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Index parity of closed geodesics and rigidity of Hopf fibrations

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Abstract. The diameter rigidity theorem of Gromoll and Grove [1987] states that a Riemannian manifold with sectional curvature ≥ 1 and diameter $\geq \pi/2$ is either homeomorphic to a sphere, locally isometric to a rank one symmetric space, or it has the cohomology ring of the Cayley plane $\mathrm{Ca}\mathbb{P}^2$. The reason that they were only able to recognize the cohomology ring of $\mathrm{Ca}\mathbb{P}^2$ is due to an exceptional case in another theorem [Gromoll and Grove, 1988]: A Riemannian submersion $\sigma\colon \mathbb{S}^m \to B^b$ with connected fibers that is defined on the Euclidean sphere $\mathbb{S}^m$ is metrically congruent to a Hopf fibration unless possibly $(m, b) = (15, 8)$. We will rule out the exceptional cases in both theorems. Our argument relies on a rather unusal application of Morse theory. For that purpose we give a general criterion which allows to decide whether the Morse index of a closed geodesic is even or odd.

1. Introduction

Gromoll and Grove [1988] proved that a metric foliation of the Euclidean sphere $\mathbb{S}^m$ is induced by a locally free isometric action of $\mathbb{R}$ or SU(2) provided that the leave dimension of the foliation is at most 3. As a consequence they could show that a Riemannian submersion $\sigma\colon \mathbb{S}^m \to B^b$ with connected fibers of dimension 1, 2 or 3 is metrically congruent to a Hopf fibration. Here and throughout this article two Riemannian submersions $\sigma_i\colon \mathbb{S}^m \to B_i$ $(i = 1, 2)$ are said to be metrically congruent, if there are isometric diffeomorphisms ι_1 and ι_2 for which the diagram

$$\begin{array}{ccc} \mathbb{S}^m & \xrightarrow{\iota_1} & \mathbb{S}^m \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ B_1 & \xrightarrow{\iota_2} & B_2 \end{array}$$

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commutes. For topological reasons a Riemannian submersion $\sigma : \mathbb{S}^m \to B^b$ has fiber dimension at most 3 unless $(m, b) = (15, 8)$, see Browder [1963]. Using different techniques we will show that in the latter case σ is metrically congruent to the Hopf fibration and hence we obtain

Theorem 1. *Consider the sphere $\mathbb{S}^m$ with its round metric of constant curvature 1, and let $\sigma : \mathbb{S}^m \to B$ be a Riemannian submersion with connected fibers of positive dimension. Then the fibration $\sigma : \mathbb{S}^m \to B$ is metrically congruent to a generalized Hopf fibration, i.e., to one of the following fibrations*

- $h : \mathbb{S}^{2n+1} \to \mathbb{CP}^n$,
- $h : \mathbb{S}^{4n+3} \to \mathbb{HP}^n$ *or*
- $h : \mathbb{S}^{15} \to \mathbb{S}^8$,

where the scaling of the metrics of the rank one symmetric spaces $\mathbb{CP}^n$, $\mathbb{HP}^n$ and $\mathbb{S}^8$ is chosen such that each space has diameter $\pi/2$.

Below we only prove that a Riemannian submersion $\sigma : \mathbb{S}^m \to B^b$ is metrically congruent to the Hopf fibration for $(m, b) = (15, 8)$. As our argument is entirely different to the approach of Gromoll and Grove [1988] which covers all other cases, it may be worth it to explain which parts of the proof carry over to all pairs (m, b). A crucial step in our proof is to show that B^8 is a C_π-manifold, i.e., all normal geodesics in B^8 have least period π. Except for some modifications in the case $(m, b) = (3, 2)$, this part can be generalized to all pairs (m, b). However, we then use a result of Weinstein to recover the volume of the base. Here a generalization is difficult, because, if the base is not a sphere, one has to show that it has the 'right' Weinstein integer. It is known though that a C_π-manifold which is homeomorphic to a rank 1 symmetric space has the same volume as its model, see Reznikov [1994]. Eventually we use the knowledge of the volume of the base to show that σ is a Hopf fibration. This part carries over immediately to all Riemannian submersions.

