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Karsten GROVE, BURKHARD WILKING & JOSEPH YEAGER

Almost non-negative curvature and rational ellipticity in cohomogeneity two

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ALMOST NON-NEGATIVE CURVATURE AND RATIONAL ELLIPTICITY IN COHOMOGENEITY TWO

by Karsten GROVE,
Burkhard WILKING & Joseph YEAGER (*)

With an appendix by Steve HALPERIN

In memory of Marcel Berger

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ABSTRACT. — An extension of a fundamental conjecture by R. Bott suggests that all simply connected closed almost non-negatively curved manifolds M are rationally elliptic, i.e., all but finitely many homotopy groups of such M are finite. We confirm this conjecture when in addition M supports an isometric action with orbits of codimension at most two. Our proof uses the geometry of the orbit space to control the topology of the homotopy fiber of the inclusion map of an orbit in M, and is applicable to more general contexts.

RÉSUMÉ. — D'après une extension d'une conjecture fondamentale de R. Bott, toute variété compacte (sans bord) simplement connexe M à courbure positive est rationellement elliptique, i.e., seul un nombre fini de groupes d'homotopie de M sont infinis. On montre cette conjecture dans le cas où M admet une action par isométries dont l'orbite principale a codimension au plus est de deux. Notre preuve utilise la géométrie de l'espace quotient pour contrôler la topologie de la fibre homotopique de l'inclusion d'une orbite dans M, et s'applique à des contextes plus généraux.

1. Introduction

Expressed in the language of rational homotopy theory a fundamental conjecture attributed to Bott (cf. [9]) can be formulated as follows:

CONJECTURE. — *Non-negatively curved simply connected closed manifolds are rationally elliptic.*

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Here curvature of a manifold M refers to its *sectional curvature*, sec M. Moreover, a simply connected closed manifold M is said to be *rationally elliptic* if and only if it has finite dimensional rational homotopy, dim $\pi_* M \otimes \mathbb{Q} < \infty$, i.e., all but finitely many homotopy groups of M are finite (in fact $\pi_k(M)$ is finite for $k > 2 \dim M - 1$, [4]), and otherwise M is said to be *rationally hyperbolic* (cf. [4]). It is a well-known simple consequence of Sullivans minimal model that M being rationally elliptic is equivalent to *polynomial growth* of the sequence of Betti numbers $\{\beta_k(\Omega M)\}$ of its loop space ΩM, the property Bott was focussed on.

Based on an iterated use of the Rauch comparison theorem for Jacobi fields an estimate for the Betti numbers of ΩM for manifolds with $0 < \delta \leqslant$ sec $M \leqslant 1$ was derived in [2]. Although, the estimate is given in terms of the pinching, δ, its growth rate is exponential.

It is well known that simply connected homogeneous spaces G/H are rationally elliptic. In [10] it was shown that simply connected manifolds M of cohomogeneity one, i.e., with 1-dimensional orbit space, are rationally elliptic as well. However, the typical cohomogeneity two manifold M is rationally hyperbolic including, e.g., $\mathbb{S}^n \times \mathbb{S}^m \# \mathbb{S}^n \times \mathbb{S}^m$.

Extending the thesis work in [28] for non-negative sectional curvature to *almost non-negative* sectional curvature, we prove:

THEOREM 1.1. — *Any simply connected almost non-negatively curved manifold M of cohomogeneity at most two is rationally elliptic.*

Here a manifold M is said to have *almost non-negative* sectional curvature if there is a sequence of Riemannian metrics g_i on M with $\sec_{g_i} M \geqslant -1$ and $\operatorname{diam}(M, g_i) \to 0$ as $i \to \infty$. By rescaling g_i so that diam $M_i = 1$, the lower bound for sec M_i approaches 0. Here we assume that all metrics are invariant under a group action with principal orbits of codimension at most two. One of the main results by Searle and Wilhelm in [26] shows that it suffices to assume that the orbit space M/G is almost non-negatively curved, since then such metrics can be lifted to G-invariant metrics on M with almost non-negative curvature. This actually supports the overall essence of the work presented here: The geometry of the obit space M/G yields control of the *horizontal* geometry of M and in turn in our case provides control of the homotopy fiber of the inclusion maps of orbits of G into M.

Recall that any compact homogeneous space $M = \mathsf{G}/\mathsf{H}$ admits a G invariant metric with sec $M \geqslant 0$. This is not the case for cohomogeneity one G manifolds in general (cf. [11]). However, all closed cohomogeneity one manifolds indeed support almost non-negative curvature [25]. The latter

provides part of our motivation for analyzing almost non-negatively curved manifolds. Another potentially more important reason is an extension of the Bott conjecture itself to almost non-negatively curved manifolds. These are expected to play a pivotal role in collapse with a lower curvature bound analogous to that of *almost flat* manifolds [8] in the collapsing theory for manifolds with bounded curvature [3]. Moreover, as our method here might indicate, an approach towards such an extension of the Bott conjecture by induction on dimension is intriguing.

The strategy in our proof of Theorem 1.1 is to determine the possible structures of orbit spaces M/G for simply connected M with almost non-negative curvature. Even without any curvature assumption the possible structures fall into two types (Section 2): For one of them M is the union of tubular neighborhoods of two orbits and we can appeal to the work in [10] (that said, we will apply a more geometric strategy here, and thus provide a uniform proof for all cases, including cohomogeneity 1). For the other, M/G admits a metric so that M/G tiles one of the hyperbolic plane, the euclidean plan, or the unit two sphere $\mathbb{S}^2$ (when M is almost negatively curved only the non-negatively curved space forms appear here). Moreover, this new metric comes from a G invariant metric on M (Section 3). A Morse theoretic argument (as in [12]) based on the elliptic geometry of M/G combined with the fact the orbits of G are elliptic then leads to a proof for these types (Section 4).

