Differential Geometry and its Applications 13 (2000) 129–165
North-Holland
www.elsevier.com/locate/difgeo

On fundamental groups of manifolds of nonnegative curvature

Burkhard Wilking[1]
Westfälische Wilhelms–Universität, Einsteinstr. 62, 48149 Münster, Germany

Communicated by E. Heintze
Received 21 April 1999

Abstract: We will characterize the fundamental groups of compact manifolds of (almost) nonnegative Ricci curvature and also the fundamental groups of manifolds that admit bounded curvature collapses to manifolds of nonnegative sectional curvature. Actually it turns out that the known necessary conditions on these groups are sufficient as well. Furthermore, we reduce the Milnor problem—are the fundamental groups of open manifolds of nonnegative Ricci curvature finitely generated?—to manifolds with abelian fundamental groups. Moreover, we prove for each positive integer n that there are only finitely many non-cyclic, finite, simple groups acting effectively on some complete n-manifold of nonnegative Ricci curvature. Finally, sharping a result of Cheeger and Gromoll [6], we show for a compact Riemannian manifold (M, g_0) of nonnegative Ricci curvature that there is a continuous family of metrics (g_λ), $\lambda \in [0, 1]$ such that the universal covering spaces of (M, g_λ) are mutually isometric and (M, g_1) is finitely covered by a Riemannian product $N \times T^d$, where T^d is a torus and N is simply connected.

Keywords: Fundamental groups, nonnegative curvature, groups of polynomial growth.

MS classification: 53C20, 57S30.

1. Introduction

The paper is divided in six sections. In the beginning of each of the following sections we state its main results which can be understood independently from each other. The corresponding proofs can be found at the end of each section. In Section 2 we give algebraic characterizations of the fundamental groups of several classes of Riemannian manifolds. At first, we consider compact manifolds of nonnegative Ricci curvature. Using the fact that the fundamental group of such a manifold acts cocompactly as the deck transformation group on the universal covering space $\tilde{M}$ of M, Cheeger and Gromoll [5] have shown that $\tilde{M}$ is compact, unless it contains a line. Combining this observation with their deep splitting theorem they proved that $\tilde{M}$ is isometric to a Riemannian product $\mathbb{R}^d \times K$ where K is a compact manifold. As a consequence they deduced that $\pi_1(M)$ acts discontinuously and cocompactly on $\mathbb{R}^d$, and therefore it contains a finite normal subgroup E such that $\pi_1(M)/\mathsf{E}$ is isomorphic to a crystallographic group, i.e., to a discrete, cocompact subgroup of $\mathbb{R}^d \rtimes \mathrm{O}(d)$ ($d = 0$ is allowed).

It is natural to ask whether conversely all abstract groups satisfying this condition occur as fundamental groups of compact manifolds of nonnegative Ricci curvature. In Theorem 2.1 we

[1] *E-mail*: wilking@math.uni-muenster.de.

0926-2245/00/$ – see frontmatter ©2000 Published by Elsevier Science B.V. All rights reserved
PII S0926-2245(00)00030-9

give a positive answer to this question. Actually we prove the stronger result that these groups occur as fundamental groups of compact manifolds of nonnegative sectional curvature.

The fundamental group of an n-dimensional, complete manifold M of nonnegative Ricci curvature was first investigated by Milnor [14]. He used the polynomial volume growth of the universal covering space of M to show that any finitely generated subgroup of $\pi_1(M)$ has polynomial growth of order $\leqslant n$. Gromov [9] has given an algebraic characterization of the growth condition: A finitely generated group has polynomial growth if and only if it contains a nilpotent subgroup of finite index. We shall briefly say that such a group is nilpotent up to finite index.

There are two more classes of Riemannian manifolds for which the corresponding fundamental groups are nilpotent up to finite index: Fukaya and Yamaguchi [8] have shown that for any positive integer n there exists a constant $\varepsilon(n) > 0$ such that the fundamental group of any n-dimensional Riemannian manifold M with sectional curvature bounded below by $-\varepsilon(n)/\mathrm{diam}(M)^2$ is nilpotent up to finite index. Recently Cheeger and Colding [3] proved the Gromov conjecture which is the Ricci version of Fukaya's and Yamaguchi's theorem: For any n there is a constant $\tilde{\varepsilon}(n) > 0$ such that the fundamental group of any n-dimensional Riemannian manifold M with Ricci curvature bounded below by $-\tilde{\varepsilon}(n)/\mathrm{diam}(M)^2$ is nilpotent up to finite index.

A compact n-dimensional manifold M is said to have almost nonnegative sectional curvature (resp. Ricci curvature) if its sectional curvature (resp. Ricci curvature) is bounded below by $-\varepsilon(n)/\mathrm{diam}(M)^2$ (resp. $-\tilde{\varepsilon}(n)/\mathrm{diam}(M)^2$).

Again one can ask whether any finitely generated group that is nilpotent up to finite index occurs as a fundamental group in each of the considered classes. We give a positive answer to this question in Theorem 2.3.

For the class of complete manifolds of nonnegative Ricci curvature a partial answer was already known before, and this result also plays a crucial role in the proof of Theorem 2.3: For a nilpotent Lie group N Wei [21] constructed a complete metric of positive Ricci curvature on $M = \mathbb{R}^p \times \mathsf{N}$, where p is a sufficiently large integer, in such a way that the natural action of N on M is still isometric. Since, according to Malcev, any finitely generated, torsion free, nilpotent group can be realized as a lattice in a connected, simply connected, nilpotent Lie group, she obtained as a corollary that any finitely generated, torsion free, nilpotent group is the fundamental group of some complete manifold of positive Ricci curvature.

The concept of a finitely generated, nilpotent group has a natural generalization, the concept of a polycyclic group, see Section 2 for the definition. Professor E. Heintze posed the question whether there is a class of manifolds for which the corresponding fundamental groups are precisely the groups that are polycyclic up to finite index. In Theorem 2.4 we give a positive answer to this question. One of the two classes we consider consists of compact manifolds that admit bounded curvature collapses to manifolds of nonnegative sectional curvature. The most difficult implication of Theorem 2.4 is actually an immediate consequence of the fiber bundle theorem of Fukaya and Yamaguchi [8].

In the Theorems 2.1, 2.3 and 2.4 we give the two equivalent algebraic conditions c) and d). Briefly stated this equivalence means that a finitely generated group is abelian (nilpotent, polycyclic) up to finite index if and only if modulo a finite subgroup it is isomorphic to a crystallographic (almost crystallographic, polycrystallographic) group. Recall that an almost crystallographic (polycrystallographic) group is a discrete, cocompact subgroup of a semidirect product

$\mathsf{N} \rtimes \mathsf{K}$, where N is a connected, simply connected, nilpotent (solvable) Lie group and K is a compact subgroup of its automorphism group. In the nilpotent case the most striking part of this equivalence, the implication d) $\Rightarrow$ c), is due to Lee [13]. The equivalence in the polycyclic case is actually an immediate consequence of [23], where several equivalent characterizations of polycrystallographic groups are given. In the proofs of the three theorems the equivalence of the conditions c) and d) is crucial to realize these groups as fundamental groups in the considered classes.

In Section 3 we will use the equivalence of the conditions c) and d) of Theorem 2.3 to generalize Gromov's polynomial growth Theorem to some extent to non-finitely generated groups. In particular, Theorem 3.1 provides an algebraic characterization of groups for which any finitely generated subgroup has polynomial growth of order $\leqslant n$. As a corollary of this theorem we prove that such a group is finitely generated if and only if any abelian subgroup is finitely generated. This result is intended to reduce one of the major open problems in the structure theory of noncompact, complete manifolds of nonnegative Ricci curvature: Is the fundamental group of such a manifold finitely generated? Since any subgroup of a fundamental group is the fundamental group of some covering space, Corollary 3.2 reduces the original problem to manifolds with abelian fundamental groups. In Section 3 we also give an example of a complete Riemannian manifold M whose fundamental group is isomorphic to $\mathbb{Q}/\mathbb{Z}$; M is obtained by a surgery construction from a sequence of homogeneous spaces, but actually it is hard to tell whether M admits a complete metric of nonnegative Ricci curvature.

Notice that Theorem 2.3 does not answer the question which of the groups that match the algebraic characterization occur as fundamental groups in the corresponding class of manifolds in a fixed dimension. For the class of compact manifolds of almost nonnegative sectional curvature Fukaya and Yamaguchi [8] have shown that in each dimension n there is a constant C_n such that the fundamental group of a manifold in this class has a solvable normal subgroup of index at most C_n. In Section 4 we will prove a weaker result for the class of complete manifolds of nonnegative Ricci curvature (Theorem 4.1): In each dimension n there is a finite collection of finite simple groups such that any finitely generated fundamental group Π has a finite subnormal series $\{e\} = \mathsf{N}_0 \subset \cdots \subset \mathsf{N}_l = \Pi$ with factor groups $\mathsf{N}_{i+1}/\mathsf{N}_i$ being either cyclic or isomorphic to a group of the finite collection. Beyond the fact that we apply strong theorems the proof of Theorem 4.1 is surprisingly easy.

In Section 5 we establish an estimate for the index of the nilradical in an almost crystallographic group. It is a well-known fact that the torsion free almost crystallographic groups are precisely the fundamental groups of Gromov's almost flat manifolds. In this context Buser and Karcher [1] proved for an almost crystallographic group Γ that its translational part Γ^* is a subgroup of index at most $2 \cdot 6^{\frac{1}{2}r(r-1)}$ where $r = \operatorname{rank}(\Gamma^*)$. We will show (Theorem 5.1) that the index divides the number $(2n)!$ where $n = \operatorname{rank}(\Gamma^*) - \operatorname{rank}([\Gamma^*, \Gamma^*])$.

The estimates of Section 5 are needed in the last section, which deals with a deformation problem. By Cheeger and Gromoll [6] a compact manifold of nonnegative Ricci curvature is finitely covered by a manifold which is diffeomorphic to a product $T^{(d)} \times N$ where N is simply connected and $T^{(d)}$ is a torus. To sharpen this result we will study normal Riemannian coverings $q_0 \colon \mathbb{R}^n \times N \to (M, g_0)$, where N is a complete Riemannian manifold with a compact isometry group. Corollary 6.3 states that such a covering can be deformed via a continuous family of Riemannian coverings $q_\lambda \colon \mathbb{R}^n \times N \to (M, g_\lambda)$, $\lambda \in [0, 1]$, such that (M, g_1) is finitely covered by a Riemannian product $T^{(d)} \times \mathbb{R}^{n-d} \times N$ where $T^{(d)}$ is a flat torus and $d \in \{0, \dots, n\}$. Moreover, we estimate the number of sheets of the finite covering.

In the case of an n-dimensional open flat manifold (M, g_0) Corollary 6.3 reduces to the following generalization of the first Bieberbach theorem (Corollary 6.4): There is a continuous family $(g_\lambda)_{\lambda\in[0,1]}$ of complete flat metrics on M such that the holonomy group of (M, g_1) is finite and its order is bounded by n^{n^2}.

2. Algebraic characterizations of fundamental groups

2.1. Basic definitions and statement of results

A crystallographic group of rank d is a discrete, cocompact subgroup of the isometry group of $\mathbb{R}^d$. The trivial group is said to be a crystallographic group of rank 0.

Theorem 2.1. *For an abstract group* Π *the following statements are equivalent.*

a) Π *is isomorphic to the fundamental group of a compact manifold of nonnegative sectional curvature.*

b) Π *is isomorphic to the fundamental group of a compact manifold of nonnegative Ricci curvature.*

c) *There is a finite group* E*, a crystallographic group* Γ *and an exact sequence*

$$\{1\} \to \mathsf{E} \xrightarrow{i} \Pi \xrightarrow{p} \Gamma \to \{1\}.$$

d) *There is a finite group* F*, an integer* $d \geqslant 0$ *and an exact sequence*

$$\{0\} \to \mathbb{Z}^d \xrightarrow{j} \Pi \xrightarrow{q} \mathsf{F} \to \{1\}.$$

e) Π *is isomorphic to a discrete, cocompact subgroup of a semidirect product* $\mathbb{R}^d \rtimes_\beta \mathsf{F}$*, where* F *is a finite group and* $\beta\colon \mathsf{F} \to \mathrm{GL}(d, \mathbb{R})$ *a homomorphism.*

The construction actually shows that these groups occur as fundamental groups of compact locally homogeneous spaces, which are finitely covered by Lie groups. It is therefore tempting to ask "Which fundamental groups occur in the homogeneous case?" We use the concept of a homogeneous space in the Riemannian sense. Hence a homogeneous space is a Riemannian manifold whose isometry group acts transitively.

Theorem 2.2. *For an abstract group* Π *the following statements are equivalent.*

a) Π *is isomorphic to the fundamental group of a compact homogeneous space.*

b) Π *is isomorphic to the fundamental group of a homogeneous space.*

c) *There is a finite group* E*, an integer* $d \geqslant 0$ *and an exact sequence*

$$\{1\} \to \mathsf{E} \xrightarrow{i} \Pi \xrightarrow{p} \mathbb{Z}^d \to \{0\}.$$

d) *The center of* Π *contains a subgroup* $\mathsf{A} \cong \mathbb{Z}^d$ *of finite index in* Π.

e) Π *is isomorphic to a discrete, cocompact subgroup of a direct product* $\mathbb{R}^d \times \mathsf{F}$*, where* F *is a finite group.*

An almost crystallographic group of rank d is a discrete, cocompact subgroup of a semidirect product $\mathsf{N} \rtimes \mathsf{K}$, where N is a connected, simply connected, d-dimensional, nilpotent Lie group, K is a compact subgroup of the automorphism group $\mathrm{Aut}(\mathsf{N})$ and where $\mathsf{N} \rtimes \mathsf{K}$ carries the natural semidirect product structure. Again the case $d = 0$ is allowed. As explained in the introduction, a Riemannian manifold is said to have almost nonnegative sectional (resp. Ricci) curvature if it satisfies the assumption of the theorem of Fukaya and Yamaguchi [8] (resp. the assumption of the theorem of Cheeger and Colding [3]).

Theorem 2.3. *For a finitely generated group Π the following statements are equivalent.*

a) Π *is isomorphic to the fundamental group of a complete manifold of positive Ricci curvature.*

b) Π *is isomorphic to the fundamental group of a complete manifold of nonnegative Ricci curvature.*

c) *There is a finite group* E, *an almost crystallographic group* Γ *and an exact sequence*

$$\{1\} \to \mathsf{E} \xrightarrow{i} \Pi \xrightarrow{p} \Gamma \to \{1\}.$$

d) *There is a finite group* F, *a torsion free, nilpotent group* L *and an exact sequence*

$$\{1\} \to \mathsf{L} \to \Pi \to \mathsf{F} \to \{1\}.$$

e) Π *is isomorphic to a discrete, cocompact subgroup of a semidirect product* $\mathsf{N} \rtimes_\beta \mathsf{F}$, *where* N *is a connected, simply connected, nilpotent Lie group,* F *is a finite group and* $\beta\colon \mathsf{F} \to \mathrm{Aut}(\mathsf{N})$ *a homomorphism.*

f) Π *is isomorphic to the fundamental group of a compact manifold of almost nonnegative sectional curvature.*

g) Π *is isomorphic to the fundamental group of a compact manifold of almost nonnegative Ricci curvature.*

The concept of a finitely generated nilpotent group has a natural generalization, the concept of a polycyclic group. Recall that a group Λ is called polycyclic if there are subgroups

$$\{e\} = \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_k = \Lambda$$

such that N_i is a normal subgroup of N_{i+1} and the factor group $\mathsf{N}_{i+1}/\mathsf{N}_i$ is cyclic. If $\mathsf{N}_{i+1}/\mathsf{N}_i \cong \mathbb{Z}$ for all i, then Λ is called a strongly polycyclic group.

The concept of an almost crystallographic group has also a natural generalization: A group Γ is called a polycrystallographic group of rank d if and only if Γ is isomorphic to a discrete cocompact subgroup of a semidirect product $\mathsf{S} \rtimes \mathsf{K}$, where S is a d-dimensional, connected, simply connected solvable Lie group and where K is a compact subgroup of $\mathrm{Aut}(\mathsf{S})$. As is shown in Wilking [23] this is equivalent to saying that there are subgroups

$$\{e\} = \Gamma_1 \subset \cdots \subset \Gamma_k = \Gamma$$

such that Γ_i is a normal subgroup of Γ_{i+1} and the factor group Γ_{i+1}/Γ_i is isomorphic to a crystallographic group. Therefore the notation “polycrystallographic” is rectified. Furthermore it is easy to see that in the above the situation the rank of Γ is given by

$$\mathrm{rank}(\Gamma) = \sum_{i=1}^{k-1} \mathrm{rank}(\Gamma_{i+1}/\Gamma_i).$$

Theorem 2.4. *Let Π be a group. Then the following statements are equivalent.*

a) Π *is isomorphic to the fundamental group of a compact manifold M satisfying: There is a sequence of Riemannian metrics g_i on M such that*

(i) *The absolute value of the sectional curvature of (M, g_i) is bounded above by* 1.

(ii) *The sequence (M, g_i) converges in the Gromov–Hausdorff sense to a (possibly lower dimensional) compact Riemannian manifold B of nonnegative sectional curvature.*

b) Π *is isomorphic to the fundamental group of a compact manifold M satisfying: There are compact manifolds $M = M_0, M_1, \ldots, M_k$ and sequences of Riemannian metrics g^i_j on M_j such that*

(i) *The sectional curvature of (M_j, g^i_j) is bounded below by* -1.

(ii) *The sequence $(M_j, g^i_j)_{i\in\mathbb{N}}$ converges in the Gromov–Hausdorff sense to (M_{j+1}, g^0_{j+1}) for $j = 0, \ldots, k-1$.*

(iii) $(M_k, g^i_k)_{i\in\mathbb{N}}$ *collapses to a single point.*

c) *There is a finite group E a polycrystallographic group Γ and an exact sequence*

$$\{1\} \to \mathsf{E} \overset{i}{\to} \Pi \overset{p}{\to} \Gamma \to \{1\}\,.$$

d) Π *is polycyclic up to finite index.*

e) Π *is isomorphic to a discrete, cocompact subgroup of a semidirect product $\mathsf{S} \rtimes_\beta \mathsf{F}$, where S is a connected, simply connected, solvable Lie group, F is a finite group and $\beta\colon \mathsf{F} \to \mathrm{Aut}(\mathsf{S})$ is a homomorphism.*

Remark 2.5. 1. The group $i(\mathsf{E}) \subset \Pi$, occurring in condition c) of each of the four theorems, is uniquely characterized as the maximal finite normal subgroup of Π. In order to prove this, we first remark that the product of two finite normal subgroups is again a finite normal subgroup. Thus Π contains at most one maximal finite normal subgroup. On the other hand, a polycrystallographic group does not contain any nontrivial finite normal subgroup, and hence $i(\mathsf{E})$ is a maximal finite normal subgroup.

2. The Theorems 2.1 and 2.3 in particular provide an algebraic characterization of (almost) crystallographic groups due to Dekimpe and Igodt [7]: An abstract finitely generated group Γ is isomorphic to an (almost) crystallographic group if and only if it contains an abelian (resp. nilpotent) subgroup of finite index and it does not contain any nontrivial finite normal subgroup. The analogue for polycrystallographic groups was proved in [23].