As a consequence of Theorem 1 we can also rule out the exceptional case in the main theorem of [Gromoll and Grove, 1987]:

Corollary 2 (Improved version of the diameter rigidity theorem). *Let M be a compact Riemannian manifold with sectional curvature ≥ 1 and diameter $\operatorname{diam}(M) \geq \pi/2$. Then M is either homeomorphic to a sphere or locally isometric to a rank one symmetric space.*

The original version of the theorem makes the additional assumption that the integral cohomology ring of M does not coincide with the cohomology ring of the Cayley plane.

Corollary 2 also improves a result of Wilhelm [1996]. He showed that the hypothesis on the cohomology ring of the manifold can be dropped, provided that the lower diameter bound is replaced by a lower radius bound.

The improved version of the diameter rigidity theorem completes the equality discussion of the diameter sphere theorem by Grove and Shiohama [1977].

A crucial ingredient for the proof of Theorem 1 is the following

Theorem 3. *Let M be an oriented Riemannian manifold all of whose geodesic are closed, and let* $c\colon [0, 1] \to M$ *be a closed geodesic. Then the index of c in the free loop space of M is odd if M is even–dimensional, and it is even if M is odd dimensional.*

Theorem 3 is a consequence of a simple criterion which allows to decide whether the Morse index of a closed geodesic in the free loop space of a Riemannian manifold is even or odd. We emphasize that this criterion applies generally and might be useful in other context as well. It is stated and proved in Sect. 3. How to derive Theorem 3 is explained after Corollary 3.6.

I would like thank Wolfgang Ziller for many useful comments, especially for bringing the article [Ballmann, Thorbergsson and Ziller, 1982] to my attention and suggesting Remark 3.7.2. I am also indebted to the referee who suggested several additions and rearrangements.

2. The proof of Theorem 1 and Corollary 2

It is known from Grove and Gromoll's proof of their diameter rigidity theorem that Theorem 1 implies Corollary 2. Therefore it is sufficient to prove Theorem 1. As explained in the introduction, it only remains to prove that a Riemannian submersion $\sigma\colon \mathbb{S}^{15} \to B^8$ with connected fibers is metrically congruent to the Hopf fibration. The fibers of σ are known to be homotopy spheres, see [Browder, 1963]. This is of course equivalent to saying that B^8 is a homotopy sphere. We will proceed in four steps.

Step 1. Every normal geodesic in the base manifold B^8 is closed with period at most π.

Step 2. If B^8 does not have a normal geodesic with period $< \pi$, then the fibration $\sigma\colon \mathbb{S}^{15} \to B^8$ is metrically congruent to the Hopf fibration.

Step 3. The energy function E on the free loop space ΛB^8 of B^8 is a Morse Bott function. The indices of all critical submanifolds with positive energy are odd.

Step 4. One of the first seven $\mathbb{Z}/2\mathbb{Z}$–Betti numbers of ΛB^8 differs from the corresponding Betti numbers of $\Lambda\mathbb{S}^8$ unless B^8 has no closed normal geodesic with period $< \pi$.

As we know B^8 to be homotopically equivalent to $\mathbb{S}^8$, the Betti numbers of ΛB^8 and $\Lambda\mathbb{S}^8$ do coincide. Hence, by Step 4 there is no exceptional closed geodesic in B^8, and by Step 2 the result follows.

Theorem 3 is used to prove Step 3. As mentioned earlier Theorem 3 is proved in Sect. 3. But at first we prove the four steps in four subsections.

2.1. Proof of Step 1. Choose an orientation on the vertical distribution $T^v\mathbb{S}^{15}$ and on the horizontal distribution $T^h\mathbb{S}^{15}$ of $T\mathbb{S}^{15}$. We consider the map $a\colon \mathbb{S}^{15} \to \mathbb{S}^{15}$, $p \mapsto -p$, and let a_* denote its differential.

