Problem 17: Polaron **(5 points)**

An electron interacts with optical phonons in an ionic crystal. The interaction part of the Hamilton operator is given by

$$\hat{H}_1 = \frac{1}{2}\sum_{\vec{k}\sigma}\sum_{\vec{q}\vec{q}'}\Bigg(\Big(A(\vec{k},\vec{q})(\hat{a}_{\vec{q}}\hat{a}_{\vec{q}'} + \hat{a}_{\vec{q}}\hat{a}^{\dagger}_{-\vec{q}'}) + B(\vec{k},\vec{q})(\hat{a}^{\dagger}_{-\vec{q}}\hat{a}_{\vec{q}} + \hat{a}^{\dagger}_{-\vec{q}}\hat{a}^{\dagger}_{-\vec{q}'}\Big) \times$$

$$\times W(\vec{q})W(\vec{q}')(\hat{c}^{\dagger}_{\vec{k}+\vec{q}+\vec{q}'\,\sigma}\hat{c}_{\vec{k}\sigma} - \hat{c}^{\dagger}_{\vec{k}+\vec{q}\sigma}\hat{c}_{\vec{k}-\vec{q}'\sigma}) + \delta_{\vec{q},\vec{q}'}|W(\vec{q})|^2\left(B(\vec{k},\vec{q}) - A(\vec{k},\vec{q})\right)\hat{c}^{\dagger}_{\vec{k}\sigma}\hat{c}_{\vec{k}\sigma}\Bigg).$$

Here,

$$A(\vec{k},\,\vec{q}) = \frac{-1}{\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} - \hbar\,\omega_{\mathrm{LO}}} \quad \text{and} \quad B(\vec{k},\vec{q}) = \frac{-1}{\varepsilon_{\vec{k}+\vec{q}} - \varepsilon_{\vec{k}} + \hbar\,\omega_{\mathrm{LO}}} \quad \text{with} \quad \varepsilon_{\vec{k}} = \frac{\hbar^2 k^2}{2\,m}\,.$$

The interaction matrix element has the property

$$W(-\vec{q}) = W(\vec{q})^* \qquad \text{with} \qquad |W(\vec{q})| = \frac{\gamma}{q}\,.$$

γ is a real constant and $\hbar\,\omega_{\mathrm{LO}}$ is the energy of the longitudinal optical phonons.

a) Calculate the contribution E_1 of $\hat{H}_1$ in first order perturbation theory. Consider a product state $|\Psi\rangle = |\Phi\rangle_{\mathrm{el}}\,|\chi\rangle_{\mathrm{ph}}$ which contains one electron with wave vector $\vec{k}$ and no phonons.

$$|\Phi\rangle_{\mathrm{el}} = \hat{c}^{\dagger}_{\vec{k}\,\sigma}\,|0\rangle_{\mathrm{el}} \qquad \text{and} \qquad |\Phi\rangle_{\mathrm{ph}} = |n_{\vec{q}} = 0\rangle_{\mathrm{ph}}\,.$$

Show that the energy can be written in the following form

$$E_1 = \sum_{\vec{q}}\frac{|W(\vec{q})|^2}{\varepsilon_{\vec{k}} - \varepsilon_{\vec{k}+\vec{q}} - \hbar\,\omega_{\mathrm{LO}}}\,.$$

b) Evaluate the remaining sum over $\vec{q}$ by integration over the *whole* $\vec{q}$ space.

c) Use a Taylor series expansion of E_1 to analyse your result in the limit of small wave vectors $\vec{k}$. Consider the quadratic term and determine the change of the electron's effective mass due to electron-phonon interaction.

Usefull integral: $$\int_0^{\infty}\frac{1}{x}\ln\left|\frac{a^2 + 2\,b\,x + x^2}{a^2 - 2\,b\,x + x^2}\right|dx = 2\,\pi\arcsin\frac{b}{a}\,.$$

Problem 18: Superconducting bar (5 points)

The current density $\vec{j}$ of superconducting electrons is related by

$$\vec{j} \,=\, -\frac{n_s\,\mathrm{e}^2}{m}\,\vec{A}\,(\vec{r}\,)$$

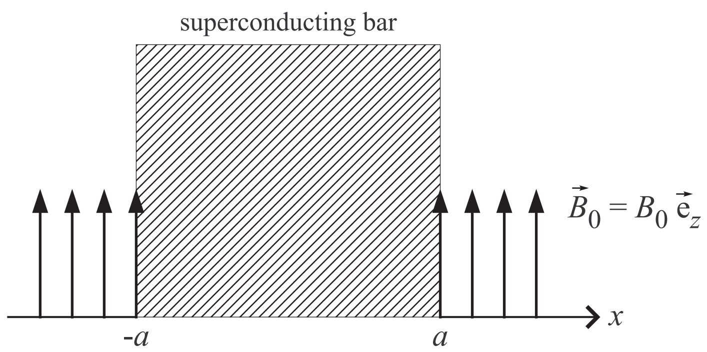
to the vector potential $\vec{A}\,(\vec{r}\,)$. Here, n_s is the particle density of the superconducting electrons.

a) Use the Maxwell equations for the static case to show that the magnetic field in a superconductor is described by the Helmholtz equation

$$\Delta\,\vec{B}\,(\vec{r}\,) \,=\, \mu_0\,\frac{n_s\,\mathrm{e}^2}{m}\,\vec{B}\,(\vec{r}\,)\ .$$

b) Solve the Helmholtz equation for a superconducting bar (with the width $2\,a$) which is in a magnetic field. Assume that a homogeneous magnetic field $\vec{B}_0 \,=\, (0,\,0,\,B_0)$ in z direction exists outside of the bar. Neglect the y and z dependence of the field inside the bar and consider the case $\vec{B} \,=\, \vec{B}\,(x)$ with $\vec{B}\,(-\vec{a}) \,=\, \vec{B}_0 \,=\, \vec{B}\,(a)$. Plot the resulting magnetic field $\vec{B}\,(x)$.

c) Calculate the y component of the current density and plot your result.



