Prof. Dr. E. Hellmann
S. Huang

SS 2019

Algebraic Geometry II

Exercise Sheet 8

Due Date: 06.06.2019

Exercise 1:

(i) Let k be a field of characteristic p and let $t \in k$ be an element that is not a p-th power. Show that the curve $X = \mathrm{Spec}\ k[X,Y]/(Y^2 - X^p + t)$ is regular (i.e. all local rings are regular) but not smooth.

(ii) Determine the set of points at which the following morphisms are smooth.

(a) $\mathrm{Spec}\,\mathbb{Z}[T]/(T^2+1) \to \mathrm{Spec}\,\mathbb{Z}$.

(b) $\mathrm{Spec}\,A[T_1,\dots,T_n]/(T_1T_2\cdots T_n - \varpi) \to \mathrm{Spec}\ A$, where A is a discrete valuation ring and $\varpi \in A$ is a uniformizer.

(iii) Let k be a field of characteristic $\mathrm{char}\, k \neq 2$ and let $X = \mathrm{Spec}\ k[T_1,T_2]/(T_2^2(1-T_1^2)-1) \subset \mathbb{A}_k^2$. Compute the set of all points where dT_2 (resp. dT_1) generates Ω_X^1/k, i.e. the set of point where T_2 (resp. T_1) is a uniformizing parameter. This is the set of points where the morphism $g : X \to \mathbb{A}_k^1$ defined by T_2 (resp. T_1) is étale. Compute the sheaf of differentials of g.

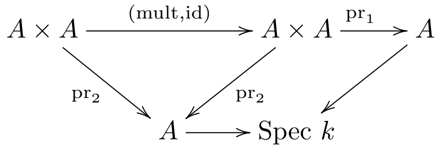
(iv) For $X = \mathrm{Spec}\ k[T_1,T_2]/(T_2^2 - T_1^2(T_1+1)) \subset \mathbb{A}_k^2$ (resp. $X = \mathrm{Spec}\ k[T_1^2,T_2^2]/(T_2^2 - T_1^3)$) let $g : \tilde{X} \to X$ denote the blow-up at $(0,0) \in X$. Show that $\tilde{X}$ is smooth over k and compute the tangent space $T_{(0,0)}X$, the morphism on tangent spaces induced by g and the differentials $\Omega^1_{\tilde{X}/X}$.

Exercise 2:

(i) Let X and Y be S-schemes and let $\mathrm{pr}_X : X \times_S Y \to X$ resp. $\mathrm{pr}_Y : X \times_S Y \to Y$ denote the canonical projections. Show that $\Omega^1_{X\times_S Y/S} \cong \mathrm{pr}_X^*\Omega^1_{X/S} \oplus \mathrm{pr}_Y^*\Omega^1_{Y/S}$.

(ii) Compute the cotangent bundle of the projective space: Let A be a ring and $\mathbb{P}_A^n \to \mathrm{Spec}\ A$ be the projective space of dimension n over A. Show that the kernel of the universal quotient $\mathcal{O}_{\mathbb{P}_A^n}^{n+1} \to \mathcal{O}(1)$ is given by $\Omega_{\mathbb{P}_A^n/A}(1)$. Especially show that $\Omega^1_{\mathbb{P}_A^1/A} \cong \mathcal{O}(-2)$.

(iii) Let $p : G \to \mathrm{Spec}\ k$ be a group scheme with unit $e : \mathrm{Spec}\ k \to G$. Let $\Omega^1_{G/k}(e)$ denote the fiber of $\Omega^1_{G/k}$ at e. Show that $\Omega^1_{G/k} \cong \mathcal{O}_G \otimes_k \Omega^1_{G/k}(e) = p^*e^*\Omega^1_{G/k}$. Especially $\Omega^1_{G/k}$ is free.
Hint: consider the commutative diagram



Exercise 3:

Let k be a field and let X be an integral k-scheme of finite type. Let $K = k(X) = \mathcal{O}_{X,\eta}$ denote its function field (here $\eta \in X$ is the generic point). Show that the following conditions are equivalent:

(i) The field extension K over k is separable.

(ii) There exists a dense open subset $U \subset X$ such that U is smooth over k.

Exercise 4:

Let R be a ring and let

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & {\scriptstyle\gamma}\downarrow & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

be a commutative diagram of R-modules with exact rows. Show that there is a canonical exact sequence

$$\ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma$$

(which is functorial in the above diagram).

Exercise 5:

Let R be a ring and let

$$0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$$

be a short exact sequence of complexes of A-modules. Show that there is a long exact sequence

$$\cdots \longrightarrow H^{n-1}(C^\bullet) \longrightarrow H^n(A^\bullet) \longrightarrow H^n(B^\bullet) \longrightarrow H^n(C^\bullet) \longrightarrow H^{n+1}(A^\bullet) \longrightarrow \dots$$

of R-modules.

Exercise 6:

(i) Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories and let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor that admits an exact additive left adjoint functor $G : \mathcal{B} \to \mathcal{A}$. Show that F preserves injective objects.

(ii) Let X be a topological space. Show that the category $\mathrm{PSh}(X, \mathbb{Z})$ of presheaves of abelian groups has enough injectives.
Hint: Let $U \subset X$ be open. Use that

$$A \longmapsto \left(\underline{A}_U : V \mapsto \begin{cases} A & \textit{if } U \subset V \\ 0 & \textit{otherwise} \end{cases} \right)$$

defines a functor from abelian groups to $\mathrm{PSh}(X, \mathbb{Z})$ *adjoint to the functor* $\Gamma(U, -)$.

Homepage: `https://www.uni-muenster.de/Arithm/hellmann/veranstaltungen`