Problem 1: Dielectric function of the electron gas (5 points)

The dielectric function of the three-dimensional electron gas has the form (cf. the expression for semiconductors as discussed in the lecture)

$$\varepsilon(\vec{q},\omega) = 1 - \frac{e^2}{\varepsilon_0\,\Omega}\frac{2}{q^2}\sum_{\substack{\vec{k}\\ |\vec{k}|\leq k_F}}\left(\frac{1}{E(\vec{k}) - E(\vec{k}+\vec{q}) + \hbar\omega} + \frac{1}{E(\vec{k}) - E(\vec{k}+\vec{q}) - \hbar\omega}\right).$$

We consider the static limit $\omega = 0$.

a) Calculate $\varepsilon(\vec{q}, 0)$. To this end, substitute the sum over $\vec{k}$ by an integral.

b) Consider the case $q \ll 2k_F$ and take terms up to the order $\frac{1}{q^2}$ into account. Calculate the screened potential in this case

$$\tilde{V}_{\text{eff}}(\vec{q}) = \frac{\tilde{V}_{\text{el}}(\vec{q})}{\varepsilon(\vec{q})} \qquad \text{with} \qquad \tilde{V}_{\text{el}}(\vec{q}) = -\frac{e^2}{\varepsilon_0\,\Omega}\frac{1}{q^2}\,.$$

Use $\tilde{V}_{\text{eff}}(\vec{q})$ to determine the screened potential $V_{\text{eff}}(\vec{r})$ in real space.

c) Discuss the behaviour of $\varepsilon(\vec{q}, 0)$ in the limit $q \to \infty$?

d) Plot $\varepsilon(\vec{q}, 0)$.

Hint:

$$\int x\ln\left|\frac{a\,x+b}{a\,x-b}\right| = \frac{b}{a}x + \frac{1}{2}\left(x^2 - \frac{b^2}{a^2}\right)\ln\left|\frac{a\,x+b}{a\,x-b}\right|\,.$$

Problem 2: Excitons in a simple model (5 points)

Consider an electron and a hole in gallium arsenide (GaAs). The hole moves in the highest valence band (effective mass $m_h = 0.5\,m_0$), the electron moves in the lowest conduction band (effective mass $m_e = 0.06\,m_0$). They interact with one another via Coulomb interaction, screened by the effective dielectric constant of GaAs ($\epsilon_r = 13$). This leads to the formation of an *exciton*, i.e. a state in which electron and hole are bound to one another. The motion of the two particles is controlled by the effective Hamiltonian of the electron ($\hat{H}_e = \hat{p}_e^2/2m_e + E_g$), that of the hole ($\hat{H}_h = -\hat{p}_h^2/2m_h$), and the screened Coulomb interaction ($\hat{V} = -e^2/\epsilon_r|\mathbf{r}_e - \mathbf{r}_h|$). The addition of E_g in $\hat{H}_e$ reflects the fact that the conduction band starts at E_g (setting the top of the valence band to zero). The total Hamiltonian $\hat{H} = \hat{H}_e - \hat{H}_h + \hat{V}$ now acts on an effective two-particle wave function $\psi(\mathbf{r}_e, \mathbf{r}_h)$ for the exciton, with $\mathbf{r}_e$ being the position of the electron and $\mathbf{r}_h$ that of the hole. The negative sign in $-\hat{H}_h$ corresponds to the fact that an electron-hole excitation involves an energy *difference* between conduction and valence band.

Show that the Schrödinger equation $\hat{H}|\psi\rangle = E|\psi\rangle$ can be solved by a separation ansatz, $\psi(\mathbf{r}_e, \mathbf{r}_h) = \chi(\mathbf{R})\phi(\mathbf{r})$, with a center-of-mass coordinate $\mathbf{R} = (m_h\mathbf{r}_h + m_e\mathbf{r}_e)/(m_h + m_e)$ and a relative coordinate $\mathbf{r} = \mathbf{r}_e - \mathbf{r}_h$, leading to separate Schrödinger equations for $|\chi\rangle$ and $|\phi\rangle$.

Show that the resulting eigenvalues can be written as $E_{n,\mathbf{K}} = E_g + \hbar^2K^2/2(m_h + m_e) - R^*/n^2$ with an effective Rydberg constant R^* and total momentum $\mathbf{K}$ of the exciton. $E_{n,\mathbf{K}}$ is thus given by center-of-mass motion (that was neglected in the lecture) plus a hydrogen-like spectrum.

How big is R^* in the present case of GaAs?