Problem 19: BCS ground state **(5 additional points)**

The BCS ground state has the form

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\vec{k}} (u_{\vec{k}} + v_{\vec{k}}\, \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}^{\dagger}_{-\vec{k}\downarrow})\, |0\rangle$$

with

$$u_{\vec{k}} = \frac{1}{2}\left(1 + \frac{\varepsilon_{\vec{k}}}{\sqrt{\varepsilon_{\vec{k}}^2 + \Delta_{\vec{k}}^2}}\right), \quad u_{\vec{k}}^2 + v_{\vec{k}}^2 = 1, \quad \Delta_k = \Delta \quad \text{for} \quad |\varepsilon_{\vec{k}}| \leq \hbar\,\omega_{\mathrm{LO}}$$

and $\Delta_{\vec{k}} = 0$ else.

a) Calculate the expectation values $\langle \hat{n}_{\vec{k}\uparrow}\rangle$ and $\langle \hat{n}_{-\vec{k}\downarrow}\rangle$ of the occupation number operators $\hat{n}_{\vec{k}\uparrow} = \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}_{\vec{k}\uparrow}$ and $\hat{n}_{-\vec{k}\downarrow} = \hat{c}^{\dagger}_{-\vec{k}\downarrow}\, \hat{c}_{-\vec{k}\downarrow}$, respectively. Plot your result as a function of $\varepsilon_{\vec{k}}$ for i) $\Delta = 0$ and ii) $\Delta \neq 0$.

b) In this part, the action of creation and annihilation operators $\hat{\alpha}^{\dagger}_{\vec{k}}$, $\hat{\beta}^{\dagger}_{\vec{k}}$, $\vec{\alpha}_{\vec{k}}$, $\beta_{\vec{k}}$ resulting from the Bogoliubov transformation should be investigated.

   i) Calculate $\hat{\alpha}_{\vec{k}}\, |\Psi_{\mathrm{BCS}}\rangle$ and discuss your result.

   ii) Proof that $\hat{a}^{\dagger}_{\vec{k}'}\, |\Psi_{\mathrm{BCS}}\rangle = \hat{c}^{\dagger}_{\vec{k}'\uparrow}\, |\tilde{\Psi}_{\vec{k}'}\rangle$ with $|\tilde{\Psi}_{\vec{k}'}\rangle = \prod_{\vec{k}\neq\vec{k}'} (u_{\vec{k}} + v_k\, \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}^{\dagger}_{-\vec{k}\downarrow})\, |0\rangle$.

   iii) Calculate $\hat{\beta}^{\dagger}_{\vec{k}'}\, \hat{\alpha}^{\dagger}_{k'}\, |\Psi_{\mathrm{BCS}}\rangle$. Express your result in terms of $|\tilde{\Psi}_{\vec{k}'}\rangle$.

c) Show that the particle number operator $\hat{N}_k = \hat{n}_{\vec{k}\uparrow} + \hat{n}_{-\vec{k}\downarrow}$ can be written as

$$\hat{N}_{\vec{k}} = 2\, v_{\vec{k}}^2 + (u_{\vec{k}}^2 - v_{\vec{k}}^2)\,(\hat{\alpha}^{\dagger}_{\vec{k}}\, \hat{\alpha}_{\vec{k}} + \hat{\beta}^{\dagger}_{\vec{k}}\, \hat{\beta}_{\vec{k}}) + 2\, u_{\vec{k}}\, v_{\vec{k}}\, (\hat{\alpha}^{\dagger}_{\vec{k}}\, \hat{\beta}^{\dagger}_{\vec{k}} + \hat{\beta}_{\vec{k}}\, \hat{\alpha}_{\vec{k}})\ .$$

   Calculate

$$\langle \hat{N}_{\vec{k}}\rangle = \langle \Psi_{\mathrm{BCS}}|\hat{N}_{\vec{k}}|\Psi_{\mathrm{BCS}}\rangle\ , \qquad \langle \hat{N}_{\vec{k}}^2\rangle = \langle \Psi_{\mathrm{BCS}}|\hat{N}_{\vec{k}}\, \hat{N}_{\vec{k}}|\Psi_{\mathrm{BCS}}\rangle$$

   and $(\Delta\, N_{\vec{k}})^2 = \langle \hat{N}_{\vec{k}}^2\rangle - \langle \hat{N}_{\vec{k}}\rangle^2$. Express your result in terms of $\varepsilon_{\vec{k}}$ and $\Delta_{\vec{k}}$. Discuss your result.