Some vanishing theorems on manifolds with almost nonnegative/ nonpositive Ricci curvature

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A classical vanishing theorem

Theorem: Let (M, g) ba a compact Riemannian manifold with nonnegative Ricci curvature and nonzero first Betti number, then the Euler number of M is zero.

Question: Can we generalize this vanishing theorem under a weaker curvature assumption, for example, almost nonnegative Ricci curvature?

If M^n admits a sequence of Riemannian metrics g_i such that

$$Ric(g_i) \geq -\frac{n-1}{i}$$

$$D(g_i) \leq 1,$$

then we say that M^n has almost nonnegative Ricci curvature

Example: Let M^4 be the manifold performing surgery along a meridian curve in T^4, i.e, removing a tubular neighborhood of the curve and attaching a copy of $D^2 \times S^2$. Anderson showed that M^4 admits a sequence of Riemannian metrics g_i such that

$$|Ric(g_i)| \leq \frac{n-1}{i},$$

$$D(g_i) \leq 1.$$

However, M^4 has nonzero Euler number.

However, the curvature operator of g_i constructed by Anderson does *not* have a uniform lower bound. M^4 in fact can *not* admit a sequence of Riemannian metrics g_i of almost nonnegative Ricci curvature with curvature operator uniformly bounded from below by the following theorem.

Theorem (X. Chen): Let M^n be a closed Riemannian manifold with almost nonnegative Ricci curvature. If the curvature operator of g_i is uniformly bounded from below by $-Id$, then

1. the Euler number of M^n is zero.

2. Moreover, for any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$, there exists some $t \in \mathbb{R}, t \neq 0$ such that $H^p(M, t\theta) = 0$ for any p, where $H^p(M, t\theta)$ is the Morse-Novikov cohomology group with respect to $t\theta$.

Remark: If we replace the assumption on curvature operator by $\frac{1}{Vol(g_i)} \int_M |\Re_i|^p dV \leq C$, where $\Re_i$ is the Riemannian curvature tensor of g_i and $p > n/2$, then the same conclusion holds.

Definitions of Morse-Novikov cohomology

Let M^n be a smooth manifold and θ a real valued closed one form on M^n. Set $\Omega^p(M^n)$ the space of real smooth p-forms and define $d_\theta : \Omega^p(M^n) \to \Omega^{p+1}(M^n)$ as $d_\theta\alpha = d\alpha + \theta \wedge \alpha$ for $\alpha \in \Omega^p(M^n)$. Then we have a complex

$$\cdots \to \Omega^{p-1}(M^n) \xrightarrow{d_\theta} \Omega^p(M^n) \xrightarrow{d_\theta} \Omega^{p+1}(M^n) \to \cdots$$

whose cohomology $H^p(M,\theta) = H^p(\Omega^*(M^n), d_\theta)$ is called the p-th Morse-Novikov cohomology group of M^n with respect to θ.

Remark: Novikov introduced a cohomology theory to generalize Morse inequalities to closed one forms. His theory turns out to be equivalent to the above definitions which was firstly given by Pazhitnov after Witten's famous work on Morse theory.

Basic properties

1. If $\theta' = \theta + df, f \in C^\infty(M^n, \mathbb{R})$, then for any p, we have $H^p(M^n, \theta') \simeq H^p(M^n, \theta)$ and the isomorphism is given by the map $[\alpha] \mapsto [e^f \alpha]$;

2. If θ_1, θ_2 are two representatives in the cohomology class $[\theta]$, then $H^p(M, \theta_1) \simeq H^p(M, \theta_2)$. Hence $H^p(M, \theta)$ only depends on the de Rham cohomology class of θ.

3. If $[\theta] \neq 0$ and M^n is connected and orientable, then $H^0(M^n, \theta)$ and $H^n(M^n, \theta)$ vanish.

4. the integration $\int : H^p(M^n, \theta) \times H^{n-p}(M^n, -\theta), (\alpha, \beta) \mapsto \int_{M^n} \alpha \wedge \beta$ induces an isomorphism $H^p(M^n, \theta) \simeq (H^{n-p}(M^n, -\theta))^*$.

