Limits of Riemannian manifolds with a Kato bound on the Ricci curvature

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Outline of the talk

1. Introduction
2. Manifolds with a (strong) Kato bound
3. Tangent cones of non-collapsed strong Kato limits
4. Some further results

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Limits of manifolds

In many problems of geometric analysis, it is often useful to make sequences of manifolds “converge”.

- blow-up sequences in studying singularities of geometric flows or area minimizing submanifolds.
- minimizing/maximizing sequences in the study of critical metrics.
- in the study of moduli spaces of critical metrics.

We focus on the notion of Gromov-Hausdorff convergence.

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Gromov's pre-compactness theorem

Theorem (Gromov, 1981)

Let $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha \in A}$ be a sequence of pointed Riemannian manifolds, $K \in \mathbb{R}$ and assume that for all $\alpha \in A$

$$\mathrm{Ric}_{g_\alpha} \geq K.$$

Then there exists a pointed metric space (X, d, o) such that, up to a sub-sequence, the manifolds $(M_\alpha, \mathrm{d}_{g_\alpha}, o_\alpha)$ converge to (X, d, o) in the pointed Gromov-Hausdorff topology.

The space (X, d, o) carries singularities.

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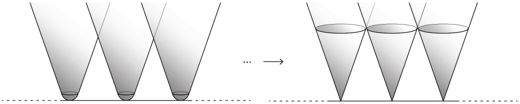

Ricci limit spaces

Definition

A pointed metric space (X, d, o) obtained as a limit of manifolds with a uniform Ricci lower bound is a Ricci limit. If moreover there exists $v > 0$ such that for all $\alpha \in A$

$$\mathrm{vol}_{g_\alpha}(B_1(o_\alpha)) \geq v,$$

we call the sequence non-collapsing and (X, d, o) non collapsed.

Under the non-collapsing assumption, pointed measured Gromov-Hausdorff convergence is usually considered:

$$(M_\alpha, \mathrm{d}_{g_\alpha}, \mathrm{vol}_{g_\alpha}, o_\alpha) \stackrel{pmGH}{\longrightarrow} (X, \mathrm{d}, \mu, o).$$

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Geometry and regularity of non collapsed Ricci limit spaces

Questions: what is the geometry of a (non-collapsed) Ricci limit? What can be said about its singularities? Are they a few, a lot?

- ⋆ Anderson, Cheeger, Colding, Tian, Naber, Jiang...
- ⋆ Lott-Sturm-Villani, Ambrosio-Gigli-Savaré : Ricci limits can also be seen as $\mathrm{RCD}(K, n)$ metric measure spaces, satisfying a generalized upper bound n for the dimension and lower bound K for the curvature.

The local geometry of a Ricci limit is described by tangent cones.

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Tangent cones and metric cones

Definition (Tangent cone)

Let (X, d) be a metric space and $x \in X$. A metric space (Y, d_Y, x) is called a tangent cone of X at x if there exists a sequence $\varepsilon \in (0, \infty)$ such that $\varepsilon \to 0$ and

$$(X, \varepsilon^{-1}\mathrm{d}, x) \xrightarrow{pGH} (Y, \mathrm{d}_Y, x).$$

A tangent cone is called "cone" because it is invariant by dilation.

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$$\mathrm{d}((s,x),(t,y))^2 = t^2 + s^2 + 2st\cos(\min\{\pi, \mathrm{d}_Z(x,y)\}),$$

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If moreover $\mathrm{Ric}_{g_\alpha} \geq K$, then

$$\mathrm{Ric}_{\varepsilon_\alpha^{-2} g_\alpha} \geq K\varepsilon_\alpha^2 \longrightarrow 0.$$

Cheeger-Colding theory relies on geometric and analytic results for manifold with $\mathrm{Ric}_g \geq 0$ and "almost positive" Ricci.

- ⋆ Bishop-Gromov and Bochner inequalities, Laplacian comparison, Hessian estimates, segment inequality, existence of good cut-off functions, splitting and almost splitting theorems...