3. There is a different, more common algebraic characterization of (almost) crystallographic groups given by L. Auslander. A finitely generated group Γ is isomorphic to an (almost) crystallographic group if and only if Γ contains an abelian (resp. nilpotent) torsion free normal subgroup Γ^* of finite index, which is maximal among all abelian (resp. nilpotent) subgroups of Π.

We use the opportunity to correct a mistake which occurs in this context in the literature, see [2, p. 74]. In Auslander's characterization of crystallographic groups the condition that Γ^* is a normal subgroup is not redundant: Let S_3 be the symmetric group of degree 3, and let $h\colon \mathsf{S}_3 \to \mathbb{Z}/2\mathbb{Z}$ be the unique epimorphism onto the additive group $\mathbb{Z}/2\mathbb{Z}$. Define Γ as the kernel of the homomorphism

$$\mathsf{S}_3 \times \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}\,, \qquad (\sigma,\, k) \mapsto h(\sigma) + \pi(k)$$

where $\pi\colon \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$ denotes the projection. Clearly, Γ is not abelian but Γ contains a cyclic subgroup Γ^* of index 3 generated by the element $(\tau, 1)$ where τ is a transposition. Thus Γ^* is a torsion free maximal abelian subgroup of finite index. Nevertheless, Γ is not isomorphic to a crystallographic group.

4. The embedding $\Pi \hookrightarrow \mathsf{S} \rtimes_\beta \mathsf{F}$ in Theorem 2.4 e) can be chosen such that any automorphism of Π can be extended uniquely to an automorphism of $\mathsf{S} \rtimes_\beta \mathsf{F}$.

The rest of this section is organized as follows: We will prove Theorem 2.1 and Theorem 2.2 in Subsection 2.2 and Subsection 2.3, respectively. For the proofs of the other two theorems and for later applications we need some elementary lemmas from group theory which we have placed in Subsection 2.4. Subsection 2.5 and Subsection 2.6 contain the proofs of Theorem 2.3 and Theorem 2.4, respectively.

2.2. Proof of Theorem 2.1

a) $\Rightarrow$ b) is trivial. b) $\Rightarrow$ c) is due to Cheeger and Gromoll [6]. The implication b) $\Rightarrow$ d) is also due to them, but we do not need it in our proof.

c) $\Rightarrow$ d) By the first Bieberbach theorem the crystallographic group Γ contains a finitely generated, free abelian normal subgroup $\Gamma^* \subset \Gamma$ of finite index (the subgroup of translations). The preimage $p^{-1}(\Gamma^*)$ is a normal subgroup of finite index in Π.

The centralizer Z of $i(\mathsf{E})$ is the kernel of the homomorphism $\Pi \to \mathrm{Aut}(i(\mathsf{E}))$ given by conjugation. Hence Π/Z is isomorphic to a subgroup of $\mathrm{Aut}(i(\mathsf{E}))$ which is finite because E is finite.

Therefore Z is a normal subgroup of finite index, too. The same is true for $\mathsf{H} := \mathsf{Z} \cap p^{-1}(\Gamma^*)$. Clearly, the finite group $\mathsf{D} := i(\mathsf{E}) \cap \mathsf{H}$ is contained in the center of H. The quotient H/D is isomorphic to the free abelian group $p(\mathsf{H}) \subset \Gamma^*$. Combining these facts we deduce that for a, $b \in \mathsf{H}$ the following relations hold:

$$
\begin{aligned}
&ba = abz && \text{for some } z \in \mathsf{D}\,,\\
&(ab)^n = a^n b^n z^{\frac{1}{2}n(n-1)} && \text{for all } n \in \mathbb{N}\,.
\end{aligned}
$$

Using the last equation for $n = 2 \cdot \mathrm{ord}(\mathsf{D})$ we find that the map $\vartheta\colon \mathsf{H} \to \mathsf{H}$, $a \mapsto a^n$ is a homomorphism. Consequently, $\mathsf{A} := \vartheta(\mathsf{H})$ is a normal subgroup of Π, and since the kernel of ϑ equals D, we conclude that $\mathsf{A} \cong \mathsf{H}/\mathsf{D} \cong p(\mathsf{H}) \cong \Gamma^*$. Thus $p(\mathsf{A})$ is of finite index in Γ. Since the kernel of p is finite, it follows that A is a subgroup of finite index in Π.

d) $\Rightarrow$ e) Let $\mathsf{A} := j(\mathbb{Z}^d)$. Via the natural inclusion $\mathbb{Z}^d \subset \mathbb{R}^d$ we have a homomorphism $\mathrm{Aut}(\mathsf{A}) \cong \mathrm{GL}(\mathbb{Z}^d) \hookrightarrow \mathrm{GL}(\mathbb{R}^d)$. So the operation of Π on A given by conjugation induces a homomorphism

$$
\alpha\colon \Pi \to \mathrm{GL}(\mathbb{Z}^d) \subset \mathrm{GL}(\mathbb{R}^d)\,.
$$

We consider Π with the discrete topology and $\mathbb{R}^d$ with the standard topology. Then the semidirect product

$$
\mathbb{R}^d \rtimes_\alpha \Pi\,, \qquad (v, g)\cdot(w, h) := \big(\alpha(g)(w) + v,\ gh\big)
$$

with its product topology becomes a Lie group, and

$$\mathsf{N} := \left\{(-v,\ j(v)) \,\middle|\, v \in \mathbb{Z}^d\right\}$$

is a discrete normal subgroup. The product $(\{0\} \times \Pi) \cdot \mathsf{N} = \mathbb{Z}^d \rtimes_\alpha \Pi$ is a discrete, cocompact subgroup of $\mathbb{R}^d \rtimes_\alpha \Pi$. The projection

$$\pi\colon \mathbb{R}^d \rtimes_\alpha \Pi \to \mathsf{G} := (\mathbb{R}^d \rtimes_\alpha \Pi)/\mathsf{N}$$

is a covering map, and the restrictions $\pi|_{\{0\}\times\Pi}$ and $\pi|_{\mathbb{R}^d\times\{1\}}$ are injective. Therefore π maps $\Pi \cong (\{0\} \times \Pi)$ isomorphically onto a discrete, cocompact subgroup of G. Moreover,

$$\mathsf{G}/\pi\big(\mathbb{R}^d \times \{1\}\big) \cong \Pi/\mathsf{A} \cong \mathsf{F}\,,$$

and hence we get an exact sequence

$$\{0\} \to \mathbb{R}^d \xrightarrow{\bar{j}} \mathsf{G} \xrightarrow{\bar{q}} \mathsf{F} \to \{1\}\,,$$

where $\bar{j}$ and $\bar{q}$ are characterized by $\bar{j}(v) = \pi(v, 1)$ and $\bar{q} \circ \pi(0, g) = q(g)$.

By the classical theory of factor systems or by cohomology theory of finite groups such a sequence splits, i.e., there is a section $h\colon \mathsf{F} \to \mathsf{G}$ that is a homomorphism, see for example [2, Chap. I.5]. Such a homomorphism can be constructed by using a set-theoretical section $s\colon \mathsf{F} \to \mathsf{G}$, i.e., a map satisfying $\bar{q} \circ s = \mathrm{id}$, as follows:

$$h(a) \;:=\; s(a) \cdot \bar{j}\left(\frac{1}{\mathrm{ord}(\mathsf{F})} \sum_{f\in\mathsf{F}} \bar{j}^{\,-1}\big(s(a)^{-1} s(f)^{-1} s(fa)\big)\right).$$

The map h can be viewed as the barycenter of the sections s_f, $f \in \mathsf{F}$ given by

$$s_f(a) = s(f)^{-1} s(fa)\,.$$

It is elementary to show that h is a homomorphism. As a consequence G is isomorphic to the semidirect product $\mathbb{R}^d \rtimes_\beta \mathsf{F}$, where $\beta\colon \mathsf{F} \to \mathrm{GL}(\mathbb{Z}^d) \subset \mathrm{GL}(\mathbb{R}^d)$ is the unique homomorphism satisfying $\alpha = \beta \circ q$. Explicitly the isomorphism is given by

$$\mathbb{R}^d \rtimes_\beta \mathsf{F} \to \mathsf{G}\,, \qquad (v,\ f) \mapsto \bar{j}\,(v) \cdot h(f)\,.$$

Since this is an isomorphism between Lie groups, we have realized Π as a discrete, cocompact subgroup of $\mathbb{R}^d \rtimes_\beta \mathsf{F}$.

e) $\Rightarrow$ a) Let Π be a discrete, cocompact subgroup of a semidirect product $\mathbb{R}^d \rtimes_\beta \mathsf{F}$, where F is finite and $\beta\colon \mathsf{F} \to \mathrm{GL}(d, \mathbb{R})$ is a real representation of F. After changing the scalar product of $\mathbb{R}^d$, we may assume that β is an orthogonal representation. We define a discontinuous, isometric action of $\Pi \subset \mathbb{R}^d \rtimes_\beta \mathsf{F}$ on $\mathbb{R}^d$

$$\Pi \times \mathbb{R}^d \to \mathbb{R}^d\,, \qquad ((v,\ f),\ w) \mapsto \beta(f)(w) + v.$$

For sufficiently large l there is an injective homomorphism $h\colon \mathsf{F} \to \mathrm{SU}(l)$. For example, one can choose $l = \mathrm{ord}(\mathsf{F})$, since F can be viewed as a subgroup of the symmetric group S_l which embeds into $\mathrm{O}(l-1) \subset \mathrm{SU}(l)$.

Consider $\mathrm{SU}(l)$ with a biinvariant metric and define an isometric action of Π on the Riemannian product $\mathrm{SU}(l) \times \mathbb{R}^d$ by

$$(v,\ f)(T,\ w) := \big(h(f)T,\ \beta(f)(w) + v\big)$$

for all $(v, f) \in \Pi$ and $(T, w) \in \mathrm{SU}(l) \times \mathbb{R}^d$. This operation is discontinuous, because the operation on the second factor is discontinuous. For $g \in \Pi \setminus \{e\}$ at least one of the corresponding isometries on the two factors is given by a nontrivial left translation, and accordingly the action is free as well. Taking into account that $\mathrm{SU}(l) \times \mathbb{R}^d$ is a connected, simply connected manifold, we see that Π is isomorphic to the fundamental group of $(\mathrm{SU}(l) \times \mathbb{R}^d)/\Pi$.

2.3. Proof of Theorem 2.2

The implication a) $\Rightarrow$ b) is trivial.

a) $\Rightarrow$ c) Let M be a compact homogeneous space with $\Pi \cong \pi_1(M)$. Clearly, we can assume that M is connected. Then the identity component $\mathrm{Iso}_0(M)$ of the isometry group of M acts transitively on M. According to Myers and Steenrod [18], the group $\mathrm{Iso}_0(M)$ is a compact Lie group. Therefore the universal covering group G of $\mathrm{Iso}_0(M)$ is a Lie group that admits a biinvariant metric, and because of that G is isomorphic to a direct product $\mathbb{R}^k \times \mathsf{K}$, where K is a compact semisimple Lie group, see [4, Prop. 3.34].

Consequently, $\mathbb{R}^k \times \mathsf{K}$ acts transitively on M. Let $\mathsf{H} \subset \mathbb{R}^k \times \mathsf{K}$ be the isotropy group of some point in M. The manifold M is diffeomorphic to $(\mathbb{R}^k \times \mathsf{K})/\mathsf{H}$ and thus $\pi_1(M) \cong \mathsf{H}/\mathsf{H}_0 =: \pi_0(\mathsf{H})$. Consider the projection $\mathrm{pr}\colon \mathsf{H} \to \mathbb{R}^k$ onto the first component and the homomorphism $\pi_0(\mathrm{pr})\colon \pi_0(\mathsf{H}) \to \pi_0(\mathrm{pr}(\mathsf{H}))$ induced by pr. Since the kernel of pr is compact, the kernel of $\pi_0(\mathrm{pr})$ is finite. The image of pr, a closed cocompact subgroup of $\mathbb{R}^k$, is isomorphic to $\mathbb{R}^l \times \mathbb{Z}^{k-l}$ for some $l \in \{0, \ldots, k\}$. For that reason the image of $\pi_0(\mathrm{pr})$ is isomorphic to $\mathbb{Z}^{k-l}$.

In summary, we can say that $\pi_1(M) \cong \pi_0(\mathsf{H})$ contains a finite normal subgroup E with a factor group $\pi_1(M)/\mathsf{E}$ isomorphic to $\mathbb{Z}^{k-l}$, as claimed.

c) $\Rightarrow$ d) From the implication c) $\Rightarrow$ d) of Theorem 2.1 we deduce that there is a normal subgroup $\mathsf{A} \cong \mathbb{Z}^d$ of finite index in Π. Since the image of p is abelian, we have the equation $p(a) = p(gag^{-1})$ for $g \in \Pi$, $a \in \mathsf{A}$. The restriction $p|_{\mathsf{A}}$ is injective, and hence $a = gag^{-1}$ for $g \in \Pi$, $a \in \mathsf{A}$. In other words, A is contained in the center of Π.

d) $\Rightarrow$ e) By the implication d) $\Rightarrow$ e) of Theorem 2.1, the group Π is isomorphic to a discrete, cocompact subgroup of a semidirect product $\mathbb{R}^d \rtimes_\beta \mathsf{F}$. The proof of Theorem 2.1 shows that in the present situation β is the trivial homomorphism; in fact, the homomorphism α occurring in the proof is trivial. It follows that $\mathbb{R}^d \rtimes_\beta \mathsf{F}$ is a direct product.

e) $\Rightarrow$ a) For sufficiently large l the group $\mathbb{R}^d \times \mathsf{F}$ can be viewed as a subgroup of $\mathbb{R}^d \times \mathrm{SU}(l)$. Consequently, Π becomes a discrete, cocompact subgroup of $\mathbb{R}^d \times \mathrm{SU}(l)$. Thus $\Pi \cong \pi_1(\mathbb{R}^d \times \mathrm{SU}(l)/\Pi)$.

b) $\Rightarrow$ a) A connected homogeneous space in the Riemannian sense is diffeomorphic to a quotient G/K, where G is a connected Lie group and K is a compact subgroup. Choose a maximal compact subgroup L of G with $\mathsf{K} \subset \mathsf{L}$. Using that G/K fibers over the contractible space G/L, we see that G/K is homotopically equivalent to the fiber L/K. In particular, $\pi_1(\mathsf{G}/\mathsf{K})$ is isomorphic to the fundamental group of the compact homogeneous space L/K.

2.4. Some group theory

Recall that a subgroup $\mathsf{G} \subset \Pi$ is called characteristic if it is invariant under all automorphisms of Π.

Lemma 2.6. *Let Π be a finitely generated group.*

a) *Define for a given positive integer n the group G as the intersection of all subgroups of Π of index at most n. Then G is a characteristic subgroup of finite index in Π.*

b) *If $\mathsf{G} \subset \Pi$ is a subgroup of finite index, then G is finitely generated, too.*

c) *Let $\mathsf{L} \subset \Pi$ be a finitely generated normal subgroup and $\mathsf{H} \subset \mathsf{L}$ a subgroup of finite index in L. Then there is a subgroup $\mathsf{H}' \subset \mathsf{H}$ of finite index that is normal in Π.*

Proof. a) Let H be a subgroup of index $k \leqslant n$. Consider the natural action of Π on the left cosets Π/H. This action induces a homomorphism $\Pi \to \mathsf{S}_k \subset \mathsf{S}_n$, where S_k and S_n denote the symmetric groups of degrees k and n. The kernel of this homomorphism is contained in H. Consequently, any subgroup of index $k \leqslant n$ contains a subgroup which is the kernel of some homomorphism $\varphi \in \mathrm{Hom}(\Pi, \mathsf{S}_n)$. Since Π is finitely generated and S_n is finite, it follows that $\mathrm{Hom}(\Pi, \mathsf{S}_n)$ is finite. Thus the group

$$\mathsf{G} \supset \bigcap_{\varphi \in \mathrm{Hom}(\Pi, \mathsf{S}_n)} \mathrm{Ker}(\varphi)$$

is a subgroup of finite index in Π. Evidently, G is a characteristic subgroup of Π as well.

b) Let $M \subset \Pi$ be a finite set that generates Π. By enlarging M if necessary, we may assume that M contains an element of each left coset of G in Π and that M is invariant under inversion. We claim that then the set

$$N := \{abc \in \mathsf{G} \mid a, b, c \in M\}$$

generates G. In fact, for $g \in \mathsf{G}$ there are elements $a_1, \ldots, a_n \in M$ such that $g = a_1 \cdots a_n$. In order to prove $g \in \langle N \rangle$ we argue by induction on n. If $n = 1$, then g itself is contained in N. If $n \geqslant 2$, we choose an element $b \in M$ such that $b \cdot a_{n-1} \cdot a_n \in \mathsf{G}$. Therefore $b \cdot a_{n-1} \cdot a_n \in N$ and $a_1 \cdots a_{n-2} \cdot b^{-1} \in \mathsf{G}$. By the induction hypothesis we can express the element $a_1 \cdots a_{n-2} \cdot b^{-1}$ as a product of elements in N. Of course, the same is valid for $g = (a_1 \cdots a_{n-2} \cdot b^{-1}) \cdot b \cdot a_{n-1} \cdot a_n$.

c) Define H' as the intersection of all subgroups of L of index at most $\mathrm{ord}(\mathsf{L}/\mathsf{H})$. According to a), the group H' is a characteristic subgroup of finite index in L. Since L is a normal subgroup of Π, it follows that H' is normal in Π, too.

Lemma 2.7. *Let Π be a group and Λ a polycyclic subgroup of finite index. Then*

a) *Any subgroup of Π is finitely generated.*

b) *There is a strongly polycyclic normal subgroup of finite index in Π.*

Proof. Part a) follows immediately from the fact that any subgroup of Λ is polycyclic. Moreover, by [19, Lemma 4.6] there is a strongly polycyclic subgroup Λ' which has finite index in Λ. Evidently, Λ' contains a subgroup of finite index which is normal in Π, and hence the statement b) is true as well.

Lemma 2.8. *Let* G, H *and* Π *be groups. Suppose that* G *and* H *are polycyclic up to finite index and assume that there is an exact sequence*

$$\{1\} \to \mathsf{H} \xrightarrow{j} \Pi \xrightarrow{q} \mathsf{G} \to \{1\}.$$

Then Π *is polycyclic up to finite index, too.*

Proof. Let $\Lambda' \subset \mathsf{H}$ be a polycyclic subgroup of finite index m. Let $\Lambda \subset \Lambda'$ be the intersection of all index m subgroups of H. By Lemma 2.6 Λ is a characteristic subgroup of finite index in H. Thus we get an exact sequence

$$\{1\} \to \mathsf{H}/\Lambda \xrightarrow{\bar{j}} \Pi/j(\Lambda) \xrightarrow{\bar{q}} \mathsf{G} \to \{1\}.$$

Using that Λ is polycyclic, we see that it is sufficient to verify that $\Pi/j(\Lambda)$ is polycyclic up to finite index. In other words, without loss of generality H is finite.