5. $\sum_{p=0}^{n} (-1)^p dim H^p(M^n, \theta)$ is equal to the Euler characteristic number of M^n;

6. If N^d be a d-dimensional manifold and γ be a closed one form on N^d, then we have $H^k(M^n \times N^d, \pi_1^*\theta + \pi_2^*\gamma) \simeq \bigoplus_{p+q=k} H^p(M^n, \theta) \bigotimes H^q(N^d, \gamma)$, where $\pi_1 : M^n \times N^d \to M^n, \pi_2 : M^n \times N^d \to N^d$ are the projection maps.

7. If $\pi : \widehat{M}^n \to M^n$ is a covering space with finite sheet, then $\pi^* : H^p(M^n, \theta) \to H^p(\widehat{M}^n, \pi^*\theta)$ is injective for any p.

8. Example: Let M^n be n-dimensional torus, then $H^p(M^n, \theta) = 0$ for any p and $[\theta] \neq 0$.

cohomology of local system

Let θ be a closed one form on M^n. Consider the following linear representation of the fundamental group of M^n:

$$\rho : \pi_1(M^n) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{\int_\gamma \theta}.$$

The representation ρ defines a complex rank one local system $\mathbb{C}_\rho$ over M^n. We denote by $H^p(M^n, \mathbb{C}_\rho)$ the p-th cohomology group of M^n with coefficients in this local system.

Theorem: $H^p(M^n, \theta) \simeq H^p(M^n, \mathbb{C}_\rho)$ for any p.

Vanishing theorems of Morse-Novikov cohomology

We always assume that M^n has nonzero first de Rham cohomology group.

1. (Măcinic, Panadima) If the fundamental group of an aspherical manifold M^n has a finitely generated nilpotent subgroup of finite index, then $H^p(M^n, \theta) = 0$ for any p and $[\theta] \neq 0$.

2. (Măcinic, Panadima) If the fundamental group of a compact manifold M^n has a finitely generated nilpotent subgroup of finite index, then $H^1(M^n, \theta) = 0$ for any $[\theta] \neq 0$.

3. (X.Chen) If M^n is an almost nilpotent space, then the Morse-Novikov cohomology $H^p(M, \theta) = 0$ for any p and $[\theta] \neq 0$.

Definition: 1. M^n is a nilpotent space if $\pi_1(M^n)$ is a nilpotent group that operates nilpotently on $\pi_k(M^n)$ for every $k \geq 2$, i.e, $V = \pi_k(M^n)$ admits a finite sequence of $\pi_1(M^n)$-invariant subgroups

$$V = V_0 \supset V_1 \supset V_2 \supset \cdots \supset V_k = 0$$

such that the induced action of $\pi_1(M^n)$ on V_j / V_{j+1} is trivial for any j.
2. M^n is an almost nilpotent space if there is a finite cover of M^n which is a nilpotent space.

Morse-Novikov cohomology and almost nonnegative sectional curvature

Definition: M^n has almost nonnegative sectional curvature if it admits a sequence of Riemannian metrics g_i such that

$$sec(g_i) \geq -\frac{1}{i}$$

$$D(g_i) \leq 1.$$

Corollary: Let M^n be a closed Riemannian manifold of almost nonnegative sectional curvature, then the Morse-Novikov cohomology $H^p(M, \theta) = 0$ for any p and $[\theta] \neq 0$. (By a theorem of Kapovitch, Petrunin, Tuschmann, a closed Riemannian manifold of almost nonnegative sectional curvature is an almost nilpotent space)

Example: If M is *not* a nilpotent space, its Morse-Novikov cohomology does not necessarily vanish as the following example shows. Let $h : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3 \times \mathbb{S}^3$ be defined by

$$h : (x, y) \to (xy, yxy).$$

Let M be the mapping torus of h. Then M has the structure of a fiber bundle:

$$\mathbb{S}^3 \times \mathbb{S}^3 \to M \to \mathbb{S}^1.$$

The induced map $h^{*,3}$ on $H^3_{dR}(\mathbb{S}^3 \times \mathbb{S}^3)$ is given by the matrix

$$A_h = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

Notice that the eigenvalues of A_h are different from 1 in absolute value. Hence M is *not* a nilpotent space.