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Regularity of non-collapsed Ricci limits

Theorem (Cheeger-Colding, 1997-2000)

Let (X, d, o) be a non-collapsed Ricci limit space.

(1) *For all $r > 0, x_\alpha \in M$ such that $x_\alpha \to x \in X$,* $\lim_\alpha \mathrm{vol}_{g_\alpha}(B_r(x_\alpha)) = \mathcal{H}^n(B_r(x))$.

(2) *For all $x \in X$, all tangent cones at x are metric cones.*

(3) *For $\mathcal{H}^n$-a.e. $x \in X$, $(\mathbb{R}^n, \mathrm{d}_e)$ is the unique tangent cone at x.*

(4) *$X = \mathcal{R} \sqcup \mathcal{S}$, where $\mathcal{R}$ is the set of points with a unique Euclidean cone and $\dim_{\mathcal{H}}(\mathcal{S}) \leq n-2$.*

(5) *X admits a **stratification** $X \supset \mathcal{S}^{n-2} \supset \ldots \supset \mathcal{S}^0$, where $\mathcal{S}^k = \{x \in X, \text{no tangent cone at } x \text{ splits } \mathbb{R}^{k+1}\}$ and $\dim_{\mathcal{H}}(\mathcal{S}^k) \leq k$.*

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Beyond Ricci bounded below

In many situations, one needs to study limits of Riemannian manifolds without having a lower bound on the Ricci curvature.

- Ricci, Kähler-Ricci, geometric flows.
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L^p-curvature bounds

Let (M^n, g) be a closed Riemannian manifold and $\rho(x) = \inf\{\mathrm{Ric}_g(v,v), v \in T_xM, g(v,v) = 1\}$. We define

$$\mathrm{Ric}_{-}(x) = \max\{-\rho(x), 0\}.$$

The L^p-integral curvature excess, $p > n/2$, is defined by

$$k_M(p) = \left(\frac{(\mathrm{diam}(M))^{2p}}{\mathrm{vol}_g(M)} \int_M \mathrm{Ric}_{-}{}^p \, d\,\mathrm{vol}_g \right)^{\frac{1}{p}}$$

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The Kato condition in $\mathbb{R}^n$

A potential $V \in L^1(\mathbb{R}^n)$ is said to be in the Kato class if

$$\lim_{t \to 0} \sup_{x \in \mathbb{R}^n} \int_0^t \int_{\mathbb{R}^n} H_{\mathbb{R}^n}(s,x,y) V(y) \mathrm{d}y \mathrm{d}s = 0,$$

where $H_{\mathbb{R}^n}(s,x,y) = (4\pi s)^{-\frac{n}{2}} \exp\left(\frac{||x-y||^2}{4s}\right)$.

The potential V is in the Kato class if it is the Laplacian of a continuous function.

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The potential V is in the Kato class if it is the Laplacian of a continuous function.

If V is in the Kato class, one can recover properties of $\Delta - V$ from properties of the Laplacian.

The Kato condition in $\mathbb{R}^n$

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Kato bounds for closed manifolds

Definition

Let (M^n, g) a closed manifold, Δ_g be the Laplacian associated to g and $\{e^{-t\Delta_g}\}_{t>0}$ its heat semi-group. For $t > 0$ we define the Kato constant

$$\mathrm{k}_t(M, g) = \sup_{x \in M} \int_0^t \int_M e^{-s\Delta_g} \operatorname{Ric}_-(y) dv_g(y) ds.$$

A bound on the $\mathrm{k}_t(M^n, g)$ implies good geometric and analytic properties, such as spectral bounds, isoperimetric inequality, bounds on the first Betti number (Carron, Güneysu, Rose, Stollmann, Voigt...)

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Pre-compactness

A uniform bound on $\mathrm{k}_t(M_\alpha, g_\alpha)$ is enough to get pre-compactness.