Our investigations mentioned above combined with one of the main results from [26] and the classification of two-dimensional orbifolds [27] yield the following:

THEOREM 1.2. — *A closed simply connected G manifold M of cohomogeneity at most two admits (an) invariant metric of almost non-negative curvature if and only if M/G is not a polygon with more than 4 sides.*

Here a polygon with 4 sides admits a metric isometric to the product of two intervals. Special examples of this arise from the product of two cohomogeneity one manifolds. A more subtle example is given by a polar T^2 action on $\mathbb{CP}^2\#\mathbb{CP}^2$ with section a flat Klein bottle (the connected sum is taken at a fixed point of the T^2 action on the $\mathbb{CP}^2$'s). The orbit space of $\mathbb{CP}^2\#\mathbb{CP}^2\#\mathbb{CP}^2$ by the T^2 action is a pentagon, and that of $\mathbb{S}^n\times\mathbb{S}^m\#\mathbb{S}^n\times\mathbb{S}^m$ by $\mathsf{SO}(n)\times\mathsf{SO}(m)$ is a hexagon (cf. Remark 2.5 and Problem 2.6).

As it turns out, all the geometric arguments we provide carry over to the more general case of *singular riemannian foliations*, $\mathcal{F}$, abbreviated SRF. In particular, if the leaves $L\in\mathcal{F}$ (or rather some finite cover thereof) of a

SRF in a simply connected, almost non-negatively curved manifold M are rationally elliptic nilpotent spaces, then M would be as well.

It is our pleasure to thank Marco Radeschi for insights and helpful suggestions (including Lemma 4.3), and Steve Halperin for providing the Appendix which combined with our geometric constructions provides the desired conclusion of rational ellipticity under much weaker hypotheses than the ones present in Theorem 1.1.

2. Orbit/leaf space structure

Although our focus in this paper is on isometric actions $\mathsf{G} \times M \to M$ of a closed connected Lie group G on a closed riemannian manifold M, several key results including the ones in this section carry over to the more general case of *singular riemannian foliations*, (abbreviated as SRF) $\mathcal{F}$ with compact leaves $L \in \mathcal{F}$ (the G-orbits in our case). For an excellent introduction to the subject of SRF we refer to the lecture notes by M. Radeschi [24].

Since the projection $M \to M/\mathcal{F}$ is a submetry, the leaf space $M/\mathcal{F}$ (orbit space M/G) with its natural metric is an Alexandrov space with lower curvature that of M. In addition, it is a Riemannian orbifold if and only if $\mathcal{F}$ is *infinitesimally polar* [19]. In the G action case this means that all isotropy representations are polar.

In general, observe that $M/\mathcal{F}$ is simply connected when M is. Since SRF of codimension at most two automatically are infinitesimally polar, we know that in our case $M/\mathcal{F}$ is a simply connected orbifold of dimension at most two. When combined with Corollary 1.7 of [17] we have:

THEOREM 2.1. — *The leaf space $M/\mathcal{F}$ of a closed singular riemannian foliation on a simply connected closed manifold M is topologically either the 2-sphere or the 2-disc. In the first case, there are at most finitely many non-principal leaves, all of which are exceptional, i.e., $\mathcal{F}$ is a regular foliation. In the second case all boundary leaves are singular, and there are no exceptional leaves.*

Here, we recall that *exceptional* leaves are non-principal leaves of the same dimension as the principal ones, whereas *singular* leaves are leaves of lower dimension.

Our key results in this section is a strengthening of Theorem 2.1, in part when our curvature assumption is made. The first, however, needs no curvature assumption.

Lemma 2.2. — *The leaf space $M/\mathcal{F}$ of a closed codimension two riemannian foliation on a simply connected closed manifold M is a 2-sphere riemannian orbifold with at most two singular points, corresponding to at most two exceptional leaves.*

Proof. — We first show that the orbifold fundamental group $\pi_1^{orb}(M/\mathcal{F})$ of $M/\mathcal{F}$ is generated by at most one element.

Consider the oriented principal $\mathsf{SO}(2)$ frame bundle $F^h(M) \to M$ of the horizontal distribution to the foliation $\mathcal{F}$. We denote the canonical lift of $\mathcal{F}$ to $F^h(M)$ by $\mathcal{F}^h$ (cf. [22]). Then $F^h(M) \to F^h(M)/\mathcal{F}^h$ is a riemannian submersion and the leaf space $F^h(M)/\mathcal{F}^h = F(M/\mathcal{F})$ is the oriented frame bundle of the orbifold $M/\mathcal{F}$. If $E\,\mathsf{SO}(2) \to B\,\mathsf{SO}(2)$ denotes the universal classifying $\mathsf{SO}(2)$ bundle, then the orbifold fundamental group $\pi_1^{orb}(M/\mathcal{F})$ is the fundamental group of the associated bundle $E\,\mathsf{SO}(2) \times_{\mathsf{SO}(2)} F(M/\mathcal{F})$ (cf. [13, 16, 18]).

Since M is simply connected $\pi_1(F^h(M))$ is trivial or generated by one element. It follows that $\pi_1(F(M/\mathcal{F}))$ is trivial or generated by one element, and hence so is $\pi_1(E\,\mathsf{SO}(2) \times_{\mathsf{SO}(2)} F(M/\mathcal{F}))$.

On the other hand, the orbifold Euler characteristic of $M/\mathcal{F}$ is given by

$$\chi^{orb}(M/\mathcal{F}) = 2 - \sum(1 - 1/m_i),$$

where $\mathbb{Z}_{m_i}$, $m_i \geqslant 2$ are the local orbifold groups at the non-smooth points of $M/\mathcal{F}$ (see [27, Chapter 13]). Moreover, if $\chi^{orb}(M/\mathcal{F}) < 0$ or $\chi^{orb}(M/\mathcal{F}) = 0$, $M/\mathcal{F}$ has the hyperbolic plane, respectively the euclidean plane as an orbifold cover (see [27, Chapter 13]). Also if $\chi^{orb}(M/\mathcal{F}) > 0$, $M/\mathcal{F}$ has $\mathbb{S}^2$ as an orbifold cover, unless it is a bad orbifold (cf. [27, Chapter 13]). In all the cases, where $M/\mathcal{F}$ has a simply connected space form as orbifold cover, $\pi_1^{orb}(M/\mathcal{F})$ is simply the group of the cover (see, e.g. [27, Chapter 13]). Since $M/\mathcal{F}$ is compact, this can only be a cyclic group when its cover is $\mathbb{S}^2$, and in this case there are exactly two non-regular points. Also, when the orbifold euler characteristic is positive and $M/\mathcal{F}$ is a bad orbifold there are at most two exceptional orbits (see [27, Chapter 13]). □

When $M/\mathcal{F}$ is a 2-disc, an *edge* is the part of boundary between two consecutive non-smooth points of (also referred to as *vertices*).