Let λ be a eigenvalue of A_h with $\lambda = e^{-t}, t \neq 0, t \in \mathbb{R}$ and θ a generator of $H^1_{dR}(M)$. We claim that $H^3(M, t\theta) \neq 0$. In fact, $t\theta$ defines a linear representation of the fundamental group of M:

$$\rho_t : \pi_1(M) \to GL(1, \mathbb{C}) = \mathbb{C}^*, [\gamma] \mapsto e^{t\int_\gamma \theta}.$$

The representation ρ_t defines a complex rank one local system $\mathbb{C}_{\rho_t}$ over M^n. Then we have $H^p(M, t\theta) \simeq H^p(M^n, \mathbb{C}_{\rho_t})$. By Wang's exact sequence, we have

$$\dim_{\mathbb{C}} H^p(M^n, \mathbb{C}_{\rho_t}) = \dim_{\mathbb{C}} \ker(h^{*,p} - e^{-t} Id) + \dim_{\mathbb{C}} coker(h^{*,p-1} - e^{-t} Id)$$

As e^{-t} is an eigenvalue of $h^{*,3}$, we see that $\dim_{\mathbb{C}} \ker(h^{*,3} - e^{-t} Id) > 0$ and $H^3(M, t\theta) \neq 0$.

What about Ricci curvature

1. If M^n admits a Riemannian metric of nonnegative Ricci curvature, then $H^p(M, \theta) = 0$ for any p and $[\theta] \neq 0$.

2. The same conclusion fails if M^n admits a sequence of Riemannian metrics g_i such that

$$Ric(g_i) \geq -\frac{n-1}{i}$$

$$D(g_i) \leq 1.$$

(We say M^n has almost nonnegative Ricci curvature)

Example: Let M^4 be the manifold performing surgery along a meridian curve in T^4, i.e, removing a tubular neighborhood of the curve and attaching a copy of $D^2 \times S^2$. Anderson showed that M^4 admits a sequence of Riemannian metrics g_i such that

$$|Ric(g_i)| \leq \frac{n-1}{i}$$

$$D(g_i) \leq 1.$$

Moreover, its fundamental group is isomorphic to $\mathbb{Z}^3$ and its Euler characteristic number is nonzero. For any $[\theta] \in H^1_{dR}(M^4), [\theta] \neq 0$, we get $H^p(M^4, \theta) = 0$ for $p \neq 2$ and $H^2(M^4, \theta) \neq 0$.

Known topological obstruction (Cheeger-Colding, Kapovitch-Wilking): The fundamental group of a Riemannian manifold with almost nonnegative Ricci curvature is almost nilpotent.

However, the curvature operator of g_i constructed by Anderson does *not* have a uniform lower bound. M^4 in fact can *not* admit a sequence of Riemannian metrics g_i of almost nonnegative Ricci curvature with curvature operator uniformly bounded from below by the following:

Theorem (X.Chen): Let M^n be a closed Riemannian manifold admiting a sequence of Riemannian metrics g_i such that

$$Ric(g_i) \geq -\frac{n-1}{i}$$

$$D(g_i) \leq 1.$$

If the curvature operator of g_i is uniformly bounded from below by $-Id$, then for any $[\theta] \in H^1_{dR}(M^n), [\theta] \neq 0$, there exists some $t \in \mathbb{R}, t \neq 0$ such that $H^p(M, t\theta) = 0$ for any p, where $H^p(M, t\theta)$ is the Morse-Novikov cohomology group with respect to $t\theta$.

An integral formula of differential forms

Let M^n be a compact Riemannian manifold and X a smooth vector field on M^n. Then for any smooth differential form u on M^n, we have the following integral formula:

$$\int_{M^n} \langle i_X du, u\rangle + \langle i_X u, \delta u\rangle = \int_{M^n} -\frac{1}{2} div(X)|u|^2 + \langle i_{\nabla_{e_i} X} u, i_{e_i} u\rangle,$$

where δ is the dual of d.

