Problem 10: Bogoliubov transformation (3 points)

The Hamilton operator of two interacting electrons has the form

$$\hat{H} = A\,(\hat{c}_1^\dagger\,\hat{c}_1 + \hat{c}_2^\dagger\,\hat{c}_2) - B\,(\hat{c}_1^\dagger\,\hat{c}_2^\dagger + \hat{c}_2\,\hat{c}_1)$$

with the constants $A,\, B > 0$. Using the new operators $\hat{\alpha}$, $\hat{\beta}$, $\hat{\alpha}^\dagger$ and $\hat{\beta}^\dagger$ with

$$\hat{c}_1 = u\,\hat{\alpha} + v\,\hat{\beta}^\dagger\,, \qquad \hat{c}_1^\dagger = u\,\hat{\alpha}^\dagger + v\,\hat{\beta}\,, \qquad \hat{c}_2 = u\,\hat{\beta} - v\,\hat{\alpha}^\dagger\,, \qquad \hat{c}_2^\dagger = u\,\hat{\beta}^\dagger - v\,\hat{\alpha}\,,$$

$\hat{H}$ can be transformed into diagonal form. Here, u and v are real constants.

a) Calculate the anticommutators

$$[\hat{\alpha},\, \hat{\alpha}^\dagger]_+\,, \qquad [\hat{\alpha},\, \hat{\beta}^\dagger]_+ \qquad \text{and} \qquad [\hat{\beta},\, \hat{\beta}^\dagger]_+$$

for the case $u^2 + v^2 = 1$.

b) Use the transformation given above to show that $\hat{H}$ can be put into the form

$$\hat{H} = F\,(\hat{\alpha}^\dagger\,\hat{\alpha} + \hat{\beta}^\dagger\,\hat{\beta}) + G$$

if u and v are chosen in an appropriate way (under the requirement $u^2 + v^2 = 1$). Here, F and G are constants.

c) Determine the ground state energy of the system.

Problem 11: Fermi-Bose model (4 points)

The simplest model for coupling a Fermi particle (electron) and a Bose particle (phonon) is given by

$$\hat{H} = \epsilon\,\hat{c}^\dagger\,\hat{c} + \omega\,\hat{b}^\dagger\,\hat{b} + \gamma\,(\hat{b}^\dagger + \hat{b})\,\hat{c}^\dagger\,\hat{c}$$

with $\hat{c}/\hat{c}^\dagger$ Fermi and $\hat{b}/\hat{b}^\dagger$ Bose annihilators/creators. Usually $\gamma < 0$.

The Hilbert space is (in second quantization) given by states $|n_f,\, n_b\rangle$ with Fermi occupation number $n_f = 0, 1$ and Bose occupation number $n_b = 0, 1, 2, \ldots$

Apparently, the Hilbert space splits into a subspace for $n_f = 0$ and one for $n_f = 1$.

a) What is the explicit form of the matrix representation of $\hat{H}$ in the two subspaces (using $|n_f,\, n_b\rangle$ as basis states)?

b) Diagonalize $\hat{H}$ by using a „quadratic addition", i. e. by turning $\hat{H}$ into a form

$$\hat{H} = \tilde{\epsilon}\,\hat{c}^\dagger\,\hat{c} + \omega\,\hat{\hat{b}}^\dagger\,\hat{\hat{b}}\,.$$

Determine $\tilde{\epsilon}$ and $\hat{\hat{b}}$. Show that $\hat{\hat{b}}/\hat{\hat{b}}^\dagger$ still fulfil Bose commutator rules. Determine the energy eigenvalues explicitly.

Problem 12: Harmonic oscillator (3 points)

Consider a one-dimensional harmonic oscillator with spring constant k, mass m, and energy eigenstates $|n\rangle$ $(n = 0, 1, 2, ...)$.

Consider a second oscillator with same parameters, but spatially shifted with respect to the first oscillator by d, with energy eigenstates $|\tilde{n}\rangle$ $(\tilde{n} = 0, 1, 2, ...)$.

a) $|\tilde{n}\rangle$ can be expanded in $|n\rangle$, i. e.

$$|\tilde{n}\rangle = \sum_n A_n^{(\tilde{n})} |n\rangle \ .$$

Calculate $A_n^{(\tilde{n})}$ for $\tilde{n} = 0$ (ground state).

b) Show that in the limit of large mass the largest coefficient $A_n^{(0)}$ is found for

$$n = \frac{1}{2}\left(\frac{d}{x_0}\right)^2$$

with $x_0 = \sqrt{\frac{\hbar}{m\,\omega}}$ being the characteristic length of the oscillator.