Lemma 2.3. — *The leaf space $M/\mathcal{F}$ of a closed singular codimension two riemannian foliation on a simply connected closed almost nonnegatively curved manifold M with a singular leaf is an orbifolds 2-disc with at most four edges. Moreover, when $M/\mathcal{F}$ has four edges, all angles are $\pi/2$, and when it has three edges the sum of angles is at least π.*

Proof. — First observe that a singular leaf necessarily corresponds to a boundary point of the leaf space $M/\mathcal{F}$. In fact otherwise, the unit normal sphere $\mathbb{S}^f$ of the singular leaf would fiber over $\mathbb{S}^1$ (the space of direction of the leaf space), which is impossible as $f \geqslant 2$.

Since the principal leaves of the infinitesimal foliation for $\mathcal{F}$ restricted to each normal sphere of a leaf corresponding to a vertex are isoparametric hypersurfaces, the possible angles α satisfy $\alpha \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$ by [23].

Suppose the orbifold $M/\mathcal{F}$ has k edges, and hence the same number of vertices. Since the edges are geodesics, the Gauss–Bonnet Theorem yields

$$\int_{M/\mathcal{F}} \mathrm{curv} = 2\pi - k\pi + \sum_i \alpha_i,$$

where α_i are the angles at the vertices of $M/\mathcal{F}$. But, $\sec M \geqslant -1$ and hence $\mathrm{curv}\, M/\mathcal{F} \geqslant -1$, so

$$\int_{M/\mathcal{F}} \mathrm{curv} \geqslant -\mathrm{Area}(M/\mathcal{F}).$$

Thus

$$\mathrm{Area}(M/\mathcal{F}) \geqslant (k-2)\pi - \sum_i \alpha_i \geqslant (k-4)\pi/2$$

since $\alpha_i \leqslant \pi/2$. But, since M collapses to a point so does $M/\mathcal{F}$ and in particular $\mathrm{Area}(M/\mathcal{F})$ approaches 0, i.e. $k \leqslant 4$. Moreover, when $k = 4$ each $\alpha_i = \pi/2$. When $k = 3$, the same reasoning combined with the inequality $\mathrm{Area}(M/\mathcal{F}) \geqslant (k-2)\pi - \sum_i \alpha_i$ implies that $\sum_i \alpha_i \geqslant \pi$. $\qquad \square$

This in particular yields the following:

COROLLARY 2.4. — *Let M be a simply connected almost non-negative curved closed manifold with a closed SRF of codimension two. Then either:*

- *$M/\mathcal{F}$ supports a constant curvature 1 or 0 metric with $M/\mathcal{F}$ tiling the unit sphere $\mathbb{S}^2$ respectively the flat plane $\mathbb{R}^2$; or,*
- *M is a double disc bundle $M = D(B_-) \cup D(B_+)$ where $B_\pm$ are leaves of $\mathcal{F}$.*

Proof. — When $M/\mathcal{F}$ is the 2-sphere pick $p_- \neq p_+ \in M/\mathcal{F}$ so that any singular point of $M/\mathcal{F}$ is in $\{p_-, p_+\}$. Clearly $M = D(B_-) \cup D(B_+)$ where $B_\pm$ are the leaves corresponding to $p_\pm$. The same reasoning applies to the case where $M/\mathcal{F}$ is a 2-disc with $0, 1$ or 2 vertex points on the boundary.

In all remaining cases $M/\mathcal{F}$ is a 2-disc with 3, or 4 vertex points on the boundary. As we have seen, in the latter case all angles must be $\pi/2$, and $M/\mathcal{F}$ is diffeomorphic to a flat square in $\mathbb{R}^2$. In the case of three vertex points, the restrictions on the angles $\alpha_i \in \{\pi/2, \pi/3, \pi/4, \pi/6\}$ leave only the following configurations possible:

- When $\sum_i \alpha_i = \pi$: $\{\pi/2, \pi/3, \pi/6\}$, $\{\pi/2, \pi/4, \pi/4\}$, $\{\pi/3, \pi/3, \pi/3\}$
- When $\sum_i \alpha_i > \pi$: $\{\pi/2, \pi/2, \pi/2\}$, $\{\pi/2, \pi/2, \pi/k\}$, $\{\pi/2, \pi/3, \pi/3\}$, $\{\pi/2, \pi/3, \pi/4\}$

When $\sum_i \alpha_i = \pi$, $M/\mathcal{F}$ is diffeomorphic to a flat triangle with the given angles, and this triangle tiles the flat plane R^2. When $\sum_i \alpha_i > \pi$, $M/\mathcal{F}$ is diffeomorphic to a constant curvature 1 triangle in $\mathbb{S}^2$ with the given angles, and this triangle tiles $\mathbb{S}^2$. $\qquad\square$

Remark 2.5. — It is interesting that all remaining 2 dimensional orbifolds arising as M/G, with M simply connected are actually hyperbolic. Thus these M/G support a constant curvature -1 metric tiling the hyperbolic plane. Also in the geodesic sense suggested above they are also "geometrically hyperbolic", i.e., the number of billiard geodesics joining two points in the tile grow exponentially with their length. In cases where for example the orbits on all boundary strata have codimension at least three our proof of Theorem 1.1 will imply that such M are rationally hyperbolic.

It would be interesting to pursue this and see if indeed

PROBLEM 2.6. — *Is it true that a simply connected closed manifold M of cohomogeneity two is rationally hyperbolic if and only if its orbit space M/G is an orbifold of hyperbolic type?*

3. Adapting the metric on M

In this section we will modify the metric on M to properly reflect the two scenarios exhibited in Corollary 2.4.

We begin with a general discussion of manifolds having the structure of a double disc bundle.

It is well known that a closed manifold M decomposes as the union of tubular neighborhoods $D(B_\pm)$ of two submanifolds $B_\pm \subset M$, i.e., $M = D(B_-) \cup D(B_+)$ if and only if M supports a Morse–Bott function f with exactly two critical values, i.e., $\min f$ and $\max f$.