Proof of the integral formula

Let $\{e_i\}$ be a local orthonormal frame and $\{\theta^i\}$ be its dual frame. Let $Y = \langle i_X u, i_{e_i} u \rangle e_i$, then Y is a global defined vector field on M^n. By direct computation, we have

$$div(Y) = \langle i_{\nabla_{e_i} X} u, i_{e_i} u \rangle + \langle \nabla_X u, u \rangle - \langle i_X du, u \rangle - \langle i_X u, \delta u \rangle$$

$$div(|u|^2 X) = \langle \nabla_{e_i}(|u|^2 X), e_i \rangle = div(X)|u|^2 + \langle \nabla_X u, u \rangle + \langle u, \nabla_X u \rangle.$$

Then we see that

$$\frac{1}{2} div(|u|^2 X) - divY = \frac{1}{2} div(X)|u|^2 - \langle i_{\nabla_{e_i} X} u, i_{e_i} u \rangle + \langle i_X du, u \rangle + \langle i_X u, \delta$$

By Stokes' theorem, we see that

$$\int_{M^n} \langle i_X du, u \rangle + \langle i_X u, \delta u \rangle = \int_{M^n} -\frac{1}{2} div(X)|u|^2 + \langle i_{\nabla_{e_i} X} u, i_{e_i} u \rangle$$

Proof of the vanishing theorem

The proof is based on Hodge theory of Morse-Novikov cohomology: Let d_θ^* be the formal L^2 adjoint of d_θ with respect to g_i and $\Delta_\theta = d_\theta d_\theta^* + d_\theta^* d_\theta$ the corresponding Laplacian. Then one obtains an orthogonal decomposition

$$\Omega^p(M^n) = \mathcal{H}^p(M^n) \oplus d_\theta(\Omega^{p-1}(M^n)) \oplus d_\theta^*(\Omega^{p+1}(M^n)),$$

where $\mathcal{H}^p(M^n)$ is the space of Δ_θ harmonic forms, which is isomorphic to $H^p(M^n, \theta)$.

By Hodge theory, we can choose a harmonic form θ_i in the cohomology class $[\theta_i]$. Let $V(g_i)$ be the volume of (M^n, g_i), X_i the dual vector field of θ_i defined by $g_i(X_i, Y) = \theta_i(Y)$. Set $t_i = (\frac{V(g_i)}{\int_{M^n} |X_i|^2 dV_i})^{1/2} > 0$. Choose a $\Delta_{t_i\theta_i}$ harmonic form α_i in $H^p(M^n, t_i\theta_i)$.
The idea is to show that $\alpha_i \equiv 0$ for sufficiently large i.

Firstly we have the following integral inequality:

$$\int_{M^n} {t_i}^2 |X_i|^2 |\alpha_i|^2 dV_i \leq C_n \int_{M^n} (t_i |\nabla X_i| + t_i |div(X_i)|) |\alpha_i|^2 dV_i$$

for some constant C_n depending only on n.
Combining the following Bochner formula

$$\int_{M^n} |\nabla X_i|^2 dV_i \leq \frac{n-1}{i} \int_{M^n} |X_i|^2 dV_i,$$

we get for sufficiently large i,

$$\int_{M^n} |\alpha_i|^2 dV_i \leq \frac{1}{2} \int_{M^n} |\alpha_i|^2 dV_i.$$

Hence $\alpha_i \equiv 0$.

Remark: 1. In the proof, we need a Poincare-Sobolev type inequality which is available due to the assumption $Ricci(g_i) \geq -\frac{n-1}{i}$ and $D(g_i) \leq 1$.

2. We also need to apply Moser's iteration to prove a mean value inequality which allows us to control the L^∞ norm of α_i in terms its L^2 norm. This was done by applying Bochner formula to α_i and we get

$$\frac{1}{2}\Delta|\alpha_i|^2 \geq |\nabla\alpha_i|^2 - |d\alpha_i|^2 - |d^*\alpha_i|^2 - C_n|\alpha_i|^2$$

for some positive constant C_n depending only on n as the curvature operator of g_i is bounded from below by $-Id$.

3. As C_n depends only on n, we can then show that those constants appearing in the mean value inequality are independent of i.

Question: Can we apply the above method to other Dirac operator? For example, can we prove that a compact spin $4n$-dimensional manifold with almost nonnegative Ricci curvature and nonzero first Betti number has vanising $\hat{A}$ genus?
There is a serious trouble which comes from the integral formula!