Proposition (G. Carron, 2016)

Let $\{(M^n_\alpha, g_\alpha)\}_{\alpha\in A}$ be a sequence of closed manifolds such that for all $\alpha \in A$

$$\mathrm{k}_1(M_\alpha, g_\alpha) \leq \frac{1}{16n}.$$

Then there exists a pointed metric space (X, d, o) and points $o_\alpha \in M_\alpha$ such that, up to a sub-sequence,

$$(M_\alpha, \mathrm{d}_{g_\alpha}, o_\alpha) \xrightarrow{pGH} (X, \mathrm{d}, o).$$

We refer to (X, d, o) as a Kato limit.

If (Y, d_Y, x) is a tangent cone of a Kato limit, then it is the limit of re-scaled manifolds $(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha)$.

By the scaling properties of the heat kernel and Riemannian volume, their Kato constant satisfies

$$\mathrm{k}_1(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) = \mathrm{k}_{\varepsilon_\alpha}(M_\alpha, g_\alpha) \leq \frac{1}{16n},$$

so that tangent cones are also Kato limits.

In order to obtain more information about the regularity of the limit space a stronger bound is needed. In the spirit of the Kato class in $\mathbb{R}^n$, one needs to uniformly control the way in which

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Definition (Strong Kato bound)

We say that a sequence $\{(M_\alpha, g_\alpha)\}_{\alpha \in A}$ of closed manifolds carries a uniform strong Kato bound if there exist $\Lambda > 0$ and a non-decreasing function $f : [0,1] \to \mathbb{R}_+$ such that

$$t \in (0,1], \alpha \in A, \ \mathrm{k}_t(M_\alpha, g_\alpha) \leq f(t) \leq 1/16n,$$

and moreover

$$\int_0^1 \frac{\sqrt{f(t)}}{t} \mathrm{d}t \leq \Lambda.$$

In this case

$$\mathrm{k}_t(M_\alpha, \varepsilon_\alpha^{-2} g_\alpha) = \mathrm{k}_{\varepsilon_\alpha^2 t}(M_\alpha, g_\alpha) \leq f(\varepsilon_\alpha^2 t) \to 0 \text{ as } \alpha \to +\infty.$$

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Theorem (G. Carron, I. M., D. Tewodrose, 2021)

Let $\{(M^n_\alpha, g_\alpha, o_\alpha)\}_{\alpha\in A}$ *be a pointed sequence of manifolds satisfying a uniform strong Kato bound and such that* $\mathrm{vol}_{g_\alpha}(B_1, o_\alpha)) \geq v > 0$*. Let* (X, d, o) *be the strong non-collapsed Kato limit of* $(M_\alpha, \mathrm{d}_\alpha, o_\alpha)$*. Then:*

(a) *for any* $r > 0$ *and* $x_\alpha \in M_\alpha$ *such that* $x_\alpha \to x$ *we have* $\lim_{\alpha\to\infty} \mathrm{vol}_{g_\alpha}(B(x_\alpha, r)) = \mathcal{H}^n(B(x, r))$;

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A strong Kato bound is implied by assumptions previously considered in the literature:

- an L^p smallness assumption on Ric₋ for $p > n/2$ (C. Rose, P. Stollman, 2017, G. Carron, C. Rose, 2021);
- an L^p bound on $|\mathrm{Ric}|$ for $p > n/2$ and a non-collapsing assumption (G. Tian, Z. Zhang, 2016).

It also follows from a uniform bound on the L^p Kato constant, $p > 1$, where

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From now on we focus on the following.

Theorem (G. Carron, I. M., D. Tewodrose, 2021)

Tangent cones of non-collapsed strong Kato limits are (weakly non-collapsed $\mathrm{RCD}(0,n)$*) metric measure cones.*

An $\mathrm{RCD}(k,n)$ metric measure space (X,d,μ) is weakly non-collapsed if

$$\lim_{r\to 0}\frac{\mu(B_r(x))}{\omega_n r^n}>0,\ \omega_n=\mathcal{H}^n(\mathbb{B}_1(0)).$$

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Consequences of a Kato bound

