Problem 1: Commutator relations for fermions **(3 points)**

a) The anticommutator of the operators $\hat{A}$ and $\hat{B}$ is given by

$$[\hat{A},\, \hat{B}]_+ \,=\, \hat{A}\,\hat{B} \,+\, \hat{B}\,\hat{A}\ .$$

Show that the following relation holds for an additional operator $\hat{D}$

$$[\hat{A},\, \hat{B}\,\hat{D}]_- \,=\, [\hat{A},\, \hat{B}]_+\,\hat{D} \,-\, \hat{B}\,[\hat{A},\, \hat{D}]_+\ .$$

b) The creation and annihilation operators $\hat{c}_j^+$ and c_j have been introduced in the lecture. Use the anticommutator relations of these operators to calculate the following commutator $[\hat{A},\, \hat{B}]_- \,=\, \hat{A}\,\hat{B} \,-\, \hat{B}\,\hat{A}$

i)

$$[\hat{n}_j,\, \hat{c}_k]_- \qquad \text{and} \qquad [\hat{n}_j,\, \hat{c}_k^+]_- \qquad \text{with} \qquad \hat{n}_j \,=\, \hat{c}_j^+\,\hat{c}_j\ .$$

ii)

$$[\hat{c}_i^+\,\hat{c}_j,\, \hat{c}_l^+\,\hat{c}_m]_- \,=\, \alpha\,\cdot\,\hat{c}_i^+\,\hat{c}_m \,+\, \beta\,\hat{c}_l^+\,\hat{c}_j\ .$$

Calculate α and β.

iii)

$$\begin{aligned}[\hat{c}_i^+\,\hat{c}_j\,\hat{c}_l^+\,\hat{c}_m,\, \hat{c}_n^+\,\hat{c}_p]_- \,&=\, (\alpha\,\cdot\,\hat{c}_i^+\,\hat{c}_p \,+\, \beta\,\cdot\,\hat{c}_n^+\,\hat{c}_j)\,\hat{c}_l^+\,\hat{c}_m \\ &+\, \hat{c}_i^+\,\hat{c}_j\,(\gamma\,\cdot\,\hat{c}_l^+\,\hat{c}_p \,+\, \zeta\,\cdot\,\hat{c}_n^+\,\hat{c}_m)\ .\end{aligned}$$

Calculate α, β, γ and ζ.

Useful relation:

$$[\hat{A}\,\hat{B},\, \hat{D}]_- \,=\, [\hat{A},\, \hat{D}]_-\,\hat{B} \,+\, \hat{A}\,[\hat{B},\, \hat{D}]_-\ .$$

Problem 2: Expectation values for fermions **(2 points)**

The eigenstates of the Hamilton operator

$$\hat{H} \,=\, \sum_{j\,=\,1}^{\infty} \varepsilon_j\,\hat{c}_j^+\,\hat{c}_j$$

have the form

$$|\phi\rangle \,=\, \prod_{j\,=\,1}^{\infty} (\hat{c}_j^+)^{n_j}\,|0\rangle\ .$$

a) Calculate

$$\hat{n}_l\,|\phi\rangle \qquad \text{with} \qquad \hat{n}_l \,=\, \hat{c}_l^+\,\hat{c}_l\ .$$

b) Determine the expectation values

$$\text{a)} \qquad \langle\phi|\hat{c}_l^+\,\hat{c}_m|\phi\rangle\ ,$$

$$\text{b)} \qquad \langle\phi|\hat{c}_i^+\,\hat{c}_l^+\,\hat{c}_k\,\hat{c}_m|\phi\rangle\ .$$

Use the occupation numbers n_k and n_m to represent your result.

Problem 3: Particle density operator (2 points)

The operator of the particle density has the form (position representation)

$$\hat{\rho}(\vec{r}) = \sum_{i=1}^{N} \delta(\vec{r} - \vec{r}_i) \ .$$

a) Transform this operator into the occupation number representation $\hat{\rho}_F(\vec{r})$ with the creation and annihilation operators $\hat{c}^+_{\vec{k}\,\sigma}$ and $\hat{c}_{\vec{k}\,',\sigma'}$.
Use plane waves as single-particle basis

$$\psi_{\vec{k}\,\sigma}(\vec{r}) = \frac{1}{\sqrt{\Omega}} \cdot \mathrm{e}^{i\vec{k}\vec{r}} \cdot \chi_\sigma$$

with the volume Ω.

b) Calculate the Fourier transform of $\hat{\rho}_F(\vec{r})$

$$\tilde{\hat{\rho}}_F(\vec{q}) = \frac{1}{\Omega} \int\limits_{\Omega} \mathrm{e}^{-i\vec{q}\vec{r}}\, \hat{\rho}_F(\vec{r})\, d^3 r \ .$$

Problem 4: Two-level system (3 points)

The Hamilton operator of a system with two spin degenerate energy levels ε_a and ε_b has the form

$$\hat{H} = \varepsilon_a \left(\hat{c}^+_{a\uparrow}\hat{c}_{a\uparrow} + \hat{c}^+_{a\downarrow}\hat{c}_{a\downarrow} \right) + \varepsilon_b \left(\hat{c}^+_{b\uparrow}\hat{c}_{b\uparrow} + \hat{c}^+_{b\downarrow}\hat{c}_{b\downarrow} \right) \ .$$

a) Show that the state

$$|\phi_1\rangle = \hat{c}^+_{a\uparrow}\hat{c}^+_{b\uparrow}|0\rangle$$

is an eigenstate of the system. Which energy has the system in this state?

b) Show that the state

$$|\phi_2\rangle = \frac{1}{\sqrt{2}} \left(\hat{c}^+_{a\uparrow} + \hat{c}^+_{a\downarrow} \right) \hat{c}^+_{b\uparrow}|0\rangle$$

is normalized. Is $|\phi_2\rangle$ an eigenstate of the system?

c) Calculate for $|\phi_1\rangle$ and $|\phi_2\rangle$, respectively, the expectation values for the spin operators

$$\hat{S}_z = \frac{\hbar}{2} \sum_j \left(\hat{c}^+_{j\uparrow} c_{j\uparrow} - \hat{c}^+_{j\downarrow}\hat{c}_{j\downarrow} \right) \qquad \text{und} \qquad \hat{S}_x = \frac{\hbar}{2} \sum_j \left(\hat{c}^+_{j\uparrow}\hat{c}_{j\downarrow} + \hat{c}^+_{j\downarrow} c_{j\uparrow} \right)$$

with $j = a, b$.