For a point $p \in \mathbb{S}^{15}$ we choose a basic horizontal unit vectorfield x along the fiber $F_p := \sigma^{-1}(\sigma(p))$. The restriction $a_{|F_p}$ is then given by the holonomy map

$$\phi\colon F_p \to \sigma^{-1}(\sigma(\exp(\pi x_p))),\ q \mapsto \exp(\pi x_q) = -q.$$

Since B is simply connected, the holonomy maps preserve the orientations of fibers. Hence, a_* preserves the oriented vertical distribution. As a is an orientation preserving isometry of $\mathbb{S}^{15}$, we see that a_* preserves the oriented horizontal distribution as well. Therefore a induces an orientation preserving isometry $\bar{a}\colon B^8 \to B^8$.

We plan to prove $\bar{a} = \mathrm{id}$. Evidently, $\bar{a}^2 = \mathrm{id}$. The isometry $\bar{a}$ must have a fixed point, because otherwise $B^8/\{\mathrm{id}, \bar{a}\}$ would be, in contradiction to Synge's theorem, an even–dimensional oriented compact manifold with positive sectional curvature and with a nontrivial fundamental group.

Choose $p \in B$ with $\bar{a}(p) = p$. If the isometry $\bar{a}$ is not the identity, we could find a unit vector $\bar{x} \in T_pB$ with $\bar{a}_*(\bar{x}) = -\bar{x}$. Let x be the horizontal lift of $\bar{x}$ along $\sigma^{-1}(p)$. By construction $a_*(x_q) = -x_{-q}$. Consequently, the holonomy map

$$\tilde{\phi}\colon \sigma^{-1}(p) \to \sigma^{-1}(\exp(\pi/2 \cdot \bar{x})), \quad q \mapsto \exp(\pi/2 \cdot x_q)$$

satisfies $\tilde{\phi}(q) = \tilde{\phi}(-q)$. This is a contradiction, because holonomy maps are diffeomorphisms.

Thus $\bar{a} = \mathrm{id}$ and hence the map σ factorizes $\sigma = \bar{\sigma} \circ \mathrm{pr}$, where $\mathrm{pr}\colon \mathbb{S}^{15} \to \mathbb{RP}^{15}$ is the projection and where $\bar{\sigma}\colon \mathbb{RP}^{15} \to B^8$ is a Riemannian submersion. Since all geodesics in $\mathbb{RP}^{15}$ are closed with period π, the same is valid for B^8.

2.2. Proof of Step 2. The following lemma is a special version of a more general result of Heintze and Karcher [1978].

Lemma 2.1. *Consider the n-sphere $\mathbb{S}^n$ with its canonical metric. Suppose that $M \subset \mathbb{S}^n$ is a k-dimensional minimal compact submanifold without boundary. Then* $\mathrm{vol}(M) \geq \mathrm{vol}(\mathbb{S}^k)$, *and equality holds if and only if M is totally geodesic.*

Lemma 2.2. *Let $u \in T_pB^8$ be a unit vector, $c(t) := \exp(tu)$, $q \in \sigma^{-1}(p)$, and let $x \in T_q\mathbb{S}^{15}$ be the horizontal lift of u. Suppose that $\lambda_1, \ldots, \lambda_7$ are the eigenvalues (listed with multiplicity) of the second fundamental form of the fiber $F = \sigma^{-1}(p)$*

$$S_x\colon T_q^v\mathbb{S}^{15} \to T_q^v\mathbb{S}^{15}.$$

Then $\mathrm{arccotan}(\lambda_1), \ldots, \mathrm{arccotan}(\lambda_7) \in (0, \pi)$ *are the conjugate points of c on* $(0, \pi)$ *(listed with multiplicity). In particular, the eigenvalues $\lambda_1, \ldots, \lambda_7$*

only depend on u and not on the choice of q. Furthermore any geodesic of length π in B^8 has index 7 as a closed geodesic.