On a closed manifold (M^n, g) with $\mathrm{Ric}_g \geq 0$, a positive solution u of the heat equation satisfies the Li-Yau inequality

$$\Delta(\log u) \leq \frac{n}{2t}.$$

Proposition (Li-Yau's inequality, G. Carron 2016)

If (M^n, g) is a closed manifold such that

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Let $u : [0,1] \times M \to \mathbb{R}_+$ be a positive solution to the heat equation. Then for any $(t, x) \in [0,1] \times M$

$$e^{-8\sqrt{n\,\mathrm{k}_t(M,g)}}\frac{|du|^2}{u^2} - \frac{1}{u}\frac{\partial u}{\partial t} \leq \frac{n}{2t} e^{8\sqrt{n\,\mathrm{k}_t(M,g)}}.$$

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Doubling and Poincaré inequality

A closed manifold (M^n, g) such that $\mathrm{k}_1(M^n, g) \leq 1/16n$:

- is $\kappa(n)$-doubling at scale 1: for all $r \in (0, 1]$, $x \in M$, $\mathrm{vol}_g(B_{2r}(x)) \leq \kappa(n)\,\mathrm{vol}_g(B_r(x))$.
- carries a $\gamma(n)$-Poincaré inequality at scale 1: for all $r \in (0, 1]$, $x \in M$, $\varphi \in C^1(B_r(x))$

$$\int_{B_r(x)} |\varphi - \varphi_B|^2 \mathrm{d}\,\mathrm{vol}_g \leq \gamma(n) \int_{B_r(x)} |d\varphi|^2 \mathrm{d}\,\mathrm{vol}_g,$$

where $\varphi_B = \mathrm{vol}_g(B_r(x))^{-1} \int_{B_r(x)} \varphi \mathrm{d}\,\mathrm{vol}_g$.

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Manifolds seen as Dirichlet spaces

Let $\{(M_\alpha, g_\alpha, o_\alpha)\}_{\alpha \in A}$ be a non-collapsing sequence of closed manifolds satisfying a uniform strong Kato bound. Define the Dirichlet energies

$$\mathcal{E}_\alpha(u) = \int_M |du|^2 \mathrm{d}\,\mathrm{vol}_{g_\alpha}, \quad u \in C^1(M_\alpha).$$

Then $\{(M_\alpha, d_{g_\alpha}, \mathrm{vol}_{g_\alpha}, \mathcal{E}_\alpha, o_\alpha)\}_{\alpha \in A}$ is e sequence of uniformly PI Dirichlet spaces at scale 1.

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Improved convergence

By adapting results of Kasue-Shioya (2003), Kasue (2005) we get:

Proposition (G. Carron, I. M., D. Tewodrose 2021)

Let (X, d, o) be a non-collapsed strong Kato limit. Then there exist a doubling measure μ and a Dirichlet energy $\mathcal{E}$ on X such that $(X, \mathsf{d}, \mu, \mathcal{E})$ is a PI Dirichlet space at scale 1 and $\{(M_\alpha, d_{g_\alpha}, \mathrm{vol}_{g_\alpha}, \mathcal{E}_\alpha, o_\alpha)\}_{\alpha \in A}$ Mosco-Gromov-Hausdorff converges to $(X, \mathsf{d}, \mu, \mathcal{E}, o)$, that is

$$(M_\alpha, \mathsf{d}_{g_\alpha}, \mathrm{vol}_{g_\alpha}, o_\alpha) \xrightarrow{pmGH} (X, \mathsf{d}, \mu, o),$$

and $\mathcal{E}_\alpha$ Mosco converges to $\mathcal{E}$.