Here we exhibit a metric characterization of this property exactly as observed by Weinstein in the case of pointed Blaschke manifolds (see [1, Example 5.20 (5.21)]):

LEMMA 3.1. — *A closed manifold M admits a decomposition $M = D(B_-) \cup D(B_+)$ if and only if M admits a riemannian metric such that $R = \operatorname{dist}(B_-, B_+) = \operatorname{dist}_H(B_-, B_+)$, cut locus $\operatorname{cut}(B_\pm) = B_\mp$ and $\operatorname{inj}^\perp(B_\pm) = R$.*

Proof. — Let $E_\pm \to B_\pm$ be the normal bundles of the submanifolds $B_\pm \subset M$ relative to some Riemannian metric on M. The assumption $M = D(B_-) \cup D(B_+)$ is equivalent to the statement that the corresponding sphere bundles $SE_\pm$ are diffeomorphic.

Identify $DE_\pm - B_\pm$ with $(0, \ell) \times SE_\pm$ and equip DE_- with a riemannian metric g_{E_-} whose restriction to $(0, \ell) \times SE_-$ has the form $dt^2 + g_{E_-,t}$, where $g_{E_-,t}$ is a smooth family of Riemannian metrics on SE_-. Similarly provide DE_+ with a riemannian metric g_{E_+} whose restriction to $(0, \ell) \times SE_+$ has the form $ds^2 + g_{E_-,s}$, where $g_{E_-,s}$ is a smooth family of Riemannian metrics on SE_+.

Let $F : SE_- \to SE_+$ be the diffeomorphism with $M = DE_- \cup_F DE_+$, the identification given explicitly via

$$DE_- - B_- \ni (t, x) = (\ell - t, F(x)) \in DE_+ - B_+$$

Let $\chi : \mathbb{R} \to R$ be a smooth monotone function with $\chi(t) = 1$ for $t \leqslant \ell/3$ and $\chi(t) = 0$ for $t \geqslant 2\ell/3$. Then we obtain a smooth Riemannian metric g on M by letting $g = g_{E_-}$ near B_-, $g = g_{E_+}$ near B_+ and

$$g = \chi(t)(dt^2 + g_{E_-,t}) + (1 - \chi(t))(ds^2 + g_{E_+,\ell-t})$$
$$= dt^2 + \chi(t)g_{E_-,t} + (1 - \chi(t))g_{E_+,\ell-t}$$

on $M - (B_- \cup B_+)$, where we note that $ds = ds(t) = -dt$.

With this metric, the t-lines are minimal unit speed geodesics starting perpendicularly at B_- and ending perpendicularly at B_+.   □

Remark 3.2. — Note that if S_d is the set of points at distance d from B_- (and distance $\ell - d$ from B_+), then S_d is a smooth submanifold of M and any geodesic $c(t)$ which is perpendicular to one S_r (including $r = 0$ and $r = \ell$) will be remain perpendicular to all S_d, as $t \in \mathbb{R}$.

The map $P : M \to [0, \ell]$ given by the distance function to B_-, i.e., $P(S_d) = d$ is a submetry. Moreover, the geodesics above are horizontal lifts of geodesics on $[0, \ell]$, where the latter bounce back and forth between the end points. In fact, the family of level sets form a SRF $\mathcal{F}$ of codimension one on M.

Metrically this is the same situation as in the case of an action of cohomogeneity one with orbit space an interval.

We now turn to the case where M/G admits a metric of constant curvature. In order to treat this case as the one discussed in the above remark the following is key:

LEMMA 3.3 (Lifting). — *The constant curvature metric on a tile M/G lifts to a G invariant metric on M.*

Proof. — Let M/G be equipped with a metric so as to be one of the possible euclidean or spherical tiles. Since for each $p \in M$ the slice representation of G_p is polar, an easy application of the slice theorem yields a Riemannian metric g_p in an ϵ_p tube D_p of $\mathsf{G}p$ with D_p/G isometric to the ϵ_p ball at $\mathsf{G}p \in M/\mathsf{G}$.

Let $\{D_\alpha\}$ be an open cover of M/G by such balls (relative to the tile metric) and let $\{f_\alpha\}$ be a partition of unity subordinate to this cover. Let g_α be the corresponding lifted metrics on tubes in M and let $\{f_\alpha\}$ denote also the corresponding lifted partition of unity on M. If g is the original G invariant metric on M then $g_\alpha(u,v) = g(S_\alpha u, v)$. Consider the bundle isomorphism

$$S : TM \to TM \quad \text{defined by} \quad S^{-1} = \sum_\alpha f_\alpha S_\alpha^{-1}$$

The metric $\tilde{g}$ defined by $\tilde{g}(u,v) := g(Su,v)$ is then the desired G invariant lift of the constant curvature metric on M/G. In fact, the orbit map projection restricted to the regular part M_0 consisting of principal orbits in M is a Riemannian submersion onto M_0/G equipped with the constant curvature metric according to the Lemma below where only $P = \mathrm{id}$ is used. □

LEMMA 3.4. — *Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{R}^n$ and consider the orthogonal projection $\pi : R^n = \mathbb{R}^k \times \mathbb{R}^{n-k} \to \mathbb{R}^k$. Let P and S be symmetric positive operators on R^k and $\mathbb{R}^n$ respectively, and define new inner products on these spaces by $g_P(u,v) := \langle Pu, v \rangle$ and $g_S(u,v) := \langle Su, v \rangle$.*

The $\pi : (\mathbb{R}^n, g_S) \to (R^k, g_P)$ is a Riemannian submersion if and only if S and P are related by

$$S^{-1} = \begin{bmatrix} P^{-1} & * \\ * & * \end{bmatrix}$$

Proof. — Assume π is a Riemannian submersion with horizontal subspace say

$$H = \{(x, Ax) \,|\, x \in \mathbb{R}^k \text{ and } A : \mathbb{R}^k \to \mathbb{R}^{n=k} \text{ linear}\}$$

and vertical subspace $V = \{0\} \times \mathbb{R}^{n-k}$. Since $\langle SH, V \rangle = g_S(H, V) = 0$, $S(x, Ax) \in \mathbb{R}^k \times \{0\}$. Now with (x, Ax) and (y, Ay) being horizontal lifts of x and y respectively, we therefore have

$$\langle Px, y \rangle = g_P(x,y) = g_S((x, Ax), (y, Ay))$$
$$= \langle S(x, Ax), (y, Ay) \rangle = \langle S(x, Ax), y \rangle.$$

It follows that $S(x, Ax) = (Px, 0)$ and hence $(x, Ax) = S^{-1}(Px, 0)$. If

$$S^{-1} = \begin{bmatrix} U & * \\ W & * \end{bmatrix}$$

this means $UP = \mathrm{id}_{\mathbb{R}^k}$ and $WP = A$, in particular $U = P^{-1}$.