Choose a strongly polycyclic subgroup $\mathsf{G}' \subset \mathsf{G}$ of finite index. Clearly, we only have to check that $q^{-1}(\mathsf{G}')$ is polycyclic up to finite index. So we may assume that G itself is strongly polycyclic. There is nothing to prove if $\mathrm{rank}(\mathsf{G}) = 0$. In the case $\mathrm{rank}(\mathsf{G}) = 1$ the group Π clearly contains a cyclic subgroup of finite index. Suppose that $\mathrm{rank}(\mathsf{G}) \geqslant 2$. Choose a normal subgroup $\hat{\mathsf{G}} \subset \mathsf{G}$ with $\mathsf{G}/\hat{\mathsf{G}} \cong \mathbb{Z}$. By induction on $\mathrm{rank}(\mathsf{G})$ we may assume that $q^{-1}(\hat{\mathsf{G}})$ is polycyclic up to finite index. Choose a polycyclic characteristic subgroup $\mathsf{N} \subset q^{-1}(\hat{\mathsf{G}})$ of finite index in $\hat{\mathsf{G}}$. Then N is normal in Π, and the factor group Π/N fits in an exact sequence

$$\{1\} \to \mathsf{H} \to \Pi/\mathsf{N} \to \mathsf{G}/q(\mathsf{N}) \to \{1\}.$$

Using $\mathsf{G}/q(\mathsf{N})$ contains a cyclic subgroup of finite index and that H is finite, we see that Π/N is cyclic up to finite index.

Lemma 2.9. *Let* Π *be a finitely generated group, and let* L *be a nilpotent normal subgroup of finite index.*

a) *Then* L *is polycyclic.*

b) *The elements of finite order in* L *form a finite normal subgroup of* Π.

Proof. a) The group L is finitely generated because it is of finite index in Π. Taking into account that by [19, Theorem 2.7] any subgroup of the nilpotent group L is finitely generated, we see that L is polycyclic.

b) Clearly, the torsion elements of L are invariant under conjugation in Π, and thus we just have to check that they form a finite group. Let C be the center of L. By induction on the length of the central series we may assume that the elements of finite order in L/C form a finite group F. Evidently, the torsion elements of L are contained in $\mathsf{G} := \pi^{-1}(\mathsf{F})$, where $\pi\colon \mathsf{L} \to \mathsf{L}/\mathsf{C}$ is the projection. Therefore it is sufficient to prove the statement for G.

The finitely generated abelian group C contains a subgroup $\mathsf{A} \cong \mathbb{Z}^d$ of finite index. Notice that A is a central subgroup of finite index in G. We employ Theorem 2.2 to find a finite normal subgroup $\mathsf{E} \subset \mathsf{G}$ such that G/E is isomorphic to $\mathbb{Z}^d$. This completes the proof because E consists precisely of the torsion elements of L.

2.5. Proof of Theorem 2.3

Trivially a) implies b). The implications f) $\Rightarrow$ d) and g) $\Rightarrow$ d) are due to Fukaya and Yamaguchi [8] and Cheeger and Colding [3], respectively.

b) $\Rightarrow$ d) is due to Gromov [9] and Milnor [14]. More precisely, according to Milnor [14], a finitely generated fundamental group of a complete manifold of nonnegative Ricci curvature has polynomial growth, and a theorem of Gromov [9] states that such a group is nilpotent up to finite index. Finally, we can apply Lemma 2.9 and Lemma 2.7 in order to show that there is torsion free, nilpotent normal subgroup L of finite index.

d) $\Rightarrow$ c) was proved by Lee [13], but actually c) is also an immediate consequence of e) (see below), and therefore we do not need this implication.

d) $\Rightarrow$ a) We view L as a subgroup of Π and F as the quotient Π/L. There is a unique connected, simply connected, nilpotent Lie group N, called the Malcev completion of L, such that L is isomorphic to a lattice (a discrete, cocompact subgroup) in N, see [19, Theorem 2.18]. We identify L with a lattice in N. We plan to extend the natural action of L on N in some sense to an action of Π on the k-fold product N^k, where k is the index of L in Π.

Let $b_1, \ldots, b_k \in \Pi$ be representatives of Π/L. Since L is a normal subgroup of Π, we can find for any $g \in \Pi$ and $i \in \{1, \ldots, k\}$ a unique $\sigma_g(i) \in \{1, \ldots, k\}$ for which $b_i g b_{\sigma_g(i)}^{-1} \in \mathsf{L}$. In fact, $g \mapsto \sigma_g$ defines an anti-homomorphism from Π to the symmetric group of degree k. Notice that Π acts on the k-fold product N^k by

$$g \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix} := \begin{pmatrix} b_1 g b_{\sigma_g(1)}^{-1} x_{\sigma_g(1)} \\ \vdots \\ b_k g b_{\sigma_g(k)}^{-1} x_{\sigma_g(k)} \end{pmatrix} \tag{1}$$

for all $g \in \Pi$ and $x_1, \ldots, x_k \in \mathsf{N}$.

Now we can make use of a construction of Wei [21]. For a sufficiently large number $p \in \mathbb{N}$ she introduced on $M := \mathbb{R}^p \times \mathsf{N}$ a complete metric of positive Ricci curvature for which N still operates by isometries via left-translations on the second factor. Consider the k-fold Riemannian product M^k. We can define a discontinuous isometric action of Π on M^k by using equation (1) for all $g \in \Pi$ and $x_1, \ldots, x_k \in M$.

For sufficiently large l there is a homomorphism $h \colon \Pi \to \mathrm{SU}(l)$ with kernel L. The group Π acts on $\mathrm{SU}(l)$ by left-translation via h. Thus we get an isometric, discontinuous, free action of Π on $\mathrm{SU}(l) \times M^k$. Since $\mathrm{SU}(l) \times M^k$ is a connected, simply connected manifold, we have realized Π as the fundamental group of the orbit space $(\mathrm{SU}(l) \times M^k)/\Pi$.

d) $\Rightarrow$ e) Consider again the action of Π on N^k, which is given by (1). This operation is obviously effective. So we may identify Π with the image of the induced homomorphism

$$\Pi \to \mathsf{N}^k \rtimes \mathsf{S}_k$$

where $(\sigma, (x_1, \ldots, x_k)) \in \mathsf{N}^k \rtimes \mathsf{S}_k$ is the affine diffeomorphism

$$\mathsf{N}^k \to \mathsf{N}^k\,, \qquad (y_1, \ldots, y_k) \mapsto (x_1 \cdot y_{\sigma^{-1}(1)}, \ldots, x_k \cdot y_{\sigma^{-1}(k)})\,.$$

Let $c_i \colon \mathsf{L} \to \mathsf{L}$, $g \mapsto b_i g b_i^{-1}$, $i = 1, \ldots, k$. The automorphism c_i can be extended uniquely to a continuous automorphism of N, see [19, Theorem 2.11]. We call this map again c_i and define a subgroup $\tilde{\mathsf{N}}$ of $\mathsf{N}^k \subset \mathsf{N}^k \rtimes \mathsf{S}_k$ consisting of the elements

$$\begin{pmatrix} c_1(x) \\ \vdots \\ c_k(x) \end{pmatrix}, \qquad x \in \mathsf{N}.$$

Clearly, $\tilde{\mathsf{N}}$ is isomorphic to N, and $\Pi \subset \mathsf{N}^k \rtimes \mathsf{S}_k\mathsf{S}_k$ normalizes $\tilde{\mathsf{N}}$. Hence $\tilde{\mathsf{N}}$ is a normal subgroup of $\mathsf{G} = \tilde{\mathsf{N}} \cdot \Pi$ and $\mathsf{G}/\tilde{\mathsf{N}} \cong \Pi/\mathsf{L} = \mathsf{F}$. Consequently, we obtain an exact sequence of Lie group homomorphisms

$$\{1\} \to \mathsf{N} \overset{j}{\to} \mathsf{G} \overset{q}{\to} \mathsf{F} \to \{1\}.$$

It is sufficient to prove that this sequence splits. Let $\mathsf{C} := [\mathsf{N}, \mathsf{N}]$ be the commutator group of N. Since C is a characteristic subgroup, $j(\mathsf{C})$ is a normal subgroup of G. Thus there is an exact sequence

$$\{1\} \to \mathsf{N}/\mathsf{C} \overset{\bar{j}}{\to} \mathsf{G}/j(\mathsf{C}) \overset{\bar{q}}{\to} \mathsf{F} \to \{1\}.$$

Since N/C is vector group, this sequence splits as we have seen in the proof of Theorem 2.1. So let $\tilde{\mathsf{F}} \subset \mathsf{G}/j(\mathsf{C})$ be a subgroup that is via $\bar{q}$ isomorphic to F. For the preimage H of $\tilde{\mathsf{F}}$ under the projection $\mathsf{G} \to \mathsf{G}/j(\mathsf{C})$ we get an exact sequence

$$\{0\} \to \mathsf{C} \to \mathsf{H} \overset{q}{\to} \mathsf{F} \to \{1\},$$

and by induction we can assume that this sequence splits.

e) $\Rightarrow$ c) We can define an action of $\Pi \subset \mathsf{N} \rtimes_\beta \mathsf{F}$ on N consisting of affine diffeomorphisms by

$$(x,\ f) \cdot y = x \cdot \beta(f)(y).$$

The kernel $\mathsf{E} \subset \Pi$ of this action is clearly finite, and the quotient Π/E is an almost crystallographic group.

c) $\Rightarrow$ d) According to Auslander [19, Corollary 8.28], an almost crystallographic group Γ contains a torsion free, nilpotent normal subgroup of finite index (the subgroup of left translations). By passing from Π to a subgroup of finite index, we can assume that Γ itself is torsion free and nilpotent.

From Lemma 2.8 we infer that Π is polycyclic up to finite index, and now by Lemma 2.7 there is a torsion free normal subgroup L of finite index in Π. Taking into account that $p|_{\mathsf{L}}$ is injective we see that L is nilpotent.

e) $\Rightarrow$ f) $\wedge$ g) We identify Π with a discrete, cocompact subgroup of $\mathsf{N} \rtimes_\beta \mathsf{F}$ and define a discontinuous, cocompact action of Π on N

$$\Pi \times \mathsf{N} \to \mathsf{N}, \quad ((g,\ f),\ h) \mapsto g \cdot \beta(f)(h). \tag{2}$$

Lemma 2.10 below ensures the existence of a sequence g_μ of left invariant metrics on N for which

a) The action of Π on N, defined in (2), is isometric with respect to the metric g_μ, and the orbit space $(L, g_\mu)/\Pi$ has diameter 1.

b) The sectional curvature of (N, g_μ) is bounded below by $-1/\mu$.

For sufficiently large l there is a monomorphism $h\colon \mathsf{F} \hookrightarrow \mathrm{SU}(l)$. Let g denote the biinvariant metric on $\mathrm{SU}(l)$ normalized by $\mathrm{diam}(\mathrm{SU}(l), g) = 1$. We define an isometric action of Π on $(\mathsf{N}, g_\mu) \times (\mathrm{SU}(l), g)$ by using on the first factor the action defined in (2) and on the second factor the action given by $((v, f), A) \mapsto h(f) \cdot A$ for $A \in \mathrm{SU}(l)$, $(v, f) \in \Pi \subset \mathsf{N} \rtimes_\beta \mathsf{F}$. Clearly, this action is free and discontinuous, and thus Π is isomorphic to the fundamental group of the quotient $(M, \bar{g}_\mu) := ((\mathsf{N}, g_\mu) \times (\mathrm{SU}(l), g))/\Pi$. Finally, the diameter of this quotient has the upper bound $\sqrt{2}$, and its sectional curvature is bounded below by $-1/\mu$. Consequently, $(M, \bar{g}_\mu)$ is a compact manifold with almost nonnegative sectional (Ricci) curvature for sufficiently large μ.

Lemma 2.10. *Let F be a finite group, N a connected, simply connected, nilpotent Lie group, $\beta\colon \mathsf{F} \to \mathrm{Aut}(\mathsf{N})$ a homomorphism, and let $\Pi \subset \mathsf{N} \rtimes_\beta \mathsf{F}$ be a lattice. Then there is a sequence of left invariant metrics $(g_\mu)_{\mu\in\mathbb{N}}$ on N satisfying the following three conditions.*
(1) *The action of $\mathsf{N} \rtimes_\beta \mathsf{F}$ on (N, g_μ) given by $(g, f) \star h := g \cdot \beta(f)(h)$ is isometric.*
(2) *The diameter of the quotient $(\mathsf{N}, g_\mu)/\Pi$ is 1.*
(3) *The absolute value of the sectional curvature of (N, g_μ) is bounded by $1/\mu$.*

Actually the statement of the lemma is known for a torsion free group Γ, and the proof of this special case carries over to the present situation. However, to avoid mysteries we have included a proof.

Proof. Consider the representation of the finite group F in the Lie algebra $\mathfrak{n}$ given by $f \mapsto \beta(f)_{*e}$. For a suitable scalar product $\langle\cdot\,,\cdot\rangle$ on $\mathfrak{n}$ this representation becomes orthogonal. We identify $\langle\cdot\,,\cdot\rangle$ with the left invariant extension on N and observe that the action of $\mathsf{N} \rtimes_\beta \mathsf{F}$ on $(\mathsf{N}, \langle\cdot\,,\cdot\rangle)$ is isometric.

Let $\{0\} = \mathfrak{g}_0 \subset \cdots \subset \mathfrak{g}_k = \mathfrak{n}$ be the central series of $\mathfrak{n}$, i.e., $\mathfrak{g}_i$ is inductively defined by the property: the Lie algebra $\mathfrak{g}_{i+1}/\mathfrak{g}_i$ is the center of $\mathfrak{n}/\mathfrak{g}_i$. Define pairwise orthogonal (with respect to $\langle\cdot\,,\cdot\rangle$) vector subspaces $V_1, \ldots, V_k \subset \mathfrak{n}$ by means of $V_1 \oplus \cdots \oplus V_i = \mathfrak{g}_i$, $i = 1, \ldots, k$. Evidently, each subspace V_j is invariant under the action of F. Moreover, $[V_i, V_j] \subset \mathfrak{g}_{i-1}$ for all i, j. Set

$$g_\lambda\Big(\sum_{i=1}^k v_i, \sum_{i=1}^k w_i\Big) := \sum_{i=1}^k \lambda^{2^{k-i+1}} \langle v_i\,,\ w_i\rangle$$

for $v_i, w_i \in V_i, \lambda \in (0, 1]$. Clearly, $\mathsf{N} \rtimes_\beta \mathsf{F}$ acts isometrically on (N, g_λ). Since $g_\lambda(v, v) \leqslant \langle v, v\rangle$, it follows that

$$\mathrm{diam}((\mathsf{N},\ g_\lambda)/\Pi) \leqslant \mathrm{diam}((\mathsf{N},\ \langle\cdot\,,\ \cdot\rangle)/\Pi)\,.$$

Let $v_{i1}, \ldots, v_{ij_i}$ be an orthonormal basis of V_i with respect to $\langle\cdot\,,\cdot\rangle$. Then $v_{ij}/\lambda^{2^{k-i}}$, $i = 1, \ldots, k$, $j = 1, \ldots, j_i$ is an orthonormal basis of $\mathfrak{n}$ with respect to g_λ. Furthermore, for $i \leqslant l$

$$\begin{aligned}\left\|\left[\frac{v_{ij}}{\lambda^{2^{k-i}}}\,,\ \frac{v_{lm}}{\lambda^{2^{k-l}}}\right]\right\|^2_{g_\lambda} &\leqslant \frac{1}{\lambda^{2^{(k-i)+1}}}\cdot \|[v_{ij},\ v_{lm}]\|^2_{g_\lambda} \leqslant \lambda^{2^{k-i+2}-2^{(k-i)+1}} \cdot \|[v_{ij},\ v_{lm}]\|^2_{\langle\cdot\,,\cdot\rangle}\\ &\leqslant \lambda^2\cdot \|[v_{ij},\ v_{lm}]\|^2_{\langle\cdot\,,\cdot\rangle}.\end{aligned}$$

Now we infer from the curvature formula for left-invariant metrics on Lie groups [4, Proposition 3.18] that the sectional curvature of (N, g_λ) tends uniformly to zero provided that λ tends to 0. This completes the proof, since simultaneously the diameter of $(\mathsf{N}, g_\lambda)/\Pi$ is bounded from above.

2.6. Proof of Theorem 2.4

The implication a) $\Rightarrow$ b) is trivial.

b) $\Rightarrow$ d) This is nearly a direct consequence of the results of Fukaya and Yamaguchi [8]: Condition (iii) implies that (M_k, g_k^i) has almost nonnegative sectional curvature for sufficiently large i. Hence the fundamental group of M_k is by Fukaya and Yamaguchi [8] nilpotent up to finite index.

By induction on k we can assume that the fundamental group of the Riemannian manifold $(B, g) := (M_1, g_1^0)$ is polycyclic up to finite index. Because of the fiber bundle theorem of Fukaya and Yamaguchi [8] the manifold M fibers over B, and the fibration can be realized by a Hausdorff approximation $f_i\colon (M, g_0^i) \to (B, g)$. Thus the fibers of f_i become arbitrarily small in (M, g_0^i). By the generalized Margulis Lemma [8] there is a constant $\varepsilon > 0$, which only depends on the dimension such that for any ε-ball $B_\varepsilon(p) \subset (M, g_0^i)$ the image of the natural homomorphism $[\pi_1(B_\varepsilon(p)) \to \pi_1(M)]$ is nilpotent up to finite index.

Since for a sufficiently large i a fiber F_i of f_i is contained in some ball $B_\varepsilon(p) \subset (M, g_0^i)$, it follows that the image H of the natural homomorphism $[\pi_1(F_i) \to \pi_1(M)]$ is nilpotent up to finite index. We have an exact sequence

$$\{1\} \to \mathsf{H} \to \pi_1(M) \to \pi_1(B) \to \{1\},$$

and hence Lemma 2.8 applies.

d) $\Rightarrow$ e) By Lemma 2.7 we can find a strongly polycyclic group $\Lambda \subset \Pi$ of finite index. Clearly, Λ is a polycrystallographic group.

Theorem 5 in [23] tells us that after we have replaced Λ by a subgroup of finite index, we can view Λ as a discrete, cocompact subgroup of a connected, simply connected solvable Lie group S such that for any subgroup Λ' of finite index in Λ any automorphism of Λ' extends uniquely to an automorphism of S. Finally, we can by Lemma 2.6 assume that Λ is a characteristic subgroup of Π.