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Let (X, d, o) be a non-collapsed strong Kato limit. Then there exist a doubling measure μ and a Dirichlet energy $\mathcal{E}$ on X such that $(X, \mathrm{d}, \mu, \mathcal{E})$ is a PI Dirichlet space at scale 1 and $\{(M_\alpha, d_{g_\alpha}, \mathrm{vol}_{g_\alpha}, \mathcal{E}_\alpha, o_\alpha)\}_{\alpha \in A}$ Mosco-Gromov-Hausdorff converges to $(X, \mathrm{d}, \mu, \mathcal{E}, o)$, that is

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Consequence of a strong Kato bound

Proposition (Volume bounds, G. Carron 2016)

Let (M^n, g) be a closed manifold such that

$$\mathrm{k}_1(M^n, g) \leq \frac{1}{16n}, \quad \int_0^1 \frac{\sqrt{\mathrm{k}_t(M^n, g)}}{t} dt \leq \Lambda, .$$

Then there exists $C = C(n, \Lambda) > 0$ s.t. for all $o, x \in M$, $r \in (0, 1]$

$$C^{-d(o,x)} \mathrm{vol}_g(B_1(o)) \leq \frac{\mathrm{vol}_g(B_r(x))}{r^n} \leq C.$$

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Monotone quantities

The study of many regularity problems in geometric analysis is based on the understanding of the appropriate monotone quantity carrying rigidity and almost rigidity properties.

If (M^n, g) is such that $\mathrm{Ric}_g \geq 0$ then the Bishop-Gromov inequality implies that the volume ratio at any point $x \in M$

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Rigidity and almost rigidity

(a) When the volume ratio is constant on an interval $(0, R]$, the ball $B_R(x)$ is isometric to a ball in a metric cone.

(b) Almost volume cone implies almost metric cone: if the volume ratio is "close" to a constant, the appropriate ball is GH-close to a ball in a metric cone.

Both are true on weakly non-collapsed $\mathrm{RCD}(0, n)$ spaces.

Theorem (De Philippis, Gigli 2018)

Let (X, d, μ) be a weakly non-collapsed $\mathrm{RCD}(0, n)$ space. Assume that the volume ratio $r \mapsto \mu(B_r(x))/r^n$ is constant for some $x \in X$. Then (X, d, μ) is an n-metric measure cone at x.

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In the setting of strong Kato limits

For manifolds satisfying a strong Kato bound, the volume ratio is bounded but not monotone.

But tangent cones (Y, d_Y, x) of a strong Kato limit are weakly non-collapsed $\mathrm{RCD}(0, n)$ spaces.

Aim: Show that the volume ratio of (Y, d_Y, x) at x is constant.

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Θ-volume

Definition (Θ-volume)

Let (X, d, μ) be a m.m.s. We define for $x \in X$, $s > 0$

$$\Theta_x^X(s) = (4\pi s)^{-\frac{n}{2}} \int_X \exp\left(-\frac{\mathrm{d}(x,y)^2}{4s}\right) \mathrm{d}\mu(y).$$

- It coincides with Huisken's entropy, introduced in the study of mean curvature flow.
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Properties of the Θ-volume

Proposition

Let (X, d, μ) be a doubling metric measure space.

(1) *There exists $c > 0$ such that $\Theta_x^X(s) = c$ for all $s > 0$ if and only if $\mu(B_s(x)) = c\omega_n s^n$ for all $s > 0$.*

(2) *Θ_x^X is continuous with respect to pmGH convergence.*

We aim to show that on a tangent cone of a non-collapsed strong Kato limit, the Θ-volume is constant.

If so, (1) and De Philippis-Gigli's result allow to conclude.

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Let $(X, \mathrm{d}, \mu, \mathcal{E})$ be a Dirichlet space and define the function U s.t.

$$H(t,x,y) = (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{U(t,x,y)}{4t}\right).$$

For all $s, t > 0$ we define

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θ-volume on a manifold with $\mathrm{Ric}_g \geq 0$

Let (M^n, g) be a manifold with $\mathrm{Ric}_g \geq 0$. Consider

$$\lambda \mapsto \theta_x^M(\lambda s, \lambda t).$$

- Fix $t = 0$. Then one can write

$$\theta_x^M(\lambda s, 0) = \frac{1}{2}\int_0^\infty e^{-\frac{\rho^2}{4}} \frac{\mathrm{vol}_g(B_{\rho\sqrt{\lambda s}}(x))}{\omega_n(\rho\sqrt{\lambda s})^n} \mathrm{d}\rho.$$