Conversely, suppose S^{-1} has the above form with $U = P^{-1}$, and define A by $A = WP$. Then for any $x, y \in \mathbb{R}^k$ we have

$$g_S((x, Ax), (y, Ay)) = \langle S(x, Ax), (y, Ay) \rangle = \langle S(UPx, WPx), (y, Ay) \rangle$$
$$= \langle SS^{-1}(Px, 0), (y, Ay) \rangle = \langle (px, 0), (y, Ay) \rangle = \langle Px, y \rangle = g_P(x, y).$$

It follows that π is a Riemannian submersion with horizontal subspace $H = \{(x, Ax) \,|\, x \in \mathbb{R}^k\}$. □

Remark 3.5. — The arguments above combined with the work of Mendes [20], actually show that if M/G is an orbifold (equivalently, the G action on M is infinitesimally polar [19]), then any riemannian orbifold metric on M/G lifts to a G invariant metric on M.

From the slice theorem for SRF due to Mendes and Radeschi [21], it follows as in the proof above that also in the general case of a SRF $\mathcal{F}$ which is infinitesimally polar, any riemannian orbifold metric on $M/\mathcal{F}$ lifts to a metric on M relative to which $\mathcal{F}$ is a SRF.

4. Topology and Morse theory input

In the previous section we have established that M carries a metric which projects to a base space B exhibiting *geometrically elliptic* behavior (B is either an interval or a constant curvature tile in $\mathbb{R}^2$ or $\mathbb{S}^2$) in the sense that the number of geodesics joining two points in it grows at most polynomially as a function of their length. In M this is reflected in the following topological property of the *homotopy fiber* F of the inclusion map $f : E \to M$ of the inverse image $E \subset M$ of a point in B:

PROPOSITION 4.1 (Homotopy fiber control). — *Let M be an almost non-negatively curved simply connected closed manifold with a SRF $\mathcal{F}$ of codimension one or two. Then there is a leaf $L \in \mathcal{F}$ in M so that the Betti numbers of the homotopy fiber F of the inclusion map $L \to M$ grows at most polynomially.*

Proof. — Recall that the space M_L^I of paths $c : I = [0, 1] \to M$ with $c(0) \in L \subset M$ is homotopy equivalent to L, and $M_L^I \to M$, $c \to c(1)$

is a fibration. Thus its fiber F over any point $q \in M$ is the homotopy fiber of $L \to M$. By homotopy equivalence, we can assume that F consists of curves sufficiently regular (piecewise smooth or of Sobolev class H^1) to apply Morse theory for the energy functional $E : F \to \mathbb{R}$, given by $E(c) = 1/2 \int_I |c'|^2$.

If $M/\mathcal{F}$ is not a tile, let L be one of the leaves $B_\pm$ as chosen in the proof of Lemma 2.4, and in the remaining cases any principal leaf in M will do. Choose $q \in M - L$ and let F be the corresponding homotopy fiber of $L \to M$. Then the critical points for E on F are exactly the geodesics starting orthogonally to L and ending at q. When M is equipped with a metric as in Lemma 3.1, respectively Lemma 3.3 such geodesics are in $1-1$ correspondence with billiard geodesics in the base B. When $B = M/\mathcal{F}$ is a tile we make a generic choice of q so that the corresponding geodesics in $\mathbb{R}^2$ or $\mathbb{S}^2$ never meet a vertex of a tile (as in [12]). The focal points of these geodesics correspond to intersections with the edges of the tiles when $\dim B = 2$ and end points of B when $\dim B = 1$. The multiplicity of each focal locus corresponding to the crossing of an edge, is the co-dimension -1 of the leaf strata of the edge in M, so at least 1. Similarly, when B is an interval (tiling the real line) the multiplicity of each focal locus corresponding to the crossing an end point, is the co-dimension -1 of the corresponding leaf in M, so also at least 1.

In all scenarios, it is clear that the number of critical points in F grows at most polynomially as a function of the lengths of the geodesics, and the fact that by the Morse Index Theorem, the index grows by at least 1 at each focal points then shows that the Betti numbers of F grow at most polynomially. $\qquad\square$

Remark 4.2. — If we replace L in the Proposition above with a finite cover $\widehat{L}$, and F by the homotopy fiber $\widehat{F}$ of the composed map $\widehat{L} \to L \to M$, then clearly also the betti numbers of $\widehat{F}$ grow at most polynomially.

A result due to S. Halperin (Theorem A.1 in the appendix) then shows in particular that if a finite cover $\widehat{L}$ of the leaf L is nilpotent and $\sum_{k \geq 2} \dim \pi_k(\widehat{L}) \otimes \mathbb{Q} < \infty$, then M is rationally elliptic.

This then completes the proof of Theorem 1.1, since for any compact homogeneous space $L = \mathsf{G}/\mathsf{K}$, $\widehat{L} = \mathsf{G}/\mathsf{K}_0$ is a rationally elliptic nilpotent space. More generally this also applies to the case where $L = \mathsf{G}//\mathsf{H}$ is a biquotient.

Below we provide an alternative short direct proof of Theorem 1.1 by utilizing a trick (see Lemma 4.5 below) allowing us to assume that $\widehat{L} = \mathsf{G}/\mathsf{K}_0$ is simply connected.

Recall that taking iterated loop spaces results in a sequence of maps

$$\ldots \Omega F \to \Omega L \to \Omega M \to F \to L \to M$$

each of which are fibrations up to homotopy (cf. e.g. [14]). Here we are particularly interested in $\Omega M \to F$ with fiber ΩL, where L is a leaf, or more generally a finite cover $\widehat{L}$ of L in M. We claim that this fibration is *orientable*, if L is simply connected. From the iterative construction of the fibration sequence, the following possibly well known observation shown to us by M. Radeschi will suffice:

LEMMA 4.3. — *Given a fibration $p : E \to B$ with fiber F. Then, the associated fibration $F \to E$ with fiber ΩB is orientable if B is simply connected.*

Proof. — Given a loop γ based at $e \in E$, we will in fact see that the induced homotopy equivalence $L_\gamma : \Omega B \to \Omega B$ of the fiber, the loops of B at $b = p(e)$ is represented by $L_\gamma(\alpha) = p(\gamma^{-1}) \cdot \alpha$, $\alpha \in \Omega B$. In particular, L_γ is homotopic to the identity map when B is simply connected.