In particular, Λ is a characteristic subgroup of finite index in Π, and any automorphism of $\Lambda \subset \mathsf{S}$ extends uniquely to an automorphism of S. But these are precisely the conditions that we have used for the pair $\mathsf{L} \subset \mathsf{N}$ in the proof of the implication d) $\Rightarrow$ e) of Theorem 2.3. Hence the proof there carries over to the present situation. Thus there is a homomorphism $\beta\colon \mathsf{F} := \Pi/\Lambda \to \mathrm{Aut}(\mathsf{S})$ and an embedding $\Pi \hookrightarrow \mathsf{S} \rtimes_\beta \mathsf{F}$ that intersects each connected component of $\mathsf{S} \rtimes_\beta \mathsf{F}$ and that extends the inclusion $\Lambda \subset \mathsf{S} \subset \mathsf{S} \rtimes_\beta \mathsf{F}$. Since Λ is a characteristic subgroup, each automorphism of Π restricts to an automorphism of Λ. Consequently, we can extend each automorphism of Π uniquely to an automorphism of $\Pi \cdot \mathsf{S} = \mathsf{S} \rtimes_\beta \mathsf{F}$, so we also have proved Remark 2.5.4.

e) $\Rightarrow$ c) Let Π be a discrete, cocompact subgroup of $\mathsf{S} \rtimes_\beta \mathsf{F}$ and

$$\mathrm{pr}\colon \mathsf{S} \rtimes_\beta \mathsf{F} \longrightarrow \mathsf{S} \rtimes \beta(\mathsf{F}) \subset \mathsf{S} \rtimes \mathrm{Aut}(\mathsf{S})$$

the projection. Clearly, the image $\Gamma := \mathrm{pr}(\Pi)$ is a discrete cocompact subgroup of $\mathsf{S} \rtimes \beta(\mathsf{F})$

and therefore a polycrystallographic group. Because of the finiteness of Ker(pr) the assertion follows.

c) $\Rightarrow$ d) It is known that a polycrystallographic group is polycyclic up to finite index, see [23]. So we can infer from Lemma 2.8 that Π is polycyclic up to finite index, too.

e) $\Rightarrow$ a) Let $\mathsf{N} \subset \mathsf{S}$ be the maximal connected, nilpotent normal subgroup of S, $\mathfrak{n} \subset \mathfrak{s}$ the corresponding Lie algebras, and let $\langle \cdot , \cdot \rangle$ be a scalar product on $\mathfrak{s}$ that is invariant under the natural representation of F in $\mathfrak{s}$.

We have seen in the proof of Lemma 2.10 that there is an orthogonal decomposition $\mathfrak{n} = V_1 \oplus \cdots \oplus V_k$ satisfying

(i) $V_1 \oplus \cdots \oplus V_i$ is a characteristic Lie subalgebra of $\mathfrak{n}$, $i = 1, \ldots, k$,

(ii) $[V_i, V_j] \subset V_1 \oplus \cdots \oplus V_{i-1}$ for all i, j and

(iii) each V_i is an invariant under the natural representation of F in $\mathfrak{n}$. Let V_{k+1} be the orthogonal complement of $\mathfrak{n}$ in $\mathfrak{s}$ and define a new metric by means of

$$g_\lambda\Big(\sum_{i=1}^{k+1} v_i, \sum_{i=1}^{k+1} w_i\Big) = \sum_{i=1}^{k+1} \lambda^{2^{k+1-i}} \langle v_i\, ,\ w_i \rangle$$

for $v_i, w_i \in V_i$, $\lambda \in (0, 1)$. We know that

$$\left\| \left[\frac{v_i}{\lambda^{2^{k-i}}}\, ,\ \frac{w_j}{\lambda^{2^{k-j}}} \right] \right\|^2_{g_\lambda} \leqslant \lambda^2 \cdot \|[v_i,\ w_j]\|^2_{\langle \cdot\, ,\cdot \rangle}$$

for $v_i \in V_i$, $w_j \in V_j$ and i, $j \leqslant k$, see proof of Lemma 2.10. Moreover, $[v, w] \in \mathfrak{n}$ for v, $w \in V_{k+1}$, and accordingly $\|[v, w]\|^2_{g_\lambda} \leqslant \lambda^2 \|[v, w]\|^2_{\langle \cdot\, ,\cdot \rangle}$.

Since $V_1 \oplus \cdots \oplus V_i$ is a characteristic Lie subalgebra of $\mathfrak{n}$, it is an ideal in $\mathfrak{s}$. Thus for $v \in V_{k+1}$ and $w_i \in V_i$ ($i \leqslant k$) the Lie bracket $[v, w_i]$ is contained in $V_1 \oplus \cdots \oplus V_i$, and hence we obtain the inequality

$$\left\| \left[v,\ \frac{w_i}{\lambda^{2^{k-i}}} \right] \right\|_{g_\lambda} \leqslant \|[v,\ w_i]\|_{\langle \cdot\, ,\cdot \rangle}\, .$$

Using a curvature formula for left-invariant metrics on Lie groups [4, Prop. 3.18], it is easy to see that the sectional curvature of $(\mathsf{S}, g_\lambda)_{\lambda \in (0,1)}$ is uniformly bounded.

For each λ we have an isometric action of $\Pi \subset \mathsf{S} \rtimes_\beta \mathsf{F}$ on (S, g_λ) given by

$$(\tau,\ \tau) \star h := \tau \cdot \beta(f)(h)\, .$$

Choose an injective homomorphism $\varphi\colon \mathsf{F} \hookrightarrow \mathrm{SU}(l)$ and define an action of Π on $\mathrm{SU}(l)$ by $(\tau,\ f)A := \varphi(f)A$. Now we get a free, discontinuous, cocompact action of Π on the Riemannian product $(\mathsf{S}, g_\lambda) \times \mathrm{SU}(l)$.

Consider the Riemannian fibration

$$\mathsf{N} \longrightarrow (\mathsf{S},\ g_\lambda) \times \mathrm{SU}(l) \longrightarrow (\mathsf{S},\ g_\lambda)/\mathsf{N} \times \mathrm{SU}(l) =: \tilde{B}\, .$$

From $[\mathfrak{s}, \mathfrak{s}] \subset \mathfrak{n}$ we infer that the quotient $\mathsf{N} \setminus (\mathsf{S}, g_\lambda)$ is isometric to an Euclidean space, and the induced metric on $\tilde{B}$ does not depend on λ. Furthermore, $\tilde{B}$ has nonnegative sectional curvature.

Next we observe that the action of Π respects the fibration, and hence we get an isometric action of Π/L on $\tilde{B}$ where $\mathsf{L} := \Pi \cap \mathsf{N} \times \{1\}$. Since N/L is compact, this action is discontinuous.

Moreover, it is free, and thus we get a fibration

$$\mathsf{N}/\mathsf{L} \to \big((\mathsf{S},\, g_\lambda) \times \mathrm{SU}(l)\big)\big/\Pi \to \tilde{B}\big/(\Pi/\mathsf{L}) \;=:\; B\,.$$

Notice that diameter of the fibers in $(M, g_\lambda) := ((\mathsf{S}, g_\lambda) \times \mathrm{SU}(l))\big/\Pi$ tends uniformly to 0. Consequently, (M, g_λ) converges in the Gromov–Hausdorff sense to B for $\lambda \to 0$.

3. A generalization of Gromov's polynomial growth theorem

For an abstract group Π with a given finite generator system the growth function $g(h)$ is defined as the number of words in Π of length at most h. A finitely generated group is said to have polynomial growth of order $\leqslant n$ if and only if there is a constant C satisfying $g(h) \leqslant C \cdot h^n$ for all positive integers h. This condition is easily seen to be independent of the generator system. Gromov [9] has shown that a finitely generated group has polynomial growth, if and only if it is nilpotent up to finite index. Our aim is to prove the following generalization of Gromov's theorem.

Theorem 3.1. *For a group Π the following statements are equivalent.*

a) *There is an integer n such that any finitely generated subgroup of Π has polynomial growth of order $\leqslant n$.*

b) *There is a normal subgroup $\mathsf{T} \subset \Pi$ satisfying*

(i) *For any finitely generated subgroup Π' of Π the group $\Pi' \cap \mathsf{T}$ is finite.*

(ii) *The factor group Π/T contains a torsion free, nilpotent subgroup of finite index, which is an inductive limit of finitely generated, nilpotent groups of a fixed rank.*

Condition (ii) in the above theorem implies in particular that the factor group Π/T is countable. Moreover, the group T is by condition (i) locally finite, i.e., any finitely generated subgroup of T is finite. In fact, it is clear from the proof of Theorem 3.1 that T can be chosen as the maximal locally finite normal subgroup of Π. A deep theorem proved independently by Kargapolov and Hall and Kulatilaka states that an infinite locally finite group contains an infinite abelian subgroup, see Kegel and Wehrfritz [12]. Combining this result with Theorem 3.1 we show

Corollary 3.2. *Let Π be a group for which any finitely generated subgroup has polynomial growth of order $\leqslant n = n(\Pi)$. Then Π is finitely generated if and only if any abelian subgroup is finitely generated.*

The proof of the above quoted theorem on locally finite groups involves the celebrated Feit–Thompson Theorem, which states that any finite group of odd order is solvable. Under the stronger hypothesis that Π is the fundamental group of a complete manifold of nonnegative Ricci curvature, one can give an elementary proof for Corollary 3.2 which uses the fact that in this case any finitely generated subgroup of Π satisfies in addition the conclusion of Theorem 4.1.

The rest of this section is organized as follows: The proof of the theorem and the corollary will be given in the first two subsections. In Subsection 3.3 we draw a consequence for the fundamental groups of manifolds of nonnegative Ricci curvature. The last subsection is concerned with an example.

3.1. On the proof of the theorem

For the proof of Theorem 3.1 we need

Lemma 3.3. *Let Π be a group that has a torsion free normal subgroup L of finite index. Then the torsion elements in the centralizer of L, i.e., the elements in*

$$\mathsf{E} := \left\{ g \in \Pi \;\middle|\; gh = hg \;\; \forall h \in \mathsf{L},\; g^l = e \;\; \textit{for some } l \in \mathbb{N} \right\},$$

form the maximal finite normal subgroup of Π.

Proof. Let $C(\mathsf{L})$ denote the centralizer of L. The group $\mathsf{Z} = \mathsf{L} \cap C(\mathsf{L})$ is a torsion free central subgroup of finite index in $C(\mathsf{L})$. In order to prove that E is a group, we have to show that for $g, h \in \mathsf{E}$ the element gh^{-1} is contained in E, too. For that we consider the group Π' generated by g and h. The subgroup $\mathsf{A} := \Pi' \cap \mathsf{Z}$ has finite index in Π', and by Lemma 2.6 A is finitely generated. Moreover, A is torsion free, and hence it is a free abelian central subgroup of finite index in Π'. According to Theorem 2.2 there is a finite group E' such that Π'/E' is free abelian. Clearly, $g,\ h,\ gh^{-1} \in \mathsf{E}' \subset \mathsf{E}$.

Therefore E is a group, and since E has trivial intersection with L, it is finite as well. By its very definition E is a characteristic subgroup of $C(\mathsf{L})$, and because of that it is a finite normal subgroup of Π.

Let $\tilde{\mathsf{E}}$ be a finite normal subgroup of Π. Using that $\tilde{\mathsf{E}}$ and L are normal subgroups of Π we find that $aba^{-1}b^{-1} \in \mathsf{L} \cap \tilde{\mathsf{E}} = \{e\}$ for $a \in \tilde{\mathsf{E}}$ and $b \in \mathsf{L}$. Consequently, $\tilde{\mathsf{E}} \subset C(\mathsf{L})$ is contained in E.

Proof of Theorem 3.1. b) $\Rightarrow$ a) Let Π' be a finitely generated subgroup of Π. Then $\mathsf{E}' := \Pi' \cap \mathsf{T}$ is a finite normal subgroup of Π', and the factor group Π'/E' contains a torsion free, nilpotent subgroup of finite index. From the implication c) $\Rightarrow$ d) of Theorem 2.3 we infer that Π' contains a torsion free, nilpotent subgroup L of finite index. Moreover, the rank of L is bounded by a constant that only depends on Π. It is well-known that L has polynomial growth and that the degree of this growth is bounded by a constant only depending on $\mathrm{rank}(\mathsf{L})$. Taking into account that L is of finite index in Π', we see that Π' has polynomial growth of controlled degree, too.

a) $\Rightarrow$ b) By Gromov's theorem any finitely generated subgroup of Π contains a nilpotent subgroup of finite index. As we have seen in Section 2 this implies that any finitely generated subgroup contains a torsion free, nilpotent normal subgroup of finite index. The rank of this nilpotent subgroup is at most n. Thus we can choose a finitely generated, torsion free, nilpotent subgroup $\mathsf{L} \subset \Pi$ of maximal rank.

Set $\mathsf{L}_k := \langle \{ g^{k!} \mid g \in \mathsf{L} \} \rangle$ for $k \in \mathbb{N}$. Note that L_k is a normal subgroup of L. The factor group L/L_k is a nilpotent group which is generated by finitely many elements of finite order, and because of Lemma 2.9 it is finite.

Let $C(\mathsf{L}_k)$ denote the centralizer of L_k in Π, and let $\mathrm{Tor}(C(\mathsf{L}_k))$ be the elements of finite order in $C(\mathsf{L}_k)$.

$$\mathsf{T} := \bigcup_{k \in \mathbb{N}} \mathrm{Tor}\big(C(\mathsf{L}_k)\big).$$

We claim that for any finitely generated group $\Pi' \supset \mathsf{L}$ the set $\mathsf{T} \cap \Pi'$ is the maximal finite normal subgroup of Π'.

In order to prove this, we choose a torsion free, nilpotent normal subgroup $\mathsf{L}' \subset \Pi'$ of finite index in Π'. Then the group $\mathsf{L} \cap \mathsf{L}'$ is of finite index in L, and accordingly

$$\operatorname{rank}(\mathsf{L}') \geqslant \operatorname{rank}(\mathsf{L} \cap \mathsf{L}') = \operatorname{rank}(\mathsf{L}) .$$

Taking into account that $\operatorname{rank}(\mathsf{L})$ is maximal, we see that $\operatorname{rank}(\mathsf{L}') = \operatorname{rank}(\mathsf{L} \cap \mathsf{L}')$, and hence $\mathsf{L} \cap \mathsf{L}'$ is of finite index in L'. Put $\mathsf{L}'_l := \langle \{ g^{l!} \mid g \in \mathsf{L}' \} \rangle$, let $\mathrm{C}(\mathsf{L}'_l)$ be the centralizer of L'_l, and let $\operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l))$ denote the elements of finite order in $\mathrm{C}(\mathsf{L}'_l)$. The group L'_l is a torsion free normal subgroup of finite index in Π' for all $l > 0$. By Lemma 3.3 the set $\operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l)) \cap \Pi'$ is the maximal finite normal subgroup of Π'. In particular, $\operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l)) \cap \Pi'$ does not depend on l. For any l there is an integer k such that

$$\mathsf{L}_k \subset \mathsf{L}'_l \;\Rightarrow\; \operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l)) \subset \operatorname{Tor}(\mathrm{C}(\mathsf{L}_k)) .$$

Furthermore, for any k there is an l such that

$$\mathsf{L}'_l \subset \mathsf{L}_k \;\Rightarrow\; \operatorname{Tor}(\mathrm{C}(\mathsf{L}_k)) \subset \operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l)) .$$

Consequently,

$$\mathsf{T} \cap \Pi' \;=\; \operatorname{Tor}(\mathrm{C}(\mathsf{L}'_l)) \cap \Pi' \;=\; \operatorname{Tor}(\mathrm{C}(\mathsf{L}_k)) \cap \Pi'$$

for all l and for almost every k. Thus $\mathsf{T} \cap \Pi'$ is the maximal finite normal subgroup of Π', as claimed.

In particular, T is a normal subgroup of Π. Let $\mathrm{pr}\colon \Pi \to \Pi/\mathsf{T}$ be the projection. For any finitely generated group $\Pi' \supset \mathsf{L}$ the kernel of the restriction $\mathrm{pr}\,|_{\Pi'}$ is the maximal finite normal subgroup of Π'. According to Remark 2.5, the image of Π' is then isomorphic to an almost crystallographic group Γ' with $\operatorname{rank}(\Gamma') = \operatorname{rank}(\mathsf{L})$.

Recall that the nilradical $\operatorname{nil}(\Gamma')$ of an almost crystallographic group Γ' is the maximal nilpotent normal subgroup. It is known and will follow from Theorem 5.1 below that $\operatorname{nil}(\Gamma')$ is a torsion free subgroup of finite index in Γ' and that the index is bounded by a constant only depending on $\operatorname{rank}(\Gamma')$.

Therefore we can find a finitely generated subgroup Γ_0 of Π/T that is isomorphic to an almost crystallographic group of rank equal to $\operatorname{rank}(\mathsf{L})$ that maximizes the quantity $\operatorname{ord}(\Gamma_0/\operatorname{nil}(\Gamma_0))$. Choose a realization $\iota\colon \Gamma_0 \to \mathsf{F} \ltimes \mathsf{N}$ of Γ_0 as an almost crystallographic group, i.e., N is a connected, simply connected, nilpotent Lie group, F is a compact subgroup of $\operatorname{Aut}(\mathsf{N})$ and the homomorphism ι maps Γ_0 isomorphically onto a discrete, cocompact subgroup of $\mathsf{F} \ltimes \mathsf{N}$. Thanks to Auslander's characterization of almost crystallographic groups $\operatorname{nil}(\Gamma_0)$ is mapped onto a lattice in $\mathsf{N} \times \{1\} = \mathsf{N}$, if one is not familiar with Auslander's theorem one can employ [23, Prop. 5.1] instead. Without loss of generality F is isomorphic to the finite factor group $\Gamma_0/\operatorname{nil}(\Gamma_0)$, because we can replace F by a suitable subgroup.

Lemma 3.4. *There is a unique extension of ι to a homomorphism $h\colon \Pi/\mathsf{T} \to \mathsf{F} \ltimes \mathsf{N}$ and h is injective.*

Before verifying Lemma 3.4, we use it to complete the proof of Theorem 3.1. Since h is injective, we can and we will identify Π/T with its image. Evidently, the group $\mathsf{G} := (\Pi/\mathsf{T}) \cap \mathsf{N}$ is a torsion free, nilpotent subgroup of index $\operatorname{ord}(\mathsf{F})$ in Π/T. Furthermore, for any finitely generated subgroup $\Gamma' \subset \mathsf{G}$ the group $\Gamma' \cap \Gamma_0$ is of finite index in Γ'. Consequently, G is contained in the countable group

$$\exp(\mathrm{span}_{\mathbb{Q}}(\exp^{-1}(\mathrm{nil}(\Gamma_0)))) .$$

In summary, we can say that G is a torsion free, countable, nilpotent group, and any finitely generated subgroup has rank at most $\mathrm{rank}(\mathrm{nil}(\Gamma_0))$. But this implies that G is the inductive limit of finitely generated, nilpotent groups of a fixed rank.

Proof of Lemma 3.4. Clearly, it suffices to show: For any finitely generated subgroup $\Gamma_1 \subset \Pi/\mathsf{T}$ satisfying $\Gamma_0 \subset \Gamma_1$ there is a unique homomorphism $\varphi\colon \Gamma_1 \to \mathsf{F} \rtimes \mathsf{N}$ that extends ι, and this homomorphism is injective.

