Problem 14: BCS ground state **(5 points)**

The BCS ground state has the form

$$|\Psi_{\mathrm{BCS}}\rangle = \prod_{\vec{k}} (u_{\vec{k}} + v_{\vec{k}}\, \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}^{\dagger}_{-\vec{k}\downarrow})\, |0\rangle$$

with

$$u^2_{\vec{k}} = \frac{1}{2}\left(1 + \frac{\varepsilon_{\vec{k}}}{\sqrt{\varepsilon^2_{\vec{k}} + \Delta^2_{\vec{k}}}}\right), \quad u^2_{\vec{k}} + v^2_{\vec{k}} = 1, \quad \Delta_k = \Delta \quad \text{for} \quad |\varepsilon_{\vec{k}}| \leq \hbar\, \omega_{\mathrm{LO}}$$

and $\Delta_{\vec{k}} = 0$ else.

a) Calculate the expectation values $\langle \hat{n}_{\vec{k}\uparrow} \rangle$ and $\langle \hat{n}_{-\vec{k}\downarrow} \rangle$ of the occupation number operators $\hat{n}_{\vec{k}\uparrow} = \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}_{\vec{k}\uparrow}$ and $\hat{n}_{-\vec{k}\downarrow} = \hat{c}^{\dagger}_{-\vec{k}\downarrow}\, \hat{c}_{-\vec{k}\downarrow}$, respectively. Plot your result as a function of $\varepsilon_{\vec{k}}$ for i) $\Delta = 0$ and ii) $\Delta \neq 0$.

b) In this part, the action of creation and annihilation operators $\hat{\alpha}^{\dagger}_{\vec{k}}$, $\hat{\beta}^{\dagger}_{\vec{k}}$, $\vec{\alpha}_{\vec{k}}$, $\beta_{\vec{k}}$ resulting from the Bogoliubov transformation should be investigated.

   i) Calculate $\hat{\alpha}_{\vec{k}}\, |\Psi_{\mathrm{BCS}}\rangle$ and discuss your result.

   ii) Prove that $\hat{a}^{\dagger}_{\vec{k}'}\, |\Psi_{\mathrm{BCS}}\rangle = \hat{c}^{\dagger}_{\vec{k}'\uparrow}\, |\tilde{\Psi}_{\vec{k}'}\rangle$ with $|\tilde{\Psi}_{\vec{k}'}\rangle = \prod_{\vec{k}\neq\vec{k}'} (u_{\vec{k}} + v_k\, \hat{c}^{\dagger}_{\vec{k}\uparrow}\, \hat{c}^{\dagger}_{-\vec{k}\downarrow})\, |0\rangle$.

   iii) Calculate $\hat{\beta}^{\dagger}_{\vec{k}'}\, \hat{\alpha}^{\dagger}_{k'}\, |\Psi_{\mathrm{BCS}}\rangle$. Express your result in terms of $|\tilde{\Psi}_{\vec{k}'}\rangle$.

c) Show that the particle number operator $\hat{N}_k = \hat{n}_{\vec{k}\uparrow} + \hat{n}_{-\vec{k}\downarrow}$ can be written as

$$\hat{N}_{\vec{k}} = 2\, v^2_{\vec{k}} + (u^2_k - v^2_k)\, (\hat{\alpha}^{\dagger}_{\vec{k}}\, \hat{\alpha}_{\vec{k}} + \hat{\beta}^{\dagger}_{\vec{k}}\, \hat{\beta}_{\vec{k}}) + 2\, u_{\vec{k}}\, v_{\vec{k}}\, (\hat{\alpha}^{\dagger}_{\vec{k}}\, \hat{\beta}^{\dagger}_{\vec{k}} + \hat{\beta}_{\vec{k}}\, \hat{\alpha}_{\vec{k}})\ .$$

Calculate

$$\langle \hat{N}_{\vec{k}} \rangle = \langle \Psi_{\mathrm{BCS}} | \hat{N}_{\vec{k}} | \Psi_{\mathrm{BCS}} \rangle\ , \qquad \langle \hat{N}^2_{\vec{k}} \rangle = \langle \Psi_{\mathrm{BCS}} | \hat{N}_{\vec{k}}\, \hat{N}_{\vec{k}} | \Psi_{\mathrm{BCS}} \rangle$$

and $(\Delta\, N_{\vec{k}})^2 = \langle \hat{N}^2_{\vec{k}} \rangle - \langle \hat{N}_{\vec{k}} \rangle^2$. Express your result in terms of $\varepsilon_{\vec{k}}$ and $\Delta_{\vec{k}}$. Discuss your result.

Problem 15: Superconducting bar **(5 points)**

The current density $\vec{j}$ of superconducting electrons is related by

$$\vec{j} = -\frac{n_s \, \mathrm{e}^2}{m} \, \vec{A}\,(\vec{r})$$

to the vector potential $\vec{A}\,(\vec{r})$. Here, n_s is the particle density of the superconducting electrons.

a) Use the Maxwell equations for the static case to show that the magnetic field in a superconductor is described by the Helmholtz equation

$$\Delta\,\vec{B}\,(\vec{r}) = \mu_0 \, \frac{n_s \, \mathrm{e}^2}{m} \, \vec{B}\,(\vec{r}) \ .$$

b) Solve the Helmholtz equation for a superconducting bar (with the width $2\,a$) which is in a magnetic field. Assume that a homogeneous magnetic field $\vec{B}_0 = (0,\,0,\,B_0)$ in z direction exists outside of the bar. Neglect the y and z dependence of the field inside the bar and consider the case $\vec{B} = \vec{B}\,(x)$ with $\vec{B}\,(-\vec{a}) = \vec{B}_0 = \vec{B}\,(a)$. Plot the resulting magnetic field $\vec{B}\,(x)$.

c) Calculate the y component of the current density and plot your result.