To see this, consider the fibration $ev_0 : F' \to E$, where $F' = \{c \in E^I | c(1) \in F\}$ is homotopy equivalent to $F = p^{-1}(b)$, and $ev_0(c) = c(0)$. If $\gamma \in E^I$ is a path from p to q in E it is straightforward to see that the induced transformation T_γ (cf. [14, Proposition 4.61]) from the fiber $ev_0^{-1}(p)$ over p to the fiber $ev_0^{-1}(q)$ over q is represented by the map that concatenates γ^{-1} with elements from $ev_0^{-1}(p)$.

To complete the proof, observe that indeed $P : ev_0^{-1}(e) \to \Omega_b B$ defined by $P(c) = p \circ c$ is a homotopy equivalence with inverse a lift ρ constructed as follows: Let $P_b B = \{c \in B^I | c(o) = b\}$ and consider the homotopy $ev : P_b B \times I \to B$ given by $ev(\beta, t) = \beta(t)$. Let $\widehat{ev} : P_b B \times I \to E$ be a lift with $\widehat{ev}(\beta, 0) = e$, then $\rho(\alpha)(t) := \widehat{ev}(\alpha, t)$ for any $\alpha \in \Omega B_b \subset P_b B$. Any two such lifts result in homotopic inverses to P. Thus, if γ above is a loop at e, the action on $\alpha \in \Omega_b B$ is given by $p \circ (\gamma^{-1} \cdot \rho(\alpha)) = (p \circ \gamma^{-1}) \cdot \alpha$. □

From the Serre spectral sequence for the fibration $\Omega L \to \Omega M \to F$ we conclude from the above Proposition that

COROLLARY 4.4. — *If L in the above Proposition has finite fundamental group, and its universal cover $\widehat{L}$ is topologically elliptic. Then M is topologically elliptic when M is simply connected.*

Here we use the terminology *topologically elliptic* for M if the Betti numbers of its loop space relative to *any field of coefficients* grow at most polynomially.

Unfortunately, the assumption that L has finite fundamental group is very restrictive, ruling out, e.g., the important case where, e.g., $L = \mathsf{T}^k$ is a torus. The following easily proven *general reduction trick* will resolve this issue when the SRF $\mathcal{F}$ is homogeneous at least over the field of rational numbers $\mathbb{Q}$, and hence establish Theorem 1.1 in the introduction:

LEMMA 4.5 (Reduction). — *Let M be a closed simply connected G manifold and $\widehat{\mathsf{G}}$ a closed simply connected Lie group containing G. Let $\widehat{M} = \widehat{\mathsf{G}} \times_{\mathsf{G}} M$ be the total space over $\widehat{\mathsf{G}}/\mathsf{G}$ with fiber M associated to the principal bundle $\widehat{\mathsf{G}} \to \widehat{\mathsf{G}}/\mathsf{G}$. Then*

- *$\widehat{M}$ is simply connected*
- *$\widehat{M}/\widehat{\mathsf{G}}$ is isometric to M/G*
- *For $p \in M \subset \widehat{M}$, $\widehat{\mathsf{G}}p = \widehat{\mathsf{G}}/\mathsf{G}_p$ has finite fundamental group.*
- *M is rationally elliptic if and only if $\widehat{M}$ is.*

Since all orbits in $\widehat{M}$ have finite fundamental group and $\widehat{M}/\widehat{\mathsf{G}} = M/\mathsf{G}$ the above arguments show that $\widehat{M}$ is topologically elliptic when M/G is almost non-negatively curved and $\dim M/\mathsf{G} \leqslant 2$.

Remark 4.6. — $\widehat{\mathsf{G}}$ can be chosen as $\mathsf{SU}(n)$ for large enough n. It is clear that M topologically elliptic implies that $\widehat{M}$ is topologically elliptic, but the converse is what is needed (an oversight in the proof of Theorem 4.7 in [12]).

Remark 4.7. — Corollary 2.4 divides the possible leaf spaces in two cases. With a bit of work one can show: Each possible leaf space that falls under the second case (double disc boundle decomposition) is isomorphic, as an orbifold, to the quotient of $\mathbb{S}^3$ by a one-dimensional isometric group action. So in either case the orbit space has an elliptic orbifold metric, i.e., a metric where the number of geodesics of between two points growth polynomially. As far as we know the same could still be true for three dimensional non-negatively curved orbit spaces.

Although the horizontal geometry of SRF resembles that of isometric group actions, the vertical part can in general be completely arbitrary, in contrast to being homogeneous in the group action case. However, when the geometry of M is restricted, it is not clear how this may restrict the leaves. For example the following problems are interesting and natural in view of the work done here:

PROBLEM 4.8. — *Let $\mathcal{F}$ be a closed SRF on a closed (simply connected) Riemannian manifold M of almost nonnegative curvature. Are the leaves of $\mathcal{F}$ finitely covered by a nilpotent space, which moreover is rationally elliptic?*

A related problem, that may well have a negative answer is the following

PROBLEM 4.9. — *Let $\mathcal{F}$ be a closed SRF on a closed (simply connected) Riemannian manifold M. Is there a family of metrics on M keeping $\mathcal{F}$ a SRF which collapses to the leaf space $M/\mathcal{F}$ with a lower curvature bound?*

We mention that by work of Kapovitch, Petrunin and Tuschmann [15], any closed manifold of almost non-negative curvature, is finitely covered by a nilpotent almost non-negatively curved manifold. Moreover, such manifolds arise naturally in the context of *collapse* of manifolds with a lower sectional curvature bound.

Appendix. The rational cohomology of a fiber
(by Steve Halperin)

In this Appendix $H(Y)$ denotes the rational cohomology of a space Y, and the ground field is $\mathbb{Q}$. A graded vector space $T = T^{\geqslant 0}$ has *strong exponential growth* with respect to a constants $N, \alpha > 0$ if there is a pair of sequences (r_k, p_k) such that $p_k \leqslant r_k < r_{k+1} \leqslant r_k + N$, and for each $\beta < \alpha$ $\dim T^{p_k} > e^{\beta r_k}$ if k is sufficiently large.