Theorem (X.Chen and F.Han): Given positive numbers given positive numbers $p, n, \lambda_1, \lambda_2$ with $p > 2n$, there exists some $\epsilon = \epsilon(p, n, \lambda_1, \lambda_2) > 0$ such that for any compact $4n$-dimensional spin Rimannian manifold (M, g) with infinite isometry group and $-\lambda_1 \leq Ric(g) \leq \epsilon$, $diam(g) \leq 1$ and $\frac{1}{V(g)} \int_M |\Re_g|^p dV \leq \lambda_2$, the elliptic genera of M vanish.

Remark: Let (M, g) be a compact $4n$-dimensional spin Rimannian manifold with nonpositive Ricci curvature and infinite isometry group, then the elliptic genera of M vanish.

By the Atiyah-Singer index theorem, when the manifold M is orientable, $Ell_1(M)$ can be expressed analytically as index of the twisted Dirac operator $d_s \otimes \Theta(T_{\mathbb{C}}M) \otimes \Theta_1(T_{\mathbb{C}}M)$, where d_s is the signature operator; and further when M is spin, $Ell_2(M)$ can be expressed analytically as index of the twisted Dirac operator $D \otimes \Theta(T_{\mathbb{C}}M) \otimes \Theta_2(T_{\mathbb{C}}M))$, where D is the Atiyah-Singer spin Dirac operator on M.

Formally, we have

$$\Theta(T_{\mathbb{C}}M)\otimes\Theta_1(T_{\mathbb{C}}M) = A_0(T_{\mathbb{C}}M)+A_1(T_{\mathbb{C}}M)q+A_2(T_{\mathbb{C}}M)q^2+\cdots ,$$

$$\Theta(T_{\mathbb{C}}M)\otimes\Theta_2(T_{\mathbb{C}}M)) = B_0(T_{\mathbb{C}}M)+B_1(T_{\mathbb{C}}M)q^{\frac{1}{2}}+B_2(T_{\mathbb{C}}M)q+\cdots .$$

Three important facts used in the proof:

1. (Witten-Bott-Taubes-Liu): The twisted Dirac operator $d_s \otimes \Theta(T_{\mathbb{C}}M) \otimes \Theta_1(T_{\mathbb{C}}M)$ and $D \otimes \Theta(T_{\mathbb{C}}M) \otimes \Theta_2(T_{\mathbb{C}}M))$ are rigid.

2. $Ell_2(M)$ is determined by the index of $D \otimes B_k(T_{\mathbb{C}}M), 0 \leq k \leq [\frac{n}{2}]$.

3. $Ell_1(M)$ is determined by $Ell_2(M)$.

Let M be a closed smooth manifold and P be a Fredholm operator on M. We assume that a compact connected Lie group G acts on M nontrivially and that P is G-equivariant, by which we mean it commutes with the G action. Then the kernel and cokernel of P are finite dimensional representations of G. The equivariant index of P is the virtual character of G defined by

$$ind(P,h) = trace(h|_{kerP}) - trace(h|_{cokP}), h \in G. \qquad (1)$$

P is said to be rigid for this G action if the index of (P,h) does not depend on $h \in G$.

An integral formula for twisted harmonic spinors

Let X be a vector field on M. Suppose $s \in \Gamma(S(TM) \otimes B_k(T_{\mathbb{C}}M))$ satisfies

$$(P + \sqrt{-1}tc(X))s = 0, t \in \mathbb{R}$$

$$P = D \otimes B_k(T_{\mathbb{C}}M).$$

Then we have the following integral formula.

$$2\sqrt{-1}\int_M t|c(X)s|^2 = \int_M -2 < \nabla_X s, s > -\langle c(\nabla_{e_i} X)s, c(e_i)s\rangle.$$

We also have

$$< \nabla_X s, s > - < L_X s, s > \leq C_n |\nabla X||s|^2$$

for some constant C_n depending only on n.

If we choose X to be a Killing vector field, by rigidity of the twist Dirac operator, we can assume that

$$L_X s = 0.$$

Then we have the following integral inequality:

$$\int_M t^2 |X|^2 |s|^2 dV \leq C(n) \int_M t |\nabla X| |s|^2 dV.$$

Thank you!