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Algebraic Geometry II

Exercise Sheet 2

Due Date: 18.04.2019

Exercise 1:

(i) Let $v : k((t))^\times \to \mathbb{Z}$ denote the map $\sum_{i\in\mathbb{Z}} a_i t^i \mapsto \min\{i | a_i \neq 0\}$. Show that v is a (discrete) valuation.

(ii) Let $K = \lim_{\to n} k((t^{1/n}))$ with the obvious transition maps. Show that there exists a valuation $v : K^\times \to \mathbb{Q}$ with $v(t^{1/n}) = 1/n$. Show that the corresponing valuation ring R is not noetherian and describe $\operatorname{Spec} R$ (as a topological space).

(iii) Let

$$A = \mathbb{Z}_p\langle T\rangle = \left\{ \sum_{i\geq 0} a_i T^i \in \mathbb{Z}_p[\![T]\!] \text{ such that } v_p(a_i) \to \infty \text{ for } i \to \infty \right\},$$

where v_p is the p-adic valuation. Let $\Gamma = \mathbb{Z} \times \langle\gamma\rangle$, where $\langle\gamma\rangle$ is the (infinite) cyclic group generated by one element γ. We equipp Γ with the lexicographic order, i.e. $(m, \gamma^n) \leq (m', \gamma^{n'})$ iff $m \leq m'$ or $m = m'$ and $n \leq n'$. Show that

$$w : \sum_{i\geq 0} a_i T^i \longmapsto \min\{(v_p(a_i), \gamma^i)\}$$

extends to a valuation on the fraction field K of A. Let $R \subset K$ denote the corresponding valuation ring. Show that $\operatorname{Spec} R$ has at least three points.
(*Hint: let w' bet the composition of w with the projection $\Gamma \to \mathbb{Z}$ and show that $\{x \in K^\times | w'(x) > 0\} \cup \{0\}$ is a prime ideal in R*)

Exercise 2:
Let R be a local noetherian ring of dimension 1. Show that the following are equivalent:

(a) R is a discrete valuation ring,

(b) R is normal,

(c) R is regular.

Exercise 3:
Let X be a proper k-scheme and let C be curve over k, i.e. C is a 1-dimensional integral scheme of finite type over k. Let $P \in C$ be a closed point such that C is smooth at P (i.e. $\mathcal{O}_{C,P}$ is a discrete valuation ring). Let $f : C\backslash\{P\} \to X$ be a morphism of k-schemes. Show that there is a unique morphism $\tilde{f} : C \to X$ extending f.

Exercise 4:

(i) Let $f : X \to Y$ be a morphism of finite type and assume that Y is locally noetherian. Show that in the valuative criteria for separatedness and properness it is enough to consider

  (a) complete discrete valuation rings.

  (b) discrete valuation rings with algebraically closed residue field.

(ii) Let $f : X \to Y$ be a morphism of finite type. For $i = 1, \dots, n$ let $X_i \subset X$ and $Y_i \subset Y$ be closed subschemes such that $f|_{X_i} \to Y$ factors over $Y_i \hookrightarrow Y$ and such that $X = \bigcup_{i=1}^n X_i$. Write $f_i : X_i \to Y_i$ for the induced morphisms. Show that f is proper if and only if all the f_i are proper.

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