THEOREM A.1. — *Suppose in a fibration $F \to X \to M$ of path connected spaces that*

- *M is simply connected, $\dim H(M) < \infty$, and $H^{>N}(M) = 0$.*
- *X is nilpotent, $\dim H^1(X) < \infty$, and $\sum_{k \geqslant 2} \dim \pi_k(X) \otimes \mathbb{Q} < \infty$.*

If M is not rationally elliptic, then $H(F)$ has strong exponential growth.

The proof of this theorem relies on Sullivan models for path connected spaces, S, for which the reader is referred to [5] and [7]. Sullivan models are commutative differential graded algebras (cdga's for short) of the form $(\Lambda T, d)$ for which, in particular, $H(\Lambda T, d) \cong H(S)$. Here ΛT the free commutative graded algebra on a graded vector space $T = T^{\geqslant 1}$. Then $\Lambda T = \bigoplus_m \Lambda^m T$ with $\Lambda^m T$ denoting the linear span of monomials of length m in T; m is called the *wedge degree*. The differential in ΛT decomposes as $d = d_0 + d'$ with $d_0 : T \to T$ and $d' : T \to \Lambda^{\geqslant 2} T$; d_0 is called the *linear part*

of d and is itself a differential. Finally we identify $\Lambda W \otimes \Lambda Z = \Lambda(W \oplus Z)$ and thereby define a wedge degree in the tensor product.

The translation of the theorem to Sullivan models then proceeds as follows:

Standard rational homotopy theory gives a Sullivan model $(\Lambda W, d)$ for M in which $W = W^{\geqslant 2}$, each $\dim W^k < \infty$, and $d : W \to \Lambda^{\geqslant 2} W$. Moreover, in the terminology of [6], X is an $F-space$ and therefore has a Sullivan model $(\Lambda V, d)$ with $d_0 = 0$ and $\dim V < \infty$. Finally, corresponding to the fibration of the theorem, Theorem 5.1 in [7] yields a sequence of cdga morphisms,

$$(\Lambda W, d) \overset{\lambda}{\to} (\Lambda W \otimes \Lambda Z, d) \overset{\rho}{\to} (\Lambda Z, d_z), \lambda \Phi = \Phi \otimes 1, \rho = \mathbb{Q} \otimes_{\Lambda W} -$$

in which $(\Lambda W \otimes \Lambda Z, d)$ is also a Sullivan model for X, $(\Lambda Z, d_Z)$ is a Sullivan model for F, and $d_z : Z \to \Lambda^{\geqslant 2} Z$. Thus the linear part of the differential in $\Lambda W \otimes \Lambda Z$ satisfies

$$d_0 : W \to 0 \quad \text{and} \quad d_0 : Z \to W$$

Moreover, standard theory gives $H(W \oplus Z, d_0) \cong V$ and so $\dim H(W \oplus Z, d_0) < \infty$.

There follows for some K the following:

$$\dim Z^k < \infty, \qquad k \geqslant 1$$

(A.1)
$$d_0 : Z^k \overset{\cong}{\to} W^{k+1}, \quad k \geqslant K$$

$$\operatorname{Im} d \subset W \oplus \Lambda^{\geqslant 2}(W \oplus Z)$$

LEMMA A.2.

(1) The quotient differential in $\Lambda Z^{\geqslant K}$ (after division by $Z^{<K}$) is zero.
(2) $d_Z : Z^k \to \Lambda Z^{<k}, k \geqslant K$.

Proof.

(1). — Division by both $W^{\leqslant K}$ and by $Z^{<K}$ produces a quotient cdga $(\Lambda W^{>K} \otimes \Lambda Z^{\geqslant K}, \bar{d})$ with $\bar{d}_0 : Z^{\geqslant K} \overset{\cong}{\to} W^{>K}$. Standard theory then implies that $\bar{d}_Z = 0$.

(2). — If $z \in Z^k$ then

$$dz = \sum u_i \otimes z_i + \Phi + \Psi$$

with $u_i \in Z^1$, z_i linearly independent elements in Z^k, $\Phi \in \Lambda Z^{<k}$ and $\Psi \in W \wedge \Lambda(W \oplus Z)$. Since $d^2 z = 0$, the component of $d^2 z$ in $Z^1 \otimes W^{k+1}$ is also zero. But a straightforward computation using (A.1) shows that that component, up to sign, is $\sum d_0 z_i \otimes u_i$. By (A.1), d_0 is an injective map from Z^k to W^{k+1}, and so each $u_i = 0$. This establishes (2). $\square$

Next, choose a basis $z_1, \ldots, z_R$ of $Z^{<K}$ so that

$$(\text{A.2}) \qquad \deg z_1 \geqslant \deg z_2 \geqslant \ldots \geqslant \deg z_R \quad \text{and} \quad d_Z z_i \in \Lambda(z_{i+1}, \ldots, z_R)).$$

Division by $z_i, \ldots, z_R$ yields quotient cdga's $\Lambda(z_{i-1}, \ldots, z_1) \otimes \Lambda Z^{\geqslant K}$ and we show by induction on i that the homology of each has strong exponential growth. First, since $\dim Z^k = \dim W^{k+1}$, $k \geqslant K$, Theorems 13.5 and 13.6 of [7] provide a number $\alpha > 1$ and an infinite sequence $K \leqslant r_1 < \ldots < r_k < \ldots$ such that $r_{k+1} - r_k < N$ and for each $\beta < \alpha$, $\dim Z^{r_k} < e^{\beta r_k}$ if k is sufficiently large. Thus the sequences (r_k, r_k) exhibit $Z^{\geqslant K}$ as having strong exponential growth with respect to N and α. Since $\bar{d}_Z = 0$ in $\Lambda Z^{\geqslant K}$ the same holds for $H(\Lambda Z^{\geqslant K}) = \Lambda Z^{\geqslant K}$.

For simplicity, denote $\Lambda(z_{i-1}, \ldots, z_1) \otimes \Lambda Z^{\geqslant K}$ by $\Lambda Z(i)$, so that $\Lambda Z(1) = \Lambda Z^{\geqslant K}$. We show by induction on i that there is a sequence (p_k) with $p_k \leqslant r_k$ and such that for any $\beta < \alpha$, $\dim H^{p_k}(\Lambda Z(i)) \geqslant e^{\beta r_k}$ if k is sufficiently large. When $i = 1$, this is established above.

