Prof. Dr. E. Hellmann
S. Huang SS 2019

Algebraic Geometry II

Exercise Sheet 1

Due Date: 11.04.2019

Exercise 1:

(i) Show that a morphism $f : X \to Y$ is a monomorphism (i.e. $f \circ g = f \circ h \Rightarrow g = h$ for morphisms $g, h : T \to X$) if and only if the diagonal $\Delta_f : X \to X \times_Y X$ is an isomorphism. (In particular Δ_f is an isomorphism if f is a locally closed immersion).

(ii) Let $f : X \to Y$ and $g : Y' \to Y$ be morphisms of schemes.
We write $f' = \mathrm{pr}_{Y'} : X' = X \times_Y Y' \to Y'$ resp. $g' = \mathrm{pr}_X : X' \to X$ for the base change of f resp. g. Show that the diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle \Delta_{f'}} & & \downarrow{\scriptstyle \Delta_f} \\ X' \times_{Y'} X' & \xrightarrow{g' \times g'} & X \times_Y X \end{array}$$

is cartesian (i.e. a fiber product).

(iii) Let $f : X \to Y$ and $Y \to Z$ be morphisms of schemes. Show that the diagram

$$\begin{array}{ccc} X \times_Y X & \xrightarrow{G} & X \times_Z X \\ \downarrow & & \downarrow{\scriptstyle f \times f} \\ Y & \xrightarrow{\Delta_g} & Y \times_Z Y, \end{array}$$

where G is induced by the identity on X in both factors, is cartesian and that $\Delta_{g \circ f} = G \circ \Delta_f$.

Exercise 2:

Let A be a ring and let $p : \mathbb{P}^n_A \to \operatorname{Spec} A$ be the projection from the n-dimensional space over A to $\operatorname{Spec} A$. Show that p is separated by proving that Δ_p is a closed immersion.

Exercise 3:

(i) Let $f, g : X \to Y$ be morphisms of S-schemes and assume that X is reduced and that Y is separated over S. Assume that there is a dense open subscheme $U \subset X$ such that $f|_U = g|_U$. Show that $f = g$.
(*Hint: Show that the graphs* Γ_f *and* Γ_g *coincide.*)

(ii) Let $f : X \to Y$ be a separated morphism. Let $g : Y \to X$ be a section of f, i.e. a morphism such that $f \circ g = \mathrm{id}_Y$. Show that g is a closed immersion.

(iii) Let $f : X \to Y$ be a separated morphism and let $U, V \subset X$ be open subsets that are affine over Y (i.e. the morphisms $U \to Y$ and $V \to Y$ induced by restriction of f are affine morphisms). Show that $U \cap V$ is affine over Y.

Exercise 4: Let $\mathcal{P}$ be a property of morphisms of schemes such that:

(a) closed immersions have the property $\mathcal{P}$,

(b) the property $\mathcal{P}$ is stable under composition and base change.

Show that

(i) If $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ have $\mathcal{P}$, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ has $\mathcal{P}$.

(ii) Let $f : X \to Y$ and $g : Y \to Z$ be morphisms such that g is separated $g \circ f$ has $\mathcal{P}$. Then f has $\mathcal{P}$.

(iii) If $f : X \to Y$ has $\mathcal{P}$, then $f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ has $\mathcal{P}$.

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