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Algebraic Geometry II

Exercise Sheet 3

Due Date: 25.04.2019

Exercise 1:
Let k be an algebraically closed field and let $X = V(T_1T_2 - T_3^2) \subset \mathbb{A}_k^3$ and $Z = \{(0,0,0)\} \subset X$ viewed as a closed subscheme with the reduced scheme structure. Further let $\tilde{X} = \mathrm{Bl}_Z X$ denote the blow up of X in Z.

(i) Show that there is a morphism $f : X\backslash Z \to \mathbb{P}_k^1$ that is given by $(t_1, t_2, t_3) \mapsto (t_1 : t_3) = (t_3 : t_2)$ on k-valued points.

(ii) Show that f extends to a morphism $\tilde{f} : \tilde{X} \to \mathbb{P}_k^1$.

(iii) Show that $\tilde{f}^{-1}(U) \cong \mathbb{A}_k^1 \times U$ for all affine open subsets $U \subset \mathbb{P}_k^1$.
(*In fact* $\tilde{f} : \tilde{X} \to \mathbb{P}_k^1$ *is the geometric vector bundle associated to* $\mathcal{O}(2)$ *on* $\mathbb{P}_k^1$.)

Exercise 2+3:
Let k be an algebraically closed field and let $C = V_+(T_2^2T_3 - (T_1^3 + T_1^2T_3)) \subset \mathbb{P}_k^3$ i.e. C is the closure of the closed subscheme $C' = V(t_2^2 - (t_1^3 + t_1^2))C \cap \mathbb{A}_k^2 \subset \mathbb{A}_k^2 = D_+(T_3)$ in $\mathbb{P}_k^2$. Let $P_0 = (0,0) \in C' \subset C$.

(i) Show that the morphism $\phi' : \mathbb{A}_k^1 \to C'$ that is given by $t \mapsto (t^2 - 1, t(t^2 - 1))$ on k-valued points, extends to a unique morphism
$$\phi : \mathbb{P}_k^1 \to C$$
such that $\phi^{-1}(P_0)$ consists of two points $x, y \in \mathbb{P}_k^1$, and that the restriction of ϕ induces an isomorphism $\mathbb{P}_k^1\backslash\{x,y\} \cong C\backslash\{P_0\}$.

(ii) We fix an automorphism of $\mathbb{P}_k^1$ mapping $\{0,\infty\}$ to $\{x,y\}$ and replace ϕ by its the composition with this automorphism (so that $\phi : \mathrm{Spec}\, k[T^{\pm 1}] = \mathbb{P}_k^1\backslash\{0,\infty\} \cong C\backslash\{P_0\}$).
Given $\lambda \in k^\times$, show that the automorphism $t \mapsto \lambda t$ of $\mathrm{Spec}\, k[t^{\pm 1}]$ extends to a commutative diagram
$$\begin{array}{ccc} \mathbb{P}_k^1 & \longrightarrow & \mathbb{P}_k^1 \\ \phi \downarrow & & \downarrow \phi \\ C & \xrightarrow{\psi_\lambda} & C, \end{array}$$
where the horizontal arrows are isomorphisms.

(iii) Let $X_1 = C \times \mathbb{P}_k^1\backslash\{0\}$ and $X_2 = C \times \mathbb{P}_k^1\backslash\{\infty\}$, and $U = U_1 = U_2 = C \times \mathbb{P}_k^1\backslash\{0,\infty\} = C \times \mathrm{Spec}\, k[t^{\pm 1}]$. Show that there is an isomorphism $\psi : U_1 \to U_2$ that is given by $(c, a) \mapsto (\psi_a(c), a)$ on k-valued points.

(iv) Show that X_1 and X_2 glue along ψ to a scheme X together with a morphism $f : X \to \mathbb{P}_k^1$.
(*this morphism is by construction projective locally on the target. But it is not a projective morphism. This is not part of the exercise!*)

(v) Let $\tilde{X} \to X$ denote the blow up of X in $(\{P_0\} \times \mathbb{P}^1_k \backslash \{0\}) \cup (\{P_0\} \times \mathbb{P}^1_k \backslash \{\infty\})$. Show that the composition $\tilde{X} \to \mathbb{P}^1_k$ is a projective morphism.
(*in fact it is the projective bundle associated to* $\mathcal{O}_{\mathbb{P}^1_k} \oplus \mathcal{O}_{\mathbb{P}^1_k}(1)$.)

Exercise 4:

Let A be a noetherian domain of dimension 1 with fraction field K and let L be a finite extension of K. Let $A \subset B \subset L$ be any subring. We want to show that B is noetherian and of dimension at most 1, using the following steps:

(i) Reduce to the case $K = L$.

(ii) Let $0 \neq \mathfrak{b} \subset B$ be any ideal and let $0 \neq f \in A \cap \mathfrak{b}$. Show that B/fB is of finite length over A/fA.

(iii) Deduce that the ideal $\mathfrak{b}$ in (ii) is finitely generated and that it is of height 1 if it is a prime ideal.

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