Now assume by induction on i that there is an infinite sequence (q_k) with $q_k \leqslant r_k$ and such that for any $\gamma < \alpha$, $\dim H^{q_k}(\Lambda Z(i)) \geqslant e^{\gamma r_k}$ if k is sufficiently large. Then we use the equality $\Lambda Z(i+1) = \Lambda z_i \otimes \Lambda Z(i)$ to establish the sequence (p_k) for $\Lambda Z(i+1)$.

First suppose $\deg z_i \geqslant 2$ and extend $\Lambda Z(i+1)$ to a cdga $\Lambda Z(i+1) \otimes \Lambda x$ by setting $dx = z_i$. Thus $\deg x \geqslant 1$. Fix $\beta < \alpha$ and choose γ so that $\beta < \gamma < \alpha$. Then choose $p_k \leqslant q_k$ so that $\dim H^{p_k}(\Lambda Z(i+1)) = \max_{1 \leqslant n \leqslant q_k} \dim H^n(\Lambda Z(i))$. Division by z_i and by x is a quasi-isomorphism $\Lambda Z(i+1) \otimes \Lambda x \xrightarrow{\simeq} \Lambda Z(i)$. This with the obvious spectral sequence, gives

$$e^{\gamma r_k} \leqslant \dim H^{q_k}(\Lambda Z(i)) \leqslant \dim(H(\Lambda Z(i+1)) \otimes \Lambda x)^{q_k}$$

$$\leqslant \sum_{1 \leqslant n \leqslant q^k} \dim H^n(\Lambda Z(i+1)).$$

It follows that

$$\dim H^{p_k}(\Lambda Z(i+1)) \geqslant \frac{1}{q_k} e^{(\gamma - \beta) r_k} e^{\beta r_k} \geqslant e^{\beta r_k}$$

if k is sufficiently large.

It remains to consider the case $\deg z_i = 1$. In this case let δ denote the quotient differential in $\Lambda Z(i+1)$ and $\bar{\delta}$ the quotient differential in $\Lambda Z(i)$. Then for $\Phi \in \Lambda Z(i)$,

$$\delta(1 \otimes \Phi) = 1 \otimes \bar{\delta}\Phi + z_i \otimes \Theta(\Phi)$$

where Θ is a derivation of degree 0 in $\Lambda Z(i)$ and $\Theta\bar{\delta} = \bar{\delta}\Theta$. In particular division by z_i induces a surjection $H(\Lambda Z(i+1)) \to \ker H(\Theta)$. Moreover, it follows from the properties of Sullivan models that for each n, the restriction

$\Theta(n)$ of Θ to $[\Lambda Z(i)]^n$ is nilpotent. The key fact, to be established now, is that

(A.3) $$\Theta(n)^{l(n)} = 0, \quad \text{where} \quad l(n) = n(n+1)^i.$$

For simplicity, let S be the linear span of $z_1, \ldots, z_{i-1}$, so that $\dim S = i-1$, and denote $Z^{\geqslant K}$ by T. Thus $\Lambda Z(i) = \Lambda S \otimes \Lambda T$. Then decompose Θ as $\Theta_S + \Theta_T$, where Θ_S is a derivation vanishing on T and Θ_T is a derivation vanishing on S. Because $\Lambda z_{i+1} \otimes \Lambda S$ is preserved by δ it follows that Θ_S preserves S. But $\dim(\Lambda S)^n \leqslant (n+1)^i$ and Θ_S is nilpotent, so that $\Theta_S^{\ell_S(n)}$ vanishes in $(\Lambda S)^n$ where $\ell_S(n) = (n+1)^i$.

Further, we may take $K > N \geqslant 2$. Since $d_Z : Z \to \Lambda^{\geqslant 2} Z$ and $\deg z_i = 1$ it follows that if $z \in T$ then

$$\delta z \in z_i \wedge [\Lambda Z(i)]^{\geqslant 2} \oplus \Lambda Z(i).$$

Thus $\Theta_T(z) \in [\Lambda Z(i)]^{\geqslant 2}$. Moreover, by Lemma A.2(2), if $z \in T^n$ then $\delta z \in [\Lambda Z^{<n}(i+1)]^{n+1}$. It follows that for $z \in T^n$

$$\Theta(z) \in ((\Lambda S)^{\geqslant 2} \otimes (\Lambda T)^{\leqslant n-2}) \oplus (S^1 \otimes (\Lambda^{\geqslant 2} T)^{n-1}) \oplus (\Lambda^{\geqslant 2} T)^n$$

Again for simplicity, set $A = \Lambda Z(i) = \Lambda S \otimes \Lambda T$ and decompose A as $A = \bigoplus_k A(k)$ with

$$A(k) = \Lambda S \otimes \left(\bigoplus_{n-p=k} (\Lambda^p T)^n \right).$$

Then $A(k) = 0$, $k \leqslant 0$, and $A(k)^n = 0$, $k > n$. Now the formula above for Θ yields

$$\Theta_T : A(k)^n \to \bigoplus_{j<k} A(j)^n.$$

Since Θ_S preserves each $A(k)^n$ this gives (A.3). In particular, $H(\Theta)^{\ell(n)}$ vanishes in $H^n(\Lambda Z(i))$. Therefore

$$\dim \ker H(\Theta) \cap H^n(\Lambda Z(i)) \geqslant \frac{1}{\ell(n)} \dim H^n(\Lambda Z(i)).$$

When $n = q_k$ this gives

$$\dim H^{q_k}(\Lambda Z(i+1)) \geqslant \dim \ker H(\Theta) \cap H^{q_k}(\Lambda Z(i)) \geqslant \frac{1}{\ell(n)} e^{\gamma r_k} > e^{\beta r_k},$$

the last inequality holding for k sufficiently large. Set $p_k = q_k$ to close the induction and complete the proof of Theorem A.1.

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Karsten GROVE
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556 (USA)
kgrove2@nd.edu

Burkhard WILKING
Department of Mathematics
University of Münster
Einsteinstrasse 62
48149 Münster (Germany)
wilking@math.uni-muenster.de

Joseph YEAGER
Department of Mathematics
Howard University
204 Academic Support Building B
Washington, DC 20059 (USA)
jyeager1942@gmail.com

Steve HALPERIN
shalper@umd.edu