- Bishop-Gromov inequality: $\lambda \mapsto \theta_x^M(\lambda s, 0)$ is non-increasing.
- Li-Yau inequality: $t \mapsto (4\pi t)^{\frac{n}{2}} H(t, x, x)$ is non-decreasing.
- Fix $t > 0$ and $s = t/4$. Then

$$\theta_x^M\left(\lambda\frac{t}{4}, \lambda\frac{t}{2}\right) = (4\pi\lambda t)^{\frac{n}{2}} H(\lambda t, x, x),$$

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Let (M^n, g) be a manifold with $\mathrm{Ric}_g \geq 0$. Then for all $x \in M$, $s, t > 0$

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has the same sign as $(t - s)$ and the map $\lambda \mapsto \theta_x^M(\lambda s, \lambda t)$, is

- *monotone non-increasing for all $t \leq s$;*
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The proof relies on Li-Yau inequality and properties of U.

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has the same sign as $(t-s)$, where $\Gamma(M,g) = e^{8\sqrt{n\,\mathrm{k}_1(M,g)}} - 1$.

Monotonicity

Corollary (G. Carron, I. M., D. Tewodrose 2021)

Let $\Lambda > 0$. Let (M^n, g) be a closed manifold such that for all $t \in (0, 1]$

$$\phi(t) = \int_0^t \frac{\sqrt{\mathrm{k}_s(M,g)}}{s} ds \leq \Lambda.$$

There exist $c_n > 0$ and $\bar\lambda = \bar\lambda(s,t)$ for all $s > 0, t \in (0,1]$ s.t.

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Strong Kato bound and monotonicity

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There exists a function F_α, depending on ϕ_α, such that the map

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is monotone for all $x_\alpha \in M_\alpha$, and $F_\alpha(\lambda t) \to 1$ as $\lambda \to 0$.

In order to get the analog monotone quantity in the limit, one needs a uniform control of F_α, thus of ϕ_α.
This is ensured by a strong Kato bound

$$\forall t \in (0,1] \ \mathrm{k}_t(M_\alpha, g_\alpha) \leq f(t), \quad \int_0^1 \frac{\sqrt{f(t)}}{t} dt \leq \Lambda.$$

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There exists a function F_α, depending on ϕ_α, such that the map

$$\lambda \mapsto \theta_{x_\alpha}^{M_\alpha}(\lambda s, \lambda t) F_\alpha(\lambda t)$$

is monotone for all $x_\alpha \in M_\alpha$, and $F_\alpha(\lambda t) \to 1$ as $\lambda \to 0$.

In order to get the analog monotone quantity in the limit, one needs a uniform control of F_α, thus of ϕ_α.
This is ensured by a strong Kato bound

$$\forall t \in (0,1] \;\; \mathrm{k}_t(M_\alpha, g_\alpha) \leq f(t), \quad \int_0^1 \frac{\sqrt{f(t)}}{t} \mathrm{d}t \leq \Lambda.$$

Tangent cones are metric cones

Theorem (G. Carron, I. M., D. Tewodrose 2021)

Let $\Lambda, v > 0$ and $f : [0,1] \to \mathbb{R}_+$ be a non-decreasing function such that

$$\int_0^1 \frac{\sqrt{f(t)}}{t} dt \leq \Lambda.$$

Let $(M_\alpha, g_\alpha, o_\alpha)$ be closed manifolds such that for all α

$$\forall t \in (0,1] \;\; \mathrm{k}_t(M_\alpha, g_\alpha) \leq f(t), \quad \mathrm{vol}_{g_\alpha}(B_1(o_\alpha)) \geq v$$

Let (X, d, μ, o) be the pmGH limit of $(M_\alpha, \mathrm{d}_{g_\alpha}, \mathrm{vol}_{g_\alpha}, o_\alpha)$ Then for all $x \in X$, any tangent cone at x is a metric cone.