Notice that Γ_1 is isomorphic to an almost crystallographic group. Thus there is a connected, simply connected, nilpotent Lie group N_1 a compact subgroup $\mathsf{F}_1 \subset \mathrm{Aut}(\mathsf{N}_1)$ and a homomorphism $\iota_1\colon \Gamma_1 \to \mathsf{F}_1 \rtimes \mathsf{N}_1$ mapping Γ_1 isomorphically onto a discrete, cocompact subgroup of $\mathsf{F}_1 \rtimes \mathsf{N}_1$.

The almost crystallographic groups Γ_0 and Γ_1 have the same rank, and hence Γ_0 is a subgroup of finite index in Γ_1. Therefore $\iota_1(\Gamma_0)$ is a discrete, cocompact subgroup of $\mathsf{F}_1 \rtimes \mathsf{N}_1$, too. As above we deduce that $\iota_1(\Gamma_i) \cap \mathsf{N}_1 \times \{1\}$ coincides with the nilradical of Γ_i, $i = 1, 2$. In particular, $\mathrm{nil}(\Gamma_0) \subset \mathrm{nil}(\Gamma_1)$. Moreover, we can assume that F_1 is isomorphic to the finite factor group $\Gamma_1/\mathrm{nil}(\Gamma_1)$. Since the order of $\Gamma_0/\mathrm{nil}(\Gamma_0)$ is maximal,

$$\iota_1(\Gamma_0) \cdot (\mathsf{N}_1 \times \{1\}) = \mathsf{F}_1 \rtimes \mathsf{N}_1 . \tag{1}$$

The group $\iota_1(\mathrm{nil}(\Gamma_0))$ is a lattice in $\mathsf{N}_1 = \mathsf{N}_1 \times \{1\}$, and $\iota(\mathrm{nil}(\Gamma_0))$ is a lattice in N. Using that ι_1 and ι are injective, we conclude that there is a unique isomorphism $\psi\colon \mathsf{N}_1 \to \mathsf{N}$ satisfying $\psi \circ \iota_1|_{\Gamma_0} = \iota$, see [19, Theorem 2.11]. Because of the uniqueness of ψ we have

$$\psi\big(\iota_1(g) \cdot v \cdot \iota_1(g^{-1})\big) = \iota(g) \cdot \psi(v) \cdot \iota(g^{-1}) \qquad \text{for } g \in \Gamma_0 \text{ and } v \in \mathsf{N}_1 .$$

With this in mind it is straightforward to check that we can extend ψ to an isomorphism $\bar{\psi}\colon \mathsf{F}_1 \rtimes \mathsf{N}_1 \to \mathsf{F} \rtimes \mathsf{N}$ by defining $\bar{\psi}(\iota_1(g) \cdot v) := \iota(g) \cdot \psi(v)$ for $g \in \Gamma_0$, $v \in \mathsf{N}_1 = \mathsf{N}_1 \times \{1\}$. Now $\varphi := \bar{\psi} \circ \iota_1$ is a homomorphism that extends ι.

It remains to verify that φ is unique. Equation (3.1) yields $\Gamma_1 = \Gamma_0 \cdot \mathrm{nil}(\Gamma_1)$, and therefore it is sufficient to show that $\varphi|_{\mathrm{nil}(\Gamma_1)}\colon \mathrm{nil}(\Gamma_1) \to \mathsf{N}$ is unique. For $g \in \mathrm{nil}(\Gamma_1)$ there is a positive integer l such that $g^l \in \mathrm{nil}(\Gamma_0)$. There is precisely one element v in the simply connected nilpotent group N with $v^l = \iota(g^l)$, and clearly we have necessarily $\varphi(g) = v$.

3.2. Proof of the corollary

If Π is finitely generated, then we can apply Gromov's theorem to see that Π is nilpotent up to finite index, and then by Lemma 2.7 any subgroup of Π is finitely generated, too.

Assume conversely that Π is not finitely generated. Choose a normal subgroup T as stated in Theorem 3.1. If the locally finite group T is infinite, then it contains an infinite abelian subgroup, see [12]. Thus it suffices to consider the case of a finite group T. In particular, the quotient Π/T is not finitely generated. The group Π/T contains a torsion free, nilpotent subgroup of finite index, and evidently we can assume that Π/T itself is torsion free and nilpotent. The proof of Theorem 3.1 shows that then the group Π/T can be viewed as a cocompact subgroup of a simply connected, nilpotent Lie group N in such a way that any finitely generated subgroup of Π/T is discrete in N.

Suppose for a moment that any abelian subgroup of Π/T lies discrete in N. Since Π/T is not discrete, we could find two sequences $g_k, h_k \in \Pi/\mathsf{T} \subset \mathsf{N}$ converging to e such that g_k does not commute with h_k. Then the commutator sequence $[g_k, h_k] := g_k h_k g_k^{-1} h_k^{-1} \neq e$ also converges to e, and it follows that $[\mathsf{N}, \mathsf{N}] \cap \Pi/\mathsf{T}$ is not discrete in $[\mathsf{N}, \mathsf{N}]$. Via a simple induction argument this yields a contradiction.

Hence Π/T contains an abelian subgroup A which is not discrete and accordingly not finitely generated. Let Π' be the preimage of A under the projection $\Pi \to \Pi/\mathsf{T}$. For $h \in \Pi'$ the inner automorphism $\mathrm{Int}(h)\colon \Pi' \to \Pi'$, $g \mapsto hgh^{-1}$ leaves the left cosets of the finite group T invariant. So $\mathrm{Int}(h^{k!}) = \mathrm{id}$ for $k = \mathrm{ord}(\mathsf{T})$, and $h^{k!}$ is contained in the center of Π'. Taking into account that Π'/T is torsion free, we see that the center of Π' is not finitely generated.

3.3. A remark on fundamental groups

Remark 3.5. a) *Let Π be a discrete subgroup of a connected Lie group G and $n \in \mathbb{N}$. If any finitely generated subgroup of Π has polynomial growth of order $\leqslant n$, then Π is finitely generated.*

b) *Let M be a complete manifold of nonnegative Ricci curvature, $\tilde{M}$ the universal covering space of M and $\mathrm{Iso}(\tilde{M})$ its isometry group. If $\pi_0(\mathrm{Iso}(\tilde{M})) := \mathrm{Iso}(\tilde{M})/\mathrm{Iso}_0(\tilde{M})$ is finitely generated, then $\pi_1(M)$ is finitely generated, too.*

Proof. a) By Corollary 3.2 it is sufficient to prove that any abelian subgroup of Π is finitely generated. But according to Mostow [17], a discrete solvable subgroup of a connected Lie group is finitely generated.

b) Recall first that $\mathrm{Iso}(\tilde{M})$ is a Lie group, see [18].

A slight modification of the argument in [14] shows that any finitely generated subgroup of $\pi_0(\mathrm{Iso}(\tilde{M}))$ has polynomial growth. Thus if $\pi_0(\mathrm{Iso}(\tilde{M}))$ is finitely generated, then it is nilpotent up to finite index, and any subgroup of $\pi_0(\mathrm{Iso}(\tilde{M}))$ is finitely generated, too. The fundamental group $\pi_1(M)$ can be viewed as a discrete subgroup of $\mathrm{Iso}(\tilde{M})$. By a) the intersection N of $\pi_1(M)$ with the identity component $\mathrm{Iso}_0(\tilde{M})$ of $\mathrm{Iso}(\tilde{M})$ is finitely generated. The quotient $\pi_1(M)/\mathsf{N}$ is isomorphic to a subgroup of $\pi_0(\mathrm{Iso}(\tilde{M}))$ and consequently finitely generated. But then $\pi_1(M)$ is finitely generated, too.

3.4. An example

Example 3.6. *There is a complete Riemannian manifold M with fundamental group isomorphic to $\mathbb{Q}/\mathbb{Z}$ that is obtained by a surgery construction from a sequence of compact homogeneous spaces.*

Let $\mathsf{H}_n \subset \mathrm{SU}(3)$ be a cyclic subgroup of order $n!$, and let $M_n := \mathrm{SU}(3)/\mathsf{H}_n$, $n \geqslant 2$. For

$n \geqslant 2$ we choose two smooth regular loops $c_n, \gamma_n : [0,1] \to M_n$ satisfying

(i) The submanifolds $c_n([0,1])$ and $\gamma_n([0,1])$ are disjoint.

(ii) The loop c_n represents a generator of the fundamental group $\pi_1(M_n, c_n(0))$.

(iii) The element in $\pi_1(M_n, \gamma_n(0))$ represented by γ_n generates a cyclic subgroup of order $(n-1)!$.

Choose two disjoint open tubular neighborhoods U_n of $c_n([0,1])$ and V_n of $\gamma_n([0,1])$ with smooth boundaries ∂U_n and ∂V_n.

$$D := (M_2 \setminus U_2) \cup \bigcup_{i \geqslant 3} (M_i \setminus (U_i \cup V_i)) .$$

Let $f_n : \partial U_n \to \partial V_{n+1}$ be a diffeomorphism. Consider on D the equivalence relation generated by $p \sim f_n(p)$ for $p \in \partial U_n$, $n \geqslant 2$. The equivalence classes $M := D/\sim$ form a connected smooth manifold, and M admits a complete metric. By making iterated use of van Kampen's theorem we see that the fundamental group of M is isomorphic to $\mathbb{Q}/\mathbb{Z}$, the inductive limit of $(\mathbb{Z}/n!\,\mathbb{Z})_{n \in \mathbb{N}}$.

Furthermore, one can show that the universal covering space of M has only one end. Thus none of the presently known obstructions can be used to prove that M admits no complete metric of nonnegative Ricci curvature. However, if one uses a slightly different surgery construction, the resulting manifold has a fundamental group which contains a finitely generated subgroup of exponential growth. Clearly, this manifold does not admit a complete metric of nonnegative Ricci curvature, and from this point of view it would be rather surprising if M admits such a metric.

4. A result on fundamental groups in a fixed dimension

The goal of this section is to prove the following theorem.

Theorem 4.1. *In each dimension n there are finite simple groups* $\mathsf{F}_1, \ldots, \mathsf{F}_k$ *for which the following holds: Any finitely generated fundamental group* Π *of a complete, n-dimensional manifold M of nonnegative Ricci curvature contains subgroups*

$$\{e\} = \mathsf{N}_0 \subset \mathsf{N}_1 \subset \cdots \subset \mathsf{N}_l = \Pi$$

such that N_i *is normal in* N_{i+1} *and* $\mathsf{N}_{i+1}/\mathsf{N}_i$ *is either cyclic or isomorphic to* F_{j_i} *for some* $j_i \in \{1, \ldots, k\}$.

We will see that it is easy to reduce the statement of the theorem to the following

Proposition 4.2. *In each dimension n there are finite, simple groups* $\mathsf{F}_1, \ldots, \mathsf{F}_k$ *for which the following holds: Any non-cyclic, finite, simple group acting effectively and isometrically on some connected, n-dimensional, complete Riemannian manifold of nonnegative Ricci curvature is isomorphic to* F_j *for some* $j \in \{1, \ldots, k\}$.

In contrast to the statement of the proposition we remark that any finite or countable group F can act freely and discontinuously on a complete 2-manifold with constant curvature -1:

Therefore recall that the free group G in countable many generators acts freely and discontinuously on the hyperbolic plane $\mathbb{H}^2$. Choose an epimorphism $\varphi\colon \mathsf{G} \to \mathsf{F}$. Then the group $\mathsf{F} \cong \mathsf{G}/\mathrm{Ker}(\varphi)$ acts freely and discontinuously on the quotient $\mathbb{H}^2/\mathrm{Ker}(\varphi)$.

Proof of Theorem 4.1. Let M be a complete, n-dimensional Riemannian manifold of nonnegative Ricci curvature with a finitely generated fundamental group $\pi_1(M)$. Since $\Pi := \pi_1(M)$ is nilpotent up to finite index, there are subgroups

$$\{e\} = \mathsf{N}_0 \subset \cdots \subset \mathsf{N}_j = \Pi$$

such that N_i is a normal subgroup of N_{i+1} and the factor group $\mathsf{N}_{i+1}/\mathsf{N}_i$ is either cyclic or isomorphic to a finite, simple group. So we only have to check that there are in a given dimension only finitely many non-cyclic, finite, simple groups which can be realized in this manner. Recall that the group Π acts freely and discontinuously on the universal covering space $\tilde{M}$ of M. Moreover, the factor group $\mathsf{N}_{i+1}/\mathsf{N}_i$ acts freely and discontinuously on the orbit space $\tilde{M}/\mathsf{N}_i$. But now the assertion follows from the proposition.

Proof of Proposition 3.2. Suppose that the proposition is wrong in some dimension n. Then we could find a sequence $(\mathsf{F}_i)_{i\in\mathbb{N}}$ of non-cyclic, finite, simple groups such that

a) $\mathrm{ord}(\mathsf{F}_i) > i$,

b) F_i acts effectively and isometrically on a connected, n-dimensional, complete Riemannian manifold M_i of nonnegative Ricci curvature.

Choose a point $p_i \in M_i$ for which the coset $X_i := \mathsf{F}_i \cdot p_i$ is nontrivial. We shall think of X_i as equipped with the metric that is induced by the Riemannian distance function of M_i. After scaling the metric on M_i we have $\mathrm{diam}(X_i) = 1$.

From the Bishop–Gromov inequality we infer that X_i contains an ε-net consisting of at most $(4/\varepsilon)^n$ elements for all $\varepsilon > 0$. Thus by [9] the sequence $(X_i)_{i\in\mathbb{N}}$ has a convergent subsequence. Without loss of generality X_i itself converges to a compact metric space X_∞ with respect to the Gromov–Hausdorff distance. Furthermore, we can assume that the action of F_i on X_i also converges to an isometric action of some closed subgroup of the isometry group $\mathrm{Iso}(X_\infty)$, see [8] for the concept of equivariant Hausdorff convergence.

We define a biinvariant metric on $\mathrm{Iso}(X_\infty)$ by

$$d_\infty(\iota, \sigma) := \max\Big\{ d_\infty(\iota(p), \sigma(p)) \;\Big|\; p \in X_\infty \Big\}$$

for $\iota, \sigma \in \mathrm{Iso}(X_\infty)$. After passing once more to a subsequence if necessary, we can find an $1/i$ -almost homomorphism $\varphi_i\colon \mathsf{F}_i \to \mathrm{Iso}(X_\infty)$ for all $i \in \mathbb{N}$, i.e., a map satisfying

$$d_\infty\big(\varphi_i(ab),\, \varphi_i(a)\circ\varphi_i(b)\big) < \frac{1}{i} \qquad \text{for all } a, b \in \mathsf{F}_i\,.$$

Moreover, we can assume that the image of φ_i is $1/i$ -almost transitive, i.e., for any $x, y \in X_\infty$ there is an element $a \in \mathsf{F}_i$ with $d_\infty(\varphi(a)(x), y) < 1/i$.

The group $\mathrm{Iso}(X_\infty)$ is a compact topological group. According to [16, Theorem 2.20, p. 99], any neighborhood U of the neutral element e contains a compact normal subgroup N such that $\mathsf{G} = \mathrm{Iso}(X_\infty)/\mathsf{N}$ is a compact Lie group.

Let U be the ball of radius $\frac{1}{4}$ around e in $\mathrm{Iso}(X_\infty)$, and let $\mathsf{N} \subset U$ be as above. We consider $\mathsf{G} = \mathrm{Iso}(X_\infty)/\mathsf{N}$ with the induced metric $\bar{d}_\infty$. Clearly,

$$\psi_i := \pi \circ \varphi_i : \mathsf{F}_i \to (\mathsf{G}, \bar{d}_\infty)$$

is an $1/i$ -almost homomorphism, where $\pi : \mathrm{Iso}(X_\infty) \to \mathsf{G}$ denotes the projection.

Choose a biinvariant Riemannian metric g on G satisfying $\|[X, Y]\| \leqslant \|X\| \cdot \|Y\|$ for all X, $Y \in \mathfrak{g}$ and for which the injectivity radius of (G, g) is at least π, see [10] for existence of g. Let d denote the corresponding Riemannian distance.

Since d and $\bar{d}_\infty$ induce the same topology on G, it follows that for any $\varepsilon > 0$ there exists an integer i_0 such that ψ_i is an ε-almost homomorphism with respect to the Riemannian distance d for all $i \geqslant i_0$. By a theorem of Grove et al. [10] the almost homomorphism ψ_i then can be deformed into a homomorphism $\tilde{\psi}_i : \mathsf{F}_i \to \mathsf{G}$ with $d(\tilde{\psi}_i(a), \psi_i(a)) \leqslant \frac{3}{2}\varepsilon$ provided that $\varepsilon < \pi/6$, $i \geqslant i_0$, $a \in \mathsf{F}_i$. Using once again that d and $\bar{d}_\infty$ induce the same topology, we infer that

$$\bar{d}_\infty\big(\tilde{\psi}_i(a), \psi_i(a)\big) < \tfrac{1}{4}$$

for all $a \in \mathsf{F}_i$ and for sufficiently large i. By construction $\mathrm{diam}(X_\infty) = 1$, and taking into account that the image of φ_i is $1/i$ -almost transitive, we see that $\mathrm{diam}(\varphi_i(\mathsf{F}_i)) \geqslant \frac{3}{4}$ for $i \geqslant 4$. Moreover, the group N is contained in the ball of radius $\frac{1}{4}$ around e, and hence

$$\mathrm{diam}(\psi_i(\mathsf{F}_i), \bar{d}_\infty) \geqslant \tfrac{1}{2} \qquad \text{for } i \geqslant 4.$$

Therefore $\mathrm{diam}(\tilde{\psi}_i(\mathsf{F}_i), \bar{d}_\infty) \geqslant \frac{1}{4}$ for large i. In particular, $\tilde{\psi}_i$ is a nontrivial homomorphism for almost every i. In fact, $\tilde{\psi}_i$ is then injective because F_i is a simple group.

But a theorem of Jordan [19, Theorem 8.29] states that a finite subgroup of a compact Lie group G contains an abelian normal subgroup of index at most $m = m(\mathsf{G})$—a contradiction.

5. Estimates for groups

It is well-known that the fundamental groups of Gromov's almost flat manifolds are precisely the torsion free almost crystallographic groups. In this context Buser and Karcher [1] proved that the index of the nilradical Γ^* in an almost crystallographic group is bounded by $2 \cdot 6^{\frac{1}{2}r(r-1)}$ where $r = \mathrm{rank}(\Gamma^*)$. We will improve this estimate:

Theorem 5.1. *Let Γ be an almost crystallographic group, and let Γ^* be the nilradical of Γ. Then the factor group Γ/Γ^* is finite, and its order divides the number $(2n)!$ where $n =$* $\mathrm{rank}(\Gamma^*) - \mathrm{rank}([\Gamma^*, \Gamma^*])$.

Corollary 5.2. *Let Π be a finitely generated group that is abelian up to finite index, $d =$* $\mathrm{rank}(\Pi)$, *and let E be the maximal finite normal subgroup of Π. Then Π contains a characteristic subgroup $\mathsf{A} \cong \mathbb{Z}^d$ of index at most* $\mathrm{ord}(\mathsf{E})^{2d+1} \cdot (2d)^{2d}$.

The rest of this section is organized as follows: Subsection 5.1 is devoted to the proof of Theorem 5.1. In Subsection 5.2 we have placed a lemma stating that the order of an automorphism of a finite group F is bounded by $\mathrm{ord}(\mathsf{F})$. Finally, we prove Corollary 5.2 in Subsection 5.3.