Proof of Lemma 2.2. Let $\nu(F)$ denote the normal bundle of F. The vectors

$$\operatorname{arccotan}(\lambda_1)x, \ldots, \operatorname{arccotan}(\lambda_1)x$$

are critical points of $\exp_{|\nu(F)}$. Since the diagram

$$\begin{array}{ccc} \nu(F) & \xrightarrow{\exp} & \mathbb{S}^{15} \\ \sigma_* \downarrow & & \downarrow \sigma \\ T_p B^8 & \xrightarrow{\exp} & B^8 \end{array}$$

is commutative and the image of $\exp_{|\nu(F)*}$ contains all vertical vectors, we see that $\operatorname{arccotan}(\lambda_1)u, \ldots, \operatorname{arccotan}(\lambda_1)u$ are critical points of the exponential map of B^8 at p. Hence the result follows. □

Proof of Step 2. By hypothesis each normal geodesic in B^8 is closed with minimal period π. Weinstein [1974] proved that the volume of such a manifold is an integral multiple of the volume of a sphere of constant curvature 4. In particular

$$\operatorname{vol}(B^8) \geq \tfrac{1}{2^8} \operatorname{vol}(\mathbb{S}^8). \tag{1}$$

Applying Fubini to the standard Hopf fibration $h\colon \mathbb{S}^{15} \to \mathbb{S}^8$ gives

$$\operatorname{vol}(\mathbb{S}^{15}) = \tfrac{1}{2^8} \operatorname{vol}(\mathbb{S}^8) \cdot \operatorname{vol}(\mathbb{S}^7). \tag{2}$$

If F_p is a fiber of σ realizing the minimal volume among all fibers, then for any basic horizontal vectorfield x along F_p the equation

$$\int_{F_p} \operatorname{trace}(S_x)(q)\, dvol(q) = 0$$

holds. Because of Lemma 2.2 $\operatorname{trace}(S_x)$ is constant, and hence it is zero. In other words, F_p is a minimal submanifold. Using Fubini for the fibration $\sigma\colon \mathbb{S}^{15} \to B^8$ and substituting (1) and (2) yields

$$\begin{aligned} \tfrac{1}{2^8} \operatorname{vol}(\mathbb{S}^8) \cdot \operatorname{vol}(\mathbb{S}^7) &= \operatorname{vol}(\mathbb{S}^{15}) \\ &= \int_{B^8} \operatorname{vol}(F_q)\, dvol(q) \geq \tfrac{1}{2^8} \operatorname{vol}(\mathbb{S}^8) \cdot \operatorname{vol}(F_p). \end{aligned}$$

Thus

$$\operatorname{vol}(F_p) \leq \operatorname{vol}(\mathbb{S}^7).$$

Lemma 2.1 exhibits F_p as a totally geodesic submanifold. Accordingly $\operatorname{vol}(F_q) = \operatorname{vol}(\mathbb{S}^7)$ for each fiber F_q of σ. Hence, all fibers of σ are minimal submanifolds, and by the same argument totally geodesic.

But it is known that a Riemannian submersion $\sigma\colon \mathbb{S}^{15} \to B^8$ with totally geodesic fibers is congruent to the Hopf fibration, see Wolf [1963], Escobales [1975] or Ranjan [1985]. □

2.3. Proof of Step 3. Since all normal geodesics in B^8 are closed with common period π, the geodesic flow induces a S^1 action on the unit tangent bundle T^1B of B^8. As S^1 is a compact Lie group, we can find a metric on T^1B which turns this action into an isometric action.

Consequently, the connected components of the fixed point set of an element $g = e^{2i\varphi} \in \mathsf{S}^1$ are submanifolds. Furthermore the degree of degeneration of a fix point v, i.e., the multiplicity of the eigenvalue 1 of the orthogonal map g_{*v}, is equal to the dimension of the corresponding submanifold. It is well known that the degree of degeneration of the fix point v of g is equal to the dimension the space of φ-periodic Jacobifields along the closed geodesic $c(t) = \exp(tv)$, $t \in [0, \varphi]$; here we considered the tangent field $\dot{c}(t)$ as a periodic Jacobifield. Taking into account that the closed geodesics of length $(k/l)\pi$ can be naturally identified with the fixed point set of the element $e^{i(2k\pi/l)} \in \mathsf{S}^1$ in T^1B, we deduce that the energy function

$$E(c) := \tfrac{1}{2}\int_0^1 \|\dot{c}(t)\|^2\, dt$$

on the free loop space ΛB^8 is a Morse Bott function. Furthermore