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Sketch of proof

- On a strong Kato limit Θ_x^X and θ_x^X are well-defined.
- There exists a function $F : (0, \varepsilon] \to \mathbb{R}_+$ such that the map

$$\lambda \in (0, \varepsilon] \to \theta_x^X(\lambda s, \lambda t) F(\lambda t),$$

  is monotone, and $F(\lambda t) \to 1$ as $\lambda \to 0$.
- By the local n-Ahlfors regularity, $\theta_x^X(\lambda s, \lambda t)$ is bounded by positive constants.
- We can then define a positive number

$$\vartheta_x^X(s,t) = \lim_{\lambda \to 0^+} \theta_x^X(\lambda s, \lambda t),$$

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- A tangent cone $(Y, \mathrm{d}_Y, \mu_Y, x)$ at $x \in X$ is a limit of re-scaled spaces $(X_\alpha, \mathrm{d}_\alpha, \mu_\alpha, x) = (X, \varepsilon_\alpha^{-1}\,\mathrm{d}, \varepsilon_\alpha^{-n}\mu, x)$
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Further consequences

Proposition (G. Carron, I. M., D. Tewodrose 2021)

Let (X, d, μ, o) be a non-collapsed strong Kato limit. Then for any $x \in X$ the volume density is well-defined

$$\vartheta_X(x) = \lim_{r\to 0} \frac{\mu(B_r(x))}{\omega_n r^n},$$

and $\vartheta_X(x) \leq 1$. Moreover $\mu \leq \mathcal{H}^n$.

On a non-collapsed Ricci limit, Bishop-Gromov inequality guarantees that the volume density is well-defined in $(0, 1]$.

Moreover, $\vartheta_X(x) = 1$ if and only if $x \in \mathcal{R}$, that is, iff x admits a unique Euclidean tangent cone.

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Volume density and k-splittings

Proposition (De Philippis, Gigli 2018)

Let (X, d, μ) *be a weakly non-collapsed* $\mathrm{RCD}(0, n)$ *n-metric measure cone at* x. *Then for all* $x' \in X$ *we have* $\vartheta_X(x') \geq \vartheta_X(x)$.
There exists $k \in \mathbb{N}$ *such that the set where* $\vartheta_X(x') = \vartheta_X(x)$ *is isometric to* $\mathbb{R}^k$ *and* X *is isometric to* $\mathbb{R}^k \times C(Z)$.

The volume density detects simmetries.

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Regularity almost everywhere and stratification

Proposition (G. Carron, I. M., D. Tewodrose 2021)

The volume density is lower semi-continuous with respect to pmGH convergence in the set of non-collapsed strong Kato limits.

The previous, together with arguments of B. White, lead to

Theorem (G. Carron, I. M., D. Tewodrose 2021)

Let (X, d, μ, o) be a non-collapsed strong Kato limit.

- *regularity almost everywhere: μ-a.e. $x \in X$ has a unique Euclidean tangent cone $(\mathbb{R}^n, \mathrm{d}_e, \vartheta_X(x)\mathcal{H}^n)$;*
- *Let $S^k = \{x \in X$ s.t. no tangent cone at x splits $\mathbb{R}^{k+1}\}$. Then $\dim_{\mathcal{H}}(S^k) \leq k$.*

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Volume continuity

Theorem (G. Carron, I. M., D. Tewodrose 2021)

For a non-collapsed strong Kato limit (X, d, μ, o) *we have* $\mu = \mathcal{H}^n$.

Since we already know $\mu \leq \mathcal{H}^n$ and that μ-a.e. tangent cones are $(\mathbb{R}^n, \mathrm{d}_e, \vartheta_X(x)\mathcal{H}^n)$, it is enough to prove:

$$x \in X \text{ admits a Euclidean tangent cone} \Longrightarrow \vartheta_X(x) = 1.$$

For that we need more tools:

- good cut-off χ functions on balls, $|\nabla\chi|^2 + |\Delta\chi| \leq C$;
- Hessian estimates for harmonic functions;
- appropriate convergence of harmonic functions;
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- appropriate convergence of harmonic functions;
- existence of (ε, n)-splitting maps.

Thank you!