5.1. Proof of Theorem 5.1

A theorem of Minkowski [15] says that the least common multiple of the orders of all finite subgroups of $\mathrm{GL}(n,\mathbb{Z})$ is given by

$$v_n := \prod_{p\in\mathbf{P}} p^{\sum_{k=0}^{\infty}[n/p^k(p-1)]},$$

where $\mathbf{P}$ are the prime numbers and $[n/p^k(p-1)]$ is the Gauss bracket of $n/p^k(p-1)$. The so called Minkowski bound v_n obviously divides the number $(2n)!$. Thus we only have to prove that the factor group Γ/Γ^* is isomorphic to a subgroup of $\mathrm{GL}(n,\mathbb{Z})$.

By definition the almost crystallographic group Γ is a discrete, cocompact subgroup of a semidirect product $\mathsf{F}\ltimes\mathsf{N}$ where N is a connected, simply connected, nilpotent Lie group and F is a compact subgroup of its automorphism group $\mathrm{Aut}(\mathsf{N})$. It is known that the nilradical Γ^* of Γ is then given by $\Gamma^*=\Gamma\cap\mathsf{N}\times\{1\}$, see for example [23, Proposition 5.1]. The group Γ^* is a lattice in $\mathsf{N}=\mathsf{N}\times\{1\}$ and accordingly of finite index in Γ. By replacing F by a subgroup if necessary, we may assume that F is isomorphic to the finite factor group Γ/Γ^*. Let $[\mathsf{N},\mathsf{N}]$ be the commutator group of N. The group $\Gamma\cap[\mathsf{N},\mathsf{N}]$ is discrete and cocompact in $[\mathsf{N},\mathsf{N}]$, and the projection

$$\mathrm{pr}:\mathsf{F}\ltimes\mathsf{N}\to(\mathsf{N}/[\mathsf{N},\ \mathsf{N}])\rtimes\mathsf{F}$$

maps Γ onto a discrete, cocompact subgroup. In particular, $\mathsf{D}:=\mathrm{pr}(\Gamma)\cap(\mathsf{N}/[\mathsf{N},\mathsf{N}])$ is a lattice in the vector group $\mathsf{N}/[\mathsf{N},\mathsf{N}]$. We claim that the induced action of F on $\mathsf{N}/[\mathsf{N},\mathsf{N}]$ is effective. This implies that $\mathsf{F}\cong\Gamma/\Gamma^*$ is isomorphic to a subgroup of $\mathrm{Aut}(\mathsf{D})\cong\mathrm{GL}(n,\mathbb{Z})$.

In order to show that the action of F on $\mathsf{N}/[\mathsf{N},\mathsf{N}]$ is effective, we argue by induction on the dimension. Clearly, there is nothing to prove if N is abelian. If N is not abelian, we define inductively a sequence of subgroups by $\mathsf{N}_0=\mathsf{N}$ and $\mathsf{N}_{i+1}=[\mathsf{N},\mathsf{N}_i]$. Since N is nilpotent, there is a maximal number $i_0\geqslant 1$ for which N_{i_0} is nontrivial. Evidently, N_{i_0} is invariant under the action of F, and by our induction hypothesis it is sufficient to verify that the induced action of F on $\mathsf{N}/\mathsf{N}_{i_0}$ is effective.

Suppose that an automorphism $\iota\in\mathsf{F}$ induces the identity on $\mathsf{N}/\mathsf{N}_{i_0}$. Consider the corresponding automorphism ι_* of the Lie algebra $\mathfrak{n}$ of N. The Lie subalgebra $\mathfrak{n}_{i_0}$ corresponding to N_{i_0} is an invariant subspace of ι_*. There is an ι_*-invariant vector complement $\mathfrak{p}$ of $\mathfrak{n}_{i_0}$ in $\mathfrak{n}$ because ι_* has finite order. The automorphism of $\mathsf{N}/\mathsf{N}_{i_0}$ induced by ι is the identity and hence $\iota|_{*\mathfrak{p}}=\mathrm{id}$. Using that ι_* is a Lie algebra automorphism, we find $\iota|_{*[\mathfrak{p},\mathfrak{p}]}=\mathrm{id}$. Since $\mathfrak{n}_{i_0}$ is contained in the center of $\mathfrak{n}$, it follows that $[\mathfrak{p},\mathfrak{p}]=[\mathfrak{n},\mathfrak{n}]\supset\mathfrak{n}_{i_0}$, and accordingly $\iota_*=\iota|_{*\mathfrak{p}\oplus\mathfrak{n}_{i_0}}=\mathrm{id}$. But then trivially ι itself is the identity.

5.2. On the order of an automorphism of a finite group

We recall that for two groups $\mathsf{H}\subset\mathsf{F}$ the number $(\mathsf{F}:\mathsf{H})$ is defined as the index of H in F.

Lemma 5.3. *a) Let F be a finite group, $\sigma\in\mathrm{Aut}(\mathsf{F})$ an automorphism of F, and let $\mathsf{H}=\{g\in\mathsf{F}\mid\sigma(g)=g\}$ be the fix point subgroup. Then there is some positive integer $k\leqslant(\mathsf{F}:\mathsf{H})$ such that σ^k is an inner automorphism of F.*

b) *Let F and σ be as above. Then there is a positive integer $n\leqslant\mathrm{ord}(\mathsf{F})$ such that $\sigma^n=\mathrm{id}$.*

Although the statement of the lemma is certainly known, the author was not able to find it in the literature. Thus we will prove it:

Proof of Lemma 5.3. a) At first we consider a special case: there is a subgroup $\tilde{\mathsf{H}}$ such that $\mathsf{H} \subsetneq \tilde{\mathsf{H}} \subsetneq \mathsf{F}$ and $\sigma(\tilde{\mathsf{H}}) = \tilde{\mathsf{H}}$. By induction on $(\mathsf{F} : \mathsf{H})$ we can assume that there exists a positive integer $l \leqslant (\tilde{\mathsf{H}} : \mathsf{H})$ and an element $h_0 \in \tilde{\mathsf{H}}$ satisfying $\sigma^l(h) = h_0 h h_0^{-1}$ for all $h \in \tilde{\mathsf{H}}$. The induction hypothesis applied to $\tau(g) := h_0^{-1}\sigma^l(g)h_0$ says that τ^m is an inner automorphism of F for some positive integer $m \leqslant (\mathsf{F} : \tilde{\mathsf{H}})$. Consequently, $\sigma^{l\cdot m}$ is inner, and we are done.

Notice that for any integer l the fixpoint group of σ^l is invariant under σ, and by the above consideration we can assume that this group is either H or F. Then there exists a positive integer m such that the orbit $\{\sigma^n(g) \mid n \in \mathbb{Z}\}$ has precisely m elements for all $g \in \mathsf{F} \setminus \mathsf{H}$. Without loss of generality $m > (\mathsf{F} : \mathsf{H})$ because $\sigma^m = \mathrm{id}$. For $g \in \mathsf{F}$ we can find a positive integer $l \leqslant (\mathsf{F} : \mathsf{H}) < m$ such that $\sigma^l(g)$ and g are contained in the same right coset $\mathsf{H} \cdot g$. Therefore $\sigma^l(g) = hg$ for some $h \in \mathsf{H} \setminus \{e\}$ and

$$\sigma^l(g^{-1}\sigma(g)) = (hg)^{-1}\sigma(hg) = g^{-1}\sigma(g)\,.$$

Since H is also the fix point group of σ^l, it follows that

$$f_g := g^{-1}\sigma(g) \in \mathsf{H} \qquad \text{for all} \quad g \in \mathsf{F}\,.$$

Observe that $\sigma(ab) = af_a b f_b = ab(b^{-1} f_a b) f_b$, and thus $f_{ab} = (b^{-1} f_a b) f_b$ for all $a, b \in \mathsf{F}$. Now it is easy to see that $\mathsf{A} := \langle \{ f_g \mid g \in \mathsf{F}\} \rangle \subset \mathsf{H}$ is normal in F. Clearly, $\sigma(hg) = hg f_g$, and $f_{hg} = f_g$ for $h \in \mathsf{H}$ and $g \in \mathsf{F}$. Via $\mathsf{A}g = g\mathsf{A}$ this yields the equation $f_{gh} = f_g$ for all $h \in \mathsf{A}$. Hence

$$ghf_g = \sigma(gh) = gf_g h \implies hf_g = f_g h \qquad \forall\, h \in \mathsf{A}\,.$$

Accordingly A is abelian. As already explained we can define $f_{\mathsf{H}\cdot g} := f_g$ for any right coset of H. Since A is abelian, the element

$$c := \prod_{g \in \mathsf{H}\backslash \mathsf{F}} f_g$$

is well defined, too. Set $k := (\mathsf{F} : \mathsf{H}) = \mathrm{ord}(\mathsf{H}\backslash\mathsf{F})$. Then

$$\begin{aligned} c^{-1}gc &= \Big(\prod_{a\in \mathsf{H}\backslash\mathsf{F}} f_a\Big)^{-1} \cdot g \cdot \prod_{b\in \mathsf{H}\backslash\mathsf{F}} f_b \\ &= g\cdot \Big(\prod_{a\in \mathsf{H}\backslash\mathsf{F}} (g^{-1} f_a g)\Big)^{-1} \cdot \prod_{b\in \mathsf{H}\backslash\mathsf{F}} f_b \\ &= g\cdot \Big(\prod_{a\in \mathsf{H}\backslash\mathsf{F}} (f_{ag} f_g^{-1})\Big)^{-1} \cdot \prod_{b\in \mathsf{H}\backslash\mathsf{F}} f_b \\ &= g\cdot f_g^k \cdot \Big(\prod_{a\in \mathsf{H}\backslash\mathsf{F}} f_{ag}\Big)^{-1} \cdot \prod_{b\in \mathsf{H}\backslash\mathsf{F}} f_b \;=\; g\cdot f_g^k \\ &= \sigma^k(g). \end{aligned}$$

Thus σ^k is inner.

b) Consider the natural homomorphism

$$\varphi\colon \mathsf{F} \to \operatorname{Int}(\mathsf{F}) \subset \operatorname{Aut}(\mathsf{F}), \qquad g \mapsto [h \mapsto ghg^{-1}].$$

Notice that $\varphi(\sigma(g)) = \sigma \circ \varphi(g) \circ \sigma^{-1}$ for $g \in \mathsf{F}$. Let Z denote the cyclic group generated by σ, and let ι be a generator of the cyclic group $\mathsf{Z}' := \mathsf{Z} \cap \operatorname{Int}(\mathsf{F})$.

Evidently, σ commutes with ι, so the set $\varphi^{-1}(\iota)$ is invariant under σ. Choose $g_0 \in \varphi^{-1}(\iota)$ and consider its orbit $\mathsf{Z} \star g_0 := \{\sigma^l(g_0) \mid l \in \mathbb{Z}\}$. For $k := \operatorname{ord}(\mathsf{Z} \star g_0)$ we have $\sigma^k(g) = g$ for all $g \in \mathsf{Z} \star g_0$ and as a consequence $\sigma^k(h) = h$ for all h satisfying $g_0 h \in \mathsf{Z} \star g_0$. Since $\mathsf{Z} \star g_0 \subset \varphi^{-1}(\iota)$, we have $\varphi(h) = \mathrm{id}$ provided that $g_0 h \in \mathsf{Z} \star g_0$. Therefore the kernel of φ contains a subgroup C of order at least k which is fixed by σ^k. Clearly, the group

$$\mathsf{H} = \left\{ h \in \mathsf{F} \;\middle|\; \sigma^k(h) = h \right\}$$

contains g_0, and hence $\varphi(\mathsf{H}) \supset \mathsf{Z}'$. Consequently, $\operatorname{ord}(\mathsf{H}) \geqslant k \cdot \operatorname{ord}(\mathsf{Z}')$.

By part a) there is a number $l \leqslant (\mathsf{F} : \mathsf{H})$ such that $(\sigma^k)^l$ is an inner automorphism. Thus $\sigma^{kl} \in \mathsf{Z}'$ and $(\sigma^{k\cdot l})^{\operatorname{ord}(\mathsf{Z}')} = \mathrm{id}$. This completes the proof, since $k \cdot l \cdot \operatorname{ord}(\mathsf{Z}') \leqslant \operatorname{ord}(\mathsf{F})$.

5.3. Proof of Corollary 5.2

Let $p\colon \Pi \to \Gamma := \Pi/\mathsf{E}$ denote the projection. By Remark 2.5 Γ is isomorphic to a crystallographic group of rank d, and by Theorem 5.1 the translational part Γ^* of Γ has index at most $(2d)!$. Let $b_1, \dots, b_d$ be a basis of Γ^* and $g_i \in p^{-1}(b_i)$. The order of the automorphism $\mathsf{E} \to \mathsf{E},\ f \mapsto g_i f g_i^{-1}$ is bounded by $\operatorname{ord}(\mathsf{E})$, see Lemma 5.3. In other words, for some positive integer $k_i \leqslant \operatorname{ord}(\mathsf{E})$ the element $g_i^{k_i}$ lies in the centralizer of E. The group generated by $b_1^{k_1}$, $\dots, b_d^{k_d}$ has index at most $(2d)! \cdot \operatorname{ord}(\mathsf{E})^d$ in Γ. Accordingly the index of

$$\mathsf{H} := \left\{ g \in p^{-1}(\Gamma^*) \;\middle|\; gf = fg \ \ \forall f \in \mathsf{E} \right\} \subset \Pi$$

in Γ is bounded by $1/\operatorname{ord}(\mathsf{D}) \cdot (2d)! \cdot \operatorname{ord}(\mathsf{E})^{d+1}$, where D is the center of E. Notice that H is a characteristic subgroup of Π. From the proof of Theorem 2.1 we know that the map

$$\varphi\colon \mathsf{H} \to \mathsf{H}, \qquad h \mapsto h^{2\operatorname{ord}(\mathsf{D})}$$

is a homomorphism and the image $\mathsf{A} := \varphi(\mathsf{H})$ is a subgroup which itself is isomorphic to $\mathbb{Z}^d$. Since H is a characteristic subgroup of Π, the same is valid for A. Clearly, $(\mathsf{H} : \mathsf{A}) \leqslant \operatorname{ord}(\mathsf{D}) \cdot (2\operatorname{ord}(\mathsf{D}))^d$, and thus the index of A in Π is bounded by

$$(\mathsf{H} : \mathsf{A}) \cdot (\Pi : \mathsf{H}) \;\leqslant\; 2^d \cdot (2d)! \cdot \operatorname{ord}(\mathsf{E})^{2d+1} \;\leqslant\; (2d)^{2d} \cdot \operatorname{ord}(\mathsf{E})^{2d+1}.$$

6. Deformation of coverings

Let N be a complete manifold with a compact isometry group. Consider a normal Riemannian covering $\mathrm{q}_0\colon \mathbb{R}^n \times N \to (M, g_0)$, i.e., q_0 is bundle map of a principle Π-bundle over M for some discrete group Π acting isometrically on $\mathbb{R}^n \times N$. In this section we will study continuous deformations $\mathrm{q}_\lambda\colon \mathbb{R}^n \times N \to (M, g_\lambda)$, $\lambda \in [0, 1]$, of Riemannian coverings, where the metric of $\mathbb{R}^n \times N$ is the fixed product metric.

For the motivation we recall that by Cheeger and Gromoll [6] a complete manifold of nonnegative sectional curvature is isometric a Riemannian product $\mathbb{R}^k \times N$, where N is a manifold with a compact isometry group. Moreover, Cheeger and Gromoll [5] have shown that the universal covering space of a compact manifold of nonnegative Ricci curvature splits as $\mathbb{R}^l \times N$, where N and hence $\mathrm{Iso}(N)$ is compact.

The main results will follow from the following two theorems.

Theorem 6.1. *Let* $\mathrm{q}\colon (\tilde{M}, \tilde{g}) \to (M, g_0)$ *be a normal Riemannian covering between connected, complete Riemannian manifolds,* $\Pi \subset \mathrm{Iso}(\tilde{M})$ *the deck transformation group, and let* $\eta\colon [0,1] \times \Pi \to \mathrm{Iso}(\tilde{M})$ *be a smooth proper map such that each* $\eta_\lambda = \eta(\lambda, \cdot)$ *is a homomorphism. Assume moreover that* η_0 *coincides with the natural inclusion* $\Pi \subset \mathrm{Iso}(\tilde{M})$.

Then there is smooth family of Riemannian coverings $\mathrm{q}_\lambda\colon (\tilde{M}, \tilde{g}) \to (M, g_\lambda)$ *such that* $\mathrm{q}_0 = \mathrm{q}$ *and* $\eta_\lambda(\Pi)$ *is the deck transformation group of* q_λ.

If Π is a finitely generated group containing an abelian subgroup A of finite index, then we set $\mathrm{rank}(\Pi) := \mathrm{rank}(\mathsf{A})$.

Theorem 6.2. *Let* Π *be a finitely generated group that is abelian up to finite index,* $d = \mathrm{rank}(\Pi)$, *and let* E *be the maximal finite normal subgroup of* Π. *Let* G *be a compact Lie group of rank* r, *and let* $\psi\colon \Pi \to \mathsf{G}$ *be a homomorphism. Then there is a smooth family of homomorphisms* $(\psi)_{\lambda\in[0,1]}\colon \Pi \to \mathsf{G}$ *such that* $\psi_0 = \psi$ *and the kernel of* ψ_1 *contains a free abelian normal subgroup* $\Pi' \subset \Pi$ *satisfying*

$$(\Pi : \Pi') \leqslant (2d)^{2d(r+1)} \cdot \mathrm{ord}(\mathsf{E})^{2d+1} \cdot \mathrm{ord}(\pi_0(\mathsf{G}))\,.$$

These theorems yield several finiteness results: Cheeger and Gromoll [6] proved that a compact manifold of nonnegative Ricci curvature is finitely covered by a manifold that is diffeomorphic to product of a torus and a simply connected manifold. We can sharpen this result as follows.

Corollary 6.3. *Let* (N, g) *be a complete Riemannian manifold with a compact isometry group* $\mathrm{Iso}(N)$, *and let* $\mathrm{q}_0\colon \mathbb{R}^n \times N \to (M, g_0)$ *be a normal Riemannian covering. Then there is a continuous family* $(g_\lambda)_{\lambda\in[0,1]}$ *of metrics on* M, *a continuous family of Riemannian coverings* $\mathrm{q}_\lambda\colon \mathbb{R}^n \times N \to (M, g_\lambda)$ *and a s-sheeted normal Riemannian covering* $z_1\colon T^{(d)} \times \mathbb{R}^{n-d} \times N \to (M, g_1)$, *where* $T^{(d)}$ *is a d-dimensional, flat torus and* $T^{(d)} \times \mathbb{R}^{n-d} \times N$ *carries the product metric. Moreover,*

$$s \leqslant 2 \cdot (2d)^{2d(r+1)} \cdot \mathrm{ord}(\mathsf{E})^{2d+1} \cdot \mathrm{ord}(\pi_0(\mathrm{Iso}(N)))\,,$$

where E *is the maximal finite normal subgroup in the group of deck transformations* Π, $d = \mathrm{rank}(\Pi)$ *and* $r = \mathrm{rank}(\mathrm{Iso}(N) \times \mathrm{O}(n-d))$.

