be in the domain of the name of the forcing notion. This follows the treatment of Baumgartner in [Ba 1].

As a notational variant of Kunen’s iterated forcing construction we employ the following:

$$\Big(\big((\mathbb{P}_\gamma, \leqslant_\gamma, \mathbb{1}_\gamma)|\gamma < \alpha\big), \big((\pi_\gamma, \preccurlyeq_\gamma, \varepsilon_\gamma)|\gamma < \alpha\big)\Big) \tag{A.1}$$

which we nevertheless often abbreviate as $\mathbb{P}_\alpha$. When argueing about some specific iterated forcing construction we usually dispense with mentioning the canonical dense embeddings between the various $\mathbb{P}_\gamma$. The filter Kunen calls H_ξ in [Ku], lemma VIII.5.13 is called $F_\xi^{\xi+1}$ in this thesis—it is mentioned in the proof of theorem 4.6. Finally we sometimes need to talk about an intermediate segment of our iterated forcing construction. If $\mathbb{P}_\alpha$ is an iterated forcing construction of length α and $\gamma < \beta \leqslant \alpha$, $\frac{\mathbb{P}_\beta}{G_\gamma}$ denotes the segment of $\mathbb{P}_\alpha$ between γ and β as interpreted by the filter G_γ. $\frac{\check{\mathbb{P}}_\beta}{\Gamma_\gamma}$ shall then be our canonical $\mathbb{P}_\beta$–name for this object.

Versicherung

Ich verfasste diese Diplomarbeit selbständig,
wobei ich nur die angegebenen Quellen benutzte.