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S. Huang SS 2019

Algebraic Geometry II

Exercise Sheet 5

Due Date: 09.05.2019

Exercise 1:

Let X be a noetherian scheme, $U \subset X$ an open subscheme and $\mathscr{F}$ a coherent sheaf on U.

(i) Show that there exists a coherent sheaf $\mathscr{F}'$ on X such that $\mathscr{F}'|_U = \mathscr{F}$.

(ii) Let $\mathscr{G}$ be a coherent sheaf on X such that $\mathscr{F} \subset \mathscr{G}|_U$. Show that there exists a coherent sheaf $\mathscr{F}' \subset \mathscr{G}$ on X such that $\mathscr{F}'|_U = \mathscr{F}$.

Exercise 2:

Let $\mathscr{L}$ be a line bundle on a noetherian scheme X. Show that the following are equivalent:

(a) $\mathscr{L}$ is ample

(b) $\mathscr{L}^{\otimes n}$ is ample for all $n \geq 1$.

(c) There exists some $n \geq 1$ such that $\mathscr{L}^{\otimes n}$ is ample.

Exercise 3:

(i) For $m, n \in \mathbb{Z}$ consider the line bundle $\mathscr{L}_{(m,n)} = \mathrm{pr}_1^*\mathcal{O}(m) \otimes_{\mathcal{O}_X} \mathrm{pr}_2^*\mathcal{O}(n)$ on $X = \mathbb{P}^1_k \times \mathbb{P}^1_k$. Show that $\mathscr{L}_{(m,n)}$ is very ample if and only if $\mathscr{L}_{(m,n)}$ is ample if and only if $m, n > 0$.

(ii) Let $X = V_+(T_2^2T_3 - (T_1^3 - T_1T_3^2)) \subset \mathbb{P}^2_k$. Let $P = [(0,1,0)] \in \mathbb{P}^2_k$ and let $\mathscr{I}_P \subset \mathcal{O}_X$ be the sheaf of ideals corresponding to the closed subscheme $\{P\}$ with the reduced scheme structure. Show that $\mathscr{I}_P$ is a line bundle and let $\mathscr{L} = \mathscr{I}_P^\vee$. Show that $\mathscr{L}^{\otimes 3} \cong \mathcal{O}_{\mathbb{P}^2_k}(1)|_X$ but that $\mathscr{L}$ is not generated by global sections.
(*This shows that $\mathscr{L}$ is ample but not very ample*)

Exercise 4:

Let k be an algebraically closed field and let X be a proper k-scheme. Let $\mathscr{L}$ be a line bundle on X and let $\varphi : \mathcal{O}_X^{n+1} \to \mathscr{L}$ be a surjection corresponding to a morphism $g : X \to \mathbb{P}^n_k$. Let $V \subset \Gamma(X, \mathscr{L})$ denote the sub-k-vector space generated by the images of the standard basis of $\mathcal{O}_X^{n+1}$. Assume that

(a) for any two closed points $x \neq y \in X$ there exists $s \in V$ such that $0 = s(x) \in \mathscr{L} \otimes \kappa(x)$ and $0 \neq s(y) \in \mathscr{L} \otimes \kappa(y)$ (or vice versa).

(b) for any closed point $x \in X$ the set $\{s_x \bmod \mathfrak{m}_x^2\mathscr{L}_x \mid s \in V,\ s_x \in \mathfrak{m}_x\mathscr{L}_x\}$ spans the $\kappa(x)$-vector space $\mathfrak{m}_x\mathscr{L}_x/\mathfrak{m}_x^2\mathscr{L}_x$, where $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$ is the maximal ideal.

Show that g is a closed immersion (especially $\mathscr{L}$ is very ample).
(*Hint: In order to show that $\mathcal{O}_{\mathbb{P}^n_k, g(x)} \to \mathcal{O}_{X,x}$ is surjective for all closed points $x \in X$, show that a local homomorphism of local rings $(A, \mathfrak{m}_A) \to (B, \mathfrak{m}_B)$ is surjective, if it induces an isomorphism on residue fields, it is finite (i.e. B is finitely generated as an A-module) and the canonical morphism $\mathfrak{m}_A \to \mathfrak{m}_B/\mathfrak{m}_B^2$ is surjective.*)

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