If q_0 is the universal covering map of a (noncompact) complete flat manifold, one can restate the above corollary in a more intrinsic fashion. We recall that any noncompact, complete manifold of nonnegative sectional curvature has according to Cheeger and Gromoll [6] a totally convex compact submanifold S such that M is diffeomorphic to the normal bundle of S. The submanifold S is called a soul of M. Its dimension is determined by $\dim(S) = \max\{k \mid H_k(M, \mathbf{F}_2) \neq 0\}$, where $H_*(M, \mathbf{F}_2)$ denotes the singular homology with coefficients in $\mathbf{F}_2$.

Corollary 6.4. *Let (M, g_0) be a connected, noncompact, complete, flat manifold, and let S be a soul of M, $k = \dim(S)$, $l = \operatorname{codim}(S)$, $n = k+l = \dim(M) \geqslant 2$. There is a continuous family $(g_\lambda)_{\lambda\in[0,1]}$ of complete, flat metrics on M such that the holonomy group $\operatorname{Hol}(g_1)$ of (M, g_1) is finite and its order is bounded*

$$\operatorname{ord}(\operatorname{Hol}(g_1)) \leqslant 2 \cdot (2k)^{k(l+2)} \leqslant n^{n^2}.$$

It is an immediate consequence of Bieberbach's third theorem that there are up to affine diffeomorphisms only finitely many compact flat manifolds in each dimension. We can generalize this result:

Corollary 6.5. *Up to flat metric deformations there are only finitely many isometry classes of complete, flat manifolds in each dimension.*

A deep theorem of Tits [20] implies that a finitely generated subgroup of a connected Lie group is either solvable up to finite index or it contains a free subgroup of rank 2. As an application of this theorem, we can strengthen Theorem 6.2:

Corollary 6.6. *Let Π be a finitely generated group that does not contain a free subgroup of rank 2, and assume either that G is a compact Lie group or that $\mathsf{G} = \mathrm{GL}(n, \mathbf{K})$ where $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$. Consider the set $\operatorname{Hom}(\Pi, \mathsf{G})$ of homomorphisms $\psi\colon \Pi \to \mathsf{G}$ topologized by pointwise convergence. Then $\operatorname{Hom}(\Pi, \mathsf{G})$ has only finitely many arc-connected components and each of these components contains a homomorphism with a finite image.*

The proof of the corollary yields at least in principal estimates for the number of arc-connected components of $\operatorname{Hom}(\Pi, \mathsf{G})$ depending on Π and G. The fact that $\operatorname{Hom}(\Pi, \mathrm{GL}(n, \mathbf{K}))$ has only finitely components is known although it is not used; in fact $\operatorname{Hom}(\Pi, \mathrm{GL}(n, \mathbf{K}))$ has the structure of a real affine variety.

Now we prove the theorems and the corollaries in order of occurrence.

On the proof of Theorem 6.1

We need the following

Lemma 6.7. *Let G be a Lie group, F a finite group and $\eta_\lambda\colon \mathsf{F} \to \mathsf{G}$ a continuous family of homomorphisms, $\lambda \in [0, 1]$. Then for each $\lambda \in [0, 1]$ there exists an element $g_\lambda \in \mathsf{G}$ satisfying $\eta_\lambda(f) = g_\lambda \cdot \eta_0(f) \cdot g_\lambda^{-1}$ for all $f \in \mathsf{F}$.*

Actually the lemma is an immediate consequence of a rigidity theorem of A. Weil, see [19, Theorem 6.7]. However, in this special case there is an elementary proof. Following an idea in [1, appendix] we will work with barycenters:

Proof of Lemma 6.7. Let $\lambda_0 \in [0, 1]$, $\tilde{\mathsf{F}} := \eta_{\lambda_0}(\mathsf{F})$. Clearly, it is sufficient to show that there is a number $\delta > 0$ such that the homomorphism η_λ is conjugate to η_{λ_0} for $|\lambda - \lambda_0| < \delta$. Choose an $\mathrm{Ad}(\tilde{\mathsf{F}})$-invariant left-invariant metric on G, and consider the quotient $M := \mathsf{G}/\tilde{\mathsf{F}}$ with the induced metric. Set $p := \tilde{\mathsf{F}} \in \mathsf{G}/\tilde{\mathsf{F}}$. As usual G acts isometrically on M and $\tilde{\mathsf{F}}$ is the isotropy group of p with respect to this action.

The orbit $\eta_\lambda(\mathsf{F})p$ is contained in an arbitrarily small ball around p, provided that $|\lambda - \lambda_0|$ is sufficiently small. Thus the barycenter ζ_λ of the finite collection $(\eta_\lambda(f) \cdot p)_{f \in \mathsf{F}}$ of points is well-defined and $\lambda \to \zeta_\lambda$ is a continuous curve in a neighborhood of λ_0. Notice that $\eta_\lambda(\mathsf{F})$ is contained in the isotropy group of ζ_λ.

The natural projection $\mathsf{G} \to M$ is a covering map, and we can lift ζ_λ to a curve $\gamma\colon]\lambda_0 - \delta, \lambda_0 + \delta[\to \mathsf{G}$ with $\gamma(\lambda_0) = e$. By construction the group $\gamma(\lambda) \cdot \eta_\lambda(\mathsf{F}) \cdot \gamma(\lambda)^{-1}$ is then contained in $\tilde{\mathsf{F}}$, the isotropy group of p. Since $\tilde{\mathsf{F}}$ is finite, it follows that $\eta_\lambda(g) = \gamma(\lambda)^{-1} \cdot \eta_{\lambda_0}(g) \cdot \gamma(\lambda)$ for all $g \in \mathsf{F}$.

Proof of Theorem 6.1. The proper map η_λ induces a discontinuous action of Π on $\tilde{M}$. In particular, for any element $g \in \Pi$ of infinite order the isometry $\eta_\lambda(g)$ has no fix points. Let F be a finite subgroup of Π. From Lemma 6.7 we infer that for each $\lambda \in [0, 1]$ there is an element $g_\lambda \in G$ satisfying $\eta_\lambda(f) = g_\lambda \eta_0(f) g_\lambda^{-1}$ for all $f \in \mathsf{F}$. Since $\eta_0(f)$ is a fix point free isometry, the same is valid for $\eta_\lambda(f)$, $\lambda \in [0, 1]$, $f \in \mathsf{F} \setminus \{e\}$. We have proved that the action

$$\Pi \times \big([0,\ 1] \times \tilde{M}\big) \to [0,\ 1] \times \tilde{M}\,, \qquad g \star (\lambda,\ x) := \big(\lambda,\ \eta_\lambda(g)(x)\big)$$

is free. It is discontinuous as well, because η is a proper map. The quotient $N := ([0, 1] \times \tilde{M})/\Pi$ is a smooth manifold with boundary, and

$$\sigma\colon N \to [0,\ 1]\,, \qquad \Pi \star (\lambda,\ x) \mapsto \lambda$$

is a submersion. Thus N is an M-fiber bundle over $[0, 1]$ and therefore we can find a diffeomorphism $f\colon N \to [0, 1] \times M$ for which $f(\Pi \star (0, x)) = \mathrm{q}(x)$ and $\mathrm{pr}_1 \circ f = \sigma$, where $\mathrm{pr}_1 : [0, 1] \times M \to [0, 1]$ denotes the projection onto the first component. Let $\mathrm{pr}_2 : [0, 1] \times M \to M$ be the projection onto the second component, and let $\mathrm{q}_\lambda(x) := \mathrm{pr}_2 \circ f(\Pi \star (\lambda, x))$. Clearly, q_λ is a covering, and the deck transformation group corresponding to q_λ is $\eta_\lambda(\Pi) \subset \mathrm{Iso}(\tilde{M})$. Hence there is a unique metric g_λ on M with respect to which q_λ becomes a Riemannian covering.

On the proof of Theorem 6.2

Lemma 6.8. *Let G_0 be a connected, compact Lie group with a biinvariant metric. Let $v_i \in \mathfrak{g} - \{0\}$ be a vector with $\|v_i\| < c(v_i/\|v_i\|)$, where c denotes the cut locus function, $g_i := \exp(v_i)$, $(i = 1, 2)$. Then $g_1 g_2 = g_2 g_1$ if and only if $[v_1, v_2] = 0$.*

Proof. If $[v_1, v_2] = 0$, then g_1 and g_2 lie in a toral subgroup of G_0, and in particular $g_1 g_2 = g_2 g_1$. Assume conversely that the elements g_1 and g_2 commute. Notice that

$$\exp(\mathrm{Ad}_{g_2} v_1) = g_2 \exp(v_1) g_2^{-1} = \exp(v_1)\,.$$

Since the metric on G_0 is biinvariant, it follows that $\|\mathrm{Ad}_{g_2} v_1\| = \|v_1\|$, and by hypothesis $\mathrm{Ad}_{g_2} v_1 = v_1$.

Therefore g_2 commutes with $\exp(tv_1)$ for all $t \in \mathbb{R}$. As above this implies that

$$\mathrm{Ad}_{\exp(tv_1)} v_2 = v_2\,,$$

and accordingly $\exp(tv_1)$ commutes with $\exp(sv_2)$ for all $t, s \in \mathbb{R}$. But then $[v_1, v_2] = 0$.

Proof of Theorem 6.2. By Corollary 5.2 Π contains a normal subgroup $\Pi^* \cong \mathbb{Z}^d$ of finite index satisfying $(\Pi : \Pi^*) \leqslant (2d)^{2d} \operatorname{ord}(\mathsf{E})^{2d+1}$. Let $\Pi_1^* = \Pi^* \cap \psi^{-1}(\mathsf{G}_0)$, where G_0 is the identity component of G.

$$(\Pi : \Pi_1^*) \ \leqslant\ (2d)^{2d} \operatorname{ord}(\mathsf{E})^{2d+1} \cdot \operatorname{ord}(\pi_0(\mathsf{G})). \tag{1}$$

Choose a basis $b_1, \ldots, b_d$ of Π_1^* and a biinvariant metric on G. Then for some $k_i \leqslant 2^r$ the element $\psi(b_i^{k_i}) = \psi(b_i)^{k_i}$ is not contained in the cut locus of e in G. In order to prove this, we choose a (r-dimensional) maximal torus T_i containing $\psi(b_i)$. Let $\mathfrak{t}_i$ denote the Lie algebra of T_i. Since the maximal torus T_i is a convex submanifold of G, the cut locus function of T_i is just the restriction $c|_{\mathfrak{t}_i}$ of the cut locus function of G. Consequently, the Dirichlet fundamental-domain around 0 corresponding to the covering $\exp\colon \mathfrak{t}_i \to \mathsf{T}_i$ is given by

$$F \ :=\ \left\{ v \in \mathfrak{t}_i \;\middle|\; \|v\| \leqslant c\left(\frac{v}{\|v\|}\right) \right\}.$$

Let $\frac{1}{2}F = \{\frac{1}{2}v \mid v \in \overset{\circ}{F}\}$. Suppose now that the elements $\psi(b_i), \ldots, \psi(b_i)^l$ are contained in the cut locus of e in T_i, that is $\psi(b_i), \ldots, \psi(b_i)^l \in \exp(\partial F)$. The sets $\psi(b_i)^0 \cdot \exp(\frac{1}{2}F), \ldots, \psi(b_i)^l \cdot \exp(\frac{1}{2}F)$ are disjoint, and hence

$$\operatorname{vol}_r(\mathsf{T}_i) \ \geqslant\ (l+1)\cdot \operatorname{vol}_r\big(\exp(\tfrac{1}{2}F)\big) \ =\ \frac{l+1}{2^r} \operatorname{vol}_r(\mathsf{T}_i)\,.$$

Therefore $l < 2^r$, and the statement follows.

Choose $k_1, \ldots, k_d \leqslant 2^r$ as stated above. Consider the set

$$V \ :=\ \left\{ \psi(g b_i^{k_i} g^{-1}) \;\middle|\; i = 1, \ldots, d,\ g \in \Pi \right\} \ \subset\ \Pi_1^*\,.$$

For any $a \in V$ there is by construction a unique $v_a \in \mathfrak{g}$ of minimal norm with $\exp(v_a) = a$. We employ Lemma 6.8 to see that the set $\{v_a \mid a \in V\}$ generates an abelian Lie algebra $\mathfrak{t}$. Clearly, the image of ψ normalizes T. Thus $\Pi_2^* := \Pi_1^* \cap \psi^{-1}(\mathsf{T})$ is a normal subgroup of Π and

$$(\Pi_1^* : \Pi_2^*) \ \leqslant\ 2^{dr}. \tag{2}$$

Choose a homomorphism

$$f\colon \Pi_2^* \to \mathfrak{t} \quad \text{satisfying} \quad \exp \circ f = \psi|_{\Pi_2^*}\,.$$

Let C denote the centralizer of Π_2^* in Π. The factor group Π/C operates effectively on Π_2^*. So Π/C is isomorphic to a subgroup of $\mathrm{GL}(d, \mathbb{Z})$, and as explained in the proof of Theorem 5.1, a theorem of Minkowski [15] implies that $k := \operatorname{ord}(\Pi/\mathsf{C})$ divides the number $(2d)!$. Set

$$\mathsf{A} \ :=\ \{v^k \mid v \in \Pi_2^*\}\,.$$

For $c \in \mathsf{C}$ the endomorphism $\mathrm{Ad}_{\psi(c)}|_{\mathfrak{t}}\colon \mathfrak{t} \to \mathfrak{t}$ special is the identity. Consequently, $\mathrm{Ad}_{\psi(h)}|_{\mathfrak{t}}$ is well-defined for $h \in \Pi/\mathsf{C}$. Furthermore, the map $\Pi_2^* \to \mathsf{A}$, $v \mapsto v^k$ is an isomorphism, and hence we can define a map $\varphi\colon \mathsf{A} \to \mathfrak{t}$ by means of

$$\varphi(v^k) := \sum_{h\in\Pi/\mathsf{C}} \mathrm{Ad}_{\psi(h)}\big(f(h^{-1}vh)\big) \qquad \text{for } v \in \Pi_2^*.$$

Observe that

$$\exp\circ\varphi = \psi|_{\mathsf{A}} \tag{3}$$

and

$$\varphi(gvg^{-1}) = \mathrm{Ad}_g(v) \qquad \text{for all } v \in \mathsf{A},\ g \in \Pi. \tag{4}$$

Set $\mathsf{F} := \Pi/\mathsf{A}$. As we have seen in the proof of Theorem 2.1, we can identify Π with a subgroup of a semidirect product $\mathbb{R}^d \rtimes_\beta \mathsf{F}$ such that $\mathsf{A} = \Pi \cap \mathbb{R}^d \times \{1\}$ is lattice in $\mathbb{R}^d$. Any element in $g \in \mathbb{R}^d \rtimes_\beta \mathsf{F}$ can be written as $h\cdot\sum_{i=1}^d \lambda_i a_i$, where $h \in \Pi$, $a_i \in \mathsf{A}$ and $\lambda_i \in \mathbb{R}$. Using the equations (3) and (4), we see that the following map is well-defined.

$$\begin{aligned}\Psi\colon \mathbb{R}^d \rtimes_\beta \mathsf{F} &\to \mathsf{G},\\ h\cdot\sum_{i=1}^d \lambda_i a_i &\mapsto \psi(h)\cdot\exp\Big(\sum_{i=1}^d \lambda_i\varphi(a_i)\Big)\end{aligned}$$

for $h \in \Pi$, $a_i \in \mathsf{A}$ and $\lambda_i \in \mathbb{R}$. Moreover, Ψ is a homomorphism with $\Psi_{|\Pi} = \psi$. Thus we can define a smooth family of homomorphisms by setting

$$\psi_\lambda((v,\ f)) := \Psi((1-\lambda)v,\ f) \qquad \text{for } (v,\ f) \in \Pi \subset \mathbb{R}^d \rtimes_\beta \mathsf{F}.$$

Clearly, $\psi_0 = \psi$. Furthermore, the image $\mathsf{H} := \psi_1(\Pi_2^*)$ is a subgroup of T satisfying $h^k = e$ for all $h \in \mathsf{H}$. Consequently, $\mathrm{ord}(\mathsf{H}) \leqslant k^{\dim(\mathsf{T})}$, and if we set $\Pi_3^* := \Pi_2^* \cap \mathrm{Ker}(\psi_1)$, we obtain the inequality

$$(\Pi_2^* : \Pi_3^*) = \mathrm{ord}(\mathsf{H}) \leqslant ((2d)!)^r. \tag{5}$$

Combining the estimates (1), (2) and (5), we conclude that

$$\begin{aligned}(\Pi : \Pi_3^*) &= (\Pi_2^* : \Pi_3^*)\cdot(\Pi_1^* : \Pi_2^*)\cdot(\Pi : \Pi_1^*)\\ &\leqslant ((2d)!)^r\cdot 2^{dr}\cdot(2d)^{2d}\cdot\mathrm{ord}(\mathsf{E})^{2d+1}\cdot\mathrm{ord}(\pi_0(\mathsf{G}))\\ &\leqslant (2d)^{2d(r+1)}\cdot\mathrm{ord}(\mathsf{E})^{2d+1}\cdot\mathrm{ord}(\pi_0(\mathsf{G})).\end{aligned}$$

Proof of Corollary 6.3

Let Π be the group of deck transformations of the normal covering $\mathrm{q}_0\colon \mathbb{R}^n \times N \to M$. The isometry group of $\mathbb{R}^n \times N$ is a product $\mathrm{Iso}(\mathbb{R}^n \times N) = \mathrm{Iso}(\mathbb{R}^n)\times\mathrm{Iso}(N)$. Thus Π operates discontinuously and with a finite kernel E_1 on $\mathbb{R}^n$. Therefore the quotient $\Pi/\mathsf{E}_1 =: \Gamma'$ can be viewed as a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^n)$.

It is known that such a group acts with a finite kernel on an affine subspace of $\mathbb{R}^n$ as a crystallographic group, see [8].

After changing the origin and the canonical basis we can assume that Γ' acts discontinuously and cocompactly on the subspace $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$. The kernel E_2 of the action of Γ' on $\mathbb{R}^d \subset \mathbb{R}^n$ is finite, and Γ/E_2 is a crystallographic group. Let E be the preimage of E_2 under the projection

$\Pi \to \Gamma'$. Then Π/E is isomorphic to a crystallographic group of rank d. By Theorem 2.1 Π is abelian up to finite index, and by Remark 2.5 E is the maximal finite normal subgroup of Π.

Moreover, Π is a subgroup of $\mathrm{Iso}(\mathbb{R}^d) \times (\mathrm{O}(n-d) \times \mathrm{Iso}(N)) =: \mathrm{Iso}(\mathbb{R}^d) \times \mathsf{G}$. Let pr and ψ_0 be the projections of Π on $\mathrm{Iso}(\mathbb{R}^d)$ and G, respectively. We employ Theorem 6.2 to find a smooth deformation $(\psi_\lambda)_{\lambda\in[0,1]}$ of ψ_0 in $\mathrm{Hom}(\Pi, \mathsf{G})$ such that the kernel of ψ_1 contains a free abelian normal subgroup A satisfying

$$\begin{aligned}(\Pi : \mathsf{A}) \ &\leqslant\ (2d)^{2d(r+1)} \cdot \mathrm{ord}(\mathsf{E})^{2d+1} \cdot \mathrm{ord}\big(\pi_0(\mathrm{Iso}(N) \times \mathrm{O}(n-d))\big)\\ &=\ 2 \cdot (2d)^{2d(r+1)} \cdot \mathrm{ord}(\mathsf{E})^{2d+1} \cdot \mathrm{ord}\big(\pi_0(\mathrm{Iso}(N))\big).\end{aligned}$$

According to Theorem 6.1 there is a continuous family of metrics g_λ on M and a continuous family of normal Riemannian coverings $\mathrm{q}_\lambda\colon \mathbb{R}^n \times N \to (M, g_\lambda)$ such that the deck transformation group of q_λ is $(\mathrm{pr}\,, \psi_\lambda)(\Pi) \subset \mathrm{Iso}(\mathbb{R}^d) \times \mathsf{G}$.

Consider the covering

$$z_1\colon (\mathbb{R}^n \times N)\big/((\mathrm{pr}\,,\ \psi_1)(\mathsf{A})) \longrightarrow (M,\ g_1)\,.$$

By construction $\psi_1(\mathsf{A}) = \{e\}$. Since A is free abelian, it follows that $\mathrm{pr}(\mathsf{A})$ consists out of translations. Thus $\mathbb{R}^n \times N/(\mathrm{pr}\,, \psi_1)(\mathsf{A})$ is isometric to a Riemannian product $T^{(d)} \times \mathbb{R}^{n-d} \times N$, as claimed.

Proof of Corollary 6.4

Clearly, M is homotopy equivalent to its soul. Since S is a compact flat manifold, it follows that $\Pi := \pi_1(M) \cong \pi_1(S)$ is a torsion free crystallographic group of rank k. By applying Corollary 6.3 to the universal covering $\mathrm{q}_0\colon \mathbb{R}^{k+l} \to (M, g_0)$ of (M, g_0) we find a continuous family of Riemannian coverings $\mathrm{q}_\lambda\colon \mathbb{R}^{k+l} \to (M, g_\lambda)$ and a s-sheeted normal Riemannian covering

$$z_1\colon T^{(k)} \times \mathbb{R}^l \to (M,\ g_1)$$

with $s \leqslant 2 \cdot (2k)^{2k(r+1)}$, where $r = \mathrm{rank}(\mathrm{O}(l)) = [l/2]$. Since the holonomy group of $T^{(k)} \times \mathbb{R}^l$ is trivial, the holonomy group of (M, g_1) contains at most

$$2 \cdot (2k)^{2k(r+1)}\ \leqslant\ 2 \cdot (2k)^{k(l+2)}\ \leqslant\ n^{n^2}$$

elements.

On the proof of Corollary 6.5

To make talking easier we introduce a notation: a diffeomorphism $f\colon (M, g_0) \to (N, g)$ between two complete, flat manifolds is called flat, if there is a continuous family $(g_\lambda)_{\lambda\in[0,1]}$ of complete, flat metrics on M connecting the given metric g_0 with the pull back metric $g_1 := f^*g$. We have to prove that there are only finitely many flat diffeomorphism classes of complete, flat manifolds in each dimension. Therefore we need the following observation:

Lemma 6.9. *Let (M, g_0) and (N, g) be two complete, flat manifolds, and let $f\colon (M, g_0) \to (N, g)$ be an affine diffeomorphism. Then f is flat.*

Proof. Since f is affine, the metric $g_\lambda := \lambda g + (1-\lambda) f^*g$ is flat, too. Moreover, the metrics g_0 and g_λ have the same geodesics, and hence g_λ is also complete.

Proof of Corollary 6.5. A complete, flat manifold is isometric to the an orbit space of the form $\mathbb{R}^n/\Gamma$ where Γ is a torsion free, discrete subgroup of $\mathrm{Iso}(\mathbb{R}^n)$.

Let $\mathfrak{M}(n)$ denote the set of torsion free, discrete subgroups of $\mathrm{Iso}(\mathbb{R}^n)$. We say that two groups $\Gamma_1, \Gamma_2 \in \mathfrak{M}(n)$ are equivalent if there exists a flat diffeomorphism between the quotients $\mathbb{R}^n/\Gamma_1 \to \mathbb{R}^n/\Gamma_2$. Clearly, it is sufficient to show that $\mathfrak{M}(n)$ contains only finitely many equivalence classes.

Let $\mathfrak{N}(n) \subset \mathfrak{M}(n)$ be the subset that consists of those groups Γ for which the holonomy group of the quotient $\mathbb{R}^n/\Gamma$ has order at most n^{n^2}. By Corollary 6.4 we only have to check that $\mathfrak{N}(n)$ is finite up to equivalence.

Observe that each $\Gamma \in \mathfrak{N}(n)$ is as an abstract group isomorphic to a crystallographic group of rank $< n$. By the third Bieberbach theorem this class consists out of finitely many isomorphism classes. Thus we just have to verify that for a fixed crystallographic group $\Gamma_0 \subset \mathrm{Iso}(\mathbb{R}^d)$ of rank $d < n$ the set

$$\mathfrak{N}(n,\,\Gamma_0) := \left\{\Gamma \in \mathfrak{N}(n) \,\middle|\, \Gamma \text{ is as an abstract group isomorphic to } \Gamma_0\right\}$$

contains only finitely many equivalence classes.

As explained in the proof of Corollary 6.3 a given group $\Gamma \in \mathfrak{N}(n, \Gamma_0)$ acts on a d-dimensional affine subspace of $\mathbb{R}^n$ as a crystallographic group. By passing from Γ to an equivalent group if necessary, we can assume that Γ acts discontinuously and cocompactly on $\mathbb{R}^d \times \{0\} \subset \mathbb{R}^n$. Notice that then $\Gamma \subset \mathrm{Iso}(\mathbb{R}^d) \times \mathrm{O}(n-d) \subset \mathrm{Iso}(\mathbb{R}^n)$. Hence it remains to prove that

$$\mathfrak{L} := \left\{\Gamma \in \mathfrak{N}(n,\,\Gamma_0) \,\middle|\, \Gamma \subset \mathrm{Iso}(\mathbb{R}^d) \times \mathrm{O}(n-d)\right\}$$

is finite up to equivalence. For a group $\Gamma \in \mathfrak{L}$ the image $\mathrm{pr}(\Gamma)$ of the projection $\mathrm{pr}\colon \mathrm{Iso}(\mathbb{R}^d) \times \mathrm{O}(n-d) \to \mathrm{Iso}(\mathbb{R}^d)$ is a crystallographic group that is as an abstract group isomorphic to Γ_0. It follows from the second Bieberbach theorem that there is an element in $(v, A) \in \mathbb{R}^d \rtimes \mathrm{GL}\,(d)$ satisfying $\Gamma_0 = (v, A)\,\mathrm{pr}(\Gamma)(v, A)^{-1}$. We let (v, A) also the image of (v, A) under the natural inclusion $\mathbb{R}^d \rtimes \mathrm{GL}\,(d) \hookrightarrow \mathbb{R}^n \rtimes \mathrm{GL}\,(n)$, and define $\Gamma' = (v, A) \cdot \Gamma \cdot (v, A)^{-1}$. Clearly, $\Gamma' \in \mathfrak{L}$, and we can employ Lemma 6.9 to see that Γ' is equivalent to Γ. This consideration shows that it is sufficient to prove that the set

$$\mathfrak{K} := \left\{\Gamma \in \mathfrak{L} \,\middle|\, \Gamma \subset \Gamma_0 \times \mathrm{O}(n-d),\ \mathrm{pr}(\Gamma) = \Gamma_0\right\}$$

contains only finitely many equivalence classes.

Let $\mathrm{pr}_2\colon \Gamma_0 \times \mathrm{O}(n-d) \to \mathrm{O}(n-d)$ be the projection. For $\Gamma \in \mathfrak{K} \subset \mathfrak{N}(n)$ we have by construction that the order of $\mathrm{Hol}(\mathbb{R}^n/\Gamma)$ is bounded by n^{n^2}. Thus the order of the image $\mathsf{H} := \mathrm{pr}_2(\Gamma)$ is at most n^{n^2}. It is an elementary consequence from representation theory of finite groups that $\mathrm{O}(n-d)$ contains up to inner conjugation only finitely many groups of a given order, and hence we can think of H as a fixed group. Thus it is sufficient to check that for a given finite group H there are only finitely many subgroups $\Gamma \subset \Gamma_0 \times \mathsf{H}$ with $\mathrm{pr}(\Gamma) = \Gamma_0$. But this is an immediate consequence of the fact that a finitely generated group contains only finitely many subgroups of a given index, compare Lemma 2.6.

The Proof of Corollary 6.6

We begin with the case of a compact Lie group G. Let $\psi \in \mathrm{Hom}(\Pi, \mathsf{G})$. The image $\mathsf{H} := \psi(\Pi)$ contains no free subgroups of rank 2. By a theorem of Tits [20] this implies that H is solvable up to finite index. Then the closure $\bar{\mathsf{H}}$ of H is solvable up to finite index, too. In particular, the identity component $\bar{\mathsf{H}}_0$, a connected solvable compact Lie group, is abelian. Therefore $\bar{\mathsf{H}}$ and H are abelian up to finite index.

The homomorphism ψ factorizes $\psi = \iota \circ \psi$ where $\iota \colon \mathsf{H} \to \mathsf{G}$ is the inclusion. By Theorem 6.2 we can deform ι (and hence ψ) into a homomorphism with finite image.

According to a theorem of Jordan [19, Theorem 8.29], there is a constant $m(\mathsf{G})$ only depending on G such that any finite subgroup $\mathsf{F} \subset \mathsf{G}$ contains an abelian normal subgroup F' satisfying $(\mathsf{F} : \mathsf{F}') \leqslant m(\mathsf{G})$.

Let Π' be the intersection of all subgroups of Π of index at most $m(\mathsf{G})$. Lemma 2.6 exhibits Π' as a finitely generated subgroup of finite index in Π. Observe that for a homomorphism $\psi \in \mathrm{Hom}(\Pi, \mathsf{G})$ with finite image the group $\psi(\Pi')$ is abelian. In summary, we can say that any homomorphism $\psi \in \mathrm{Hom}(\Pi, \mathsf{G})$ can be deformed into a homomorphism $\tilde{\psi}$ with $\mathrm{Ker}(\tilde{\psi}) \supset [\Pi', \Pi']$, where $[\Pi', \Pi']$ is the commutator group of Π'. Consequently, we only have to prove that $\mathrm{Hom}(\Pi/[\Pi', \Pi'], \mathsf{G})$ has only finitely many arc-connected components.

In other words, we can assume that Π is abelian up to finite index. By Theorem 6.2 there is a constant $h = h(\Pi, \mathsf{G})$ such that any homomorphism $\psi \in \mathrm{Hom}(\Pi, \mathsf{G})$ can be deformed into a homomorphism with $\mathrm{ord}(\psi(\Pi)) \leqslant h$.

Let Π'' be the intersection of all subgroups of index at most h in Π. From Lemma 2.6 we infer that the factor group $\mathsf{F} := \Pi/\Pi''$ is finite. Similarly to above it remains to check that the set $\mathrm{Hom}(\mathsf{F}, \mathsf{G})$ has only finitely many arc-connected components. But this statement follows immediately from the fact that $\mathrm{Hom}(\mathsf{F}, \mathsf{G})$ has only finitely many conjugate classes, see [1, appendix] for quantitative estimates.

Suppose now that $\mathsf{G} = \mathrm{GL}(n, \mathbf{K})$ where $\mathbf{K} \in \{\mathbb{R}, \mathbb{C}\}$. By the first part it is sufficient to verify that a given representation $\psi \in \mathrm{Hom}(\Pi, \mathsf{G})$ can be deformed into an orthogonal representation if $\mathbf{K} = \mathbb{R}$ and into an unitary representation if $\mathbf{K} = \mathbb{C}$. This clearly can be done by showing that ψ can be deformed into a homomorphism that has a relatively compact image. We argue by induction on n.

We begin with the case of a reducible representation ψ. So there is a nontrivial $\psi(\Pi)$-invariant subspace $V \subset \mathbf{K}^n$. Without loss of generality $V = \mathbf{K}^d \times \{0\} \subset \mathbf{K}^n$ for some positive integer $d < n$. Then $\psi(g)$ is a block matrix

$$\psi(g) = \begin{pmatrix} \zeta(g) & C(g) \\ 0 & \eta(g) \end{pmatrix}$$

where $\zeta(g) \in \mathrm{GL}(d, \mathbf{K})$, $\eta(g) \in \mathrm{GL}(n-d, \mathbf{K})$ and $C(g) \in M(d \times (n-d), \mathbf{K})$. Consider the continuous family of homomorphisms

$$\psi_\lambda(g) = \begin{pmatrix} \zeta(g) & \lambda \cdot C(g) \\ 0 & \eta(g) \end{pmatrix} \qquad \text{for } \lambda \in [0,1],\ g \in \Pi.$$

Clearly, $\psi_1 = \psi$ and $\psi_0 = \zeta \oplus \eta$. By the induction hypothesis we can deform the representations ζ and η into homomorphisms with finite images, and hence we are done.

Thus we can assume that ψ is irreducible. Let Z be the Zarisky closure of $\psi(\Pi)$ in G. By Tits theorem $\psi(\Pi)$ is solvable up to finite index, so the identity component Z_0 of Z is solvable. Suppose for a moment that Z_0 is not abelian. Then the commutator group $\mathsf{U} = [\mathsf{Z}_0, \mathsf{Z}_0]$ is a nontrivial unipotent normal subgroup of Z. Because of Engel's theorem the vector space

$$V = \left\{ v \in \mathbf{K}^n \;\middle|\; Av = v \text{ for all } A \in \mathsf{U} \right\}$$

is a nontrivial subspace of $\mathbf{K}^n$. Since U is a normal subgroup of Z, V is a Z-invariant subspace which is impossible. Hence Z_0 is abelian. Let T be the maximal compact subgroup of Z_0, and let $\mathfrak{t} \subset \mathfrak{z}$ be the corresponding Lie algebras. We can find an $\mathrm{Ad}(\mathsf{Z})$-invariant complement $\mathfrak{a}$ of $\mathfrak{t}$ in $\mathfrak{z}$ because $\mathrm{Ad}(\mathsf{Z})$ is finite. The group A corresponding to $\mathfrak{a}$ is a connected, simply connected, closed, cocompact normal subgroup of Z. In particular, $\mathsf{A} \cong \mathbb{R}^l$ for a suitable integer l. Since Z has only finitely many connected components, there exists a maximal compact subgroup K and furthermore $\mathsf{K} \cdot \mathsf{A} = \mathsf{Z}$, see [11, Ch. XV, Theorems 3.1 and 3.7]. Moreover $\mathsf{A} \cap \mathsf{K} = \{e\}$ and therefore Z is isomorphic to a semidirect product $\mathbb{R}^l \rtimes_\beta \mathsf{K}$. We identify Z with $\mathbb{R}^l \rtimes_\beta \mathsf{K}$, and consider the continuous family of homomorphisms $h_\lambda(v,k) = (\lambda v, k)$ for all $(v,k) \in \mathbb{R}^l \rtimes_\beta \mathsf{K} = \mathsf{Z}$, $\lambda \in [0,1]$. For the corresponding family $\psi_\lambda := h_\lambda \circ \psi$ we have $\psi_1 = \psi$, and the image of ψ_0 is contained in the compact group K.

Acknowledgments

The paper is part of the authors dissertation [22]. I would like to thank my Ph.D. advisor Professor W.T. Meyer for his constant interest in this work and for useful discussions. Furthermore, I am indebted to Professor Heintze who posed the question whether there exists a class of manifolds for which the fundamental groups are precisely the virtually polycyclic groups. This led to the positive answer given in Theorem 2.4.

References

[1] P. Buser and H. Karcher, Gromov's almost flat manifolds, *Asterisque* **81** (1981).

[2] L.S. Charlap, *Bieberbach Groups and Flat Manifolds* (Springer, Berlin, 1986).

[3] J. Cheeger and T. Colding, Lower bounds on Ricci curvature and almost rigidity of warped products, *Ann. of Math.* **144** (1996) 189–237.

[4] J. Cheeger and D. Ebin, *Comparison Theorems in Riemannian Geometry* (North-Holland, Heildelberg, 1975).

[5] J. Cheeger and D. Gromoll, The splitting theorem for manifolds of nonnegative Ricci curvature, *J. Differential. Geom.* **6** (1971) 119–128.

[6] J. Cheeger and D. Gromoll, On the structure of complete manifolds of nonnegative curvature, *Ann. of Math.* **96** (1972) 413–443.

[7] K. Dekimpe and P. Igodt, The structure and topological meaning of almost-torsion free groups, *Comm. Algebra* **22** (1994) 2547–2558.

[8] K. Fukaya and T. Yamaguchi, The fundamental groups of almost nonnegatively curved manifolds, *Ann. of Math.* **136** (1992) 253–333.

[9] M. Gromov, Groups of polynomial growth and expanding maps, *Publ. Math. IHES* **53** (1981) 53–73.

[10] K. Grove, H. Karcher and E.A. Ruh, Group actions and curvature, *Invent. Math.* **23** (1974) 31–48.

[11] G. Hochschild, *The Structure of Lie Groups* (Holden–Day, London, 1965).

[12] O. Kegel and B. Wehrfritz, *Locally Finite Groups* (North-Holland, Heidelberg, 1973).

[13] K.B. Lee, There are only finitely many infranilmanifolds under each nilmanifold, *Quart. J. Math. Oxford Ser. (2)* **39** (1988) 61–66.

[14] J. Milnor, A note on curvature and fundamental group, *J. Differential Geom.* **2** (1968) 1–7.

[15] H. Minkowski, *Gesammelte Abhandlungen* (Teubner, Leipzig, 1911).

[16] D. Montgomery and L. Zippin, *Topological Transformation Groups* (Interscience, New York, 1955).

[17] G.D. Mostow, On the fundamental group of a homogeneous space, *Ann. of Math.* **66** (1957) 249–255.

[18] S.B. Myers and N.E. Steenrod, The group of isometries of a Riemannian manifold, *Ann. of Math.* **40** (1939) 400–416.

[19] M.S. Raghunathan, *Discrete Subgroups of Lie Groups* (Springer, Berlin, 1972).

[20] J. Tits, Free subgroups in linear groups, *J. Algebra* **20** (1972) 250–270.

[21] G. Wei, Examples of complete manifolds of positive Ricci curvature with nilpotent isometry group, *Bull. Amer. Math. Soc.* **19** (1988) 311–313.

[22] B. Wilking, Group actions on manifold of nonnegative sectional curvature and generalized Bieberbach theorems, Thesis, Univ. Münster, 1998.

[23] B. Wilking, Rigidity of group actions on solvable Lie groups, *Math. Ann.* **317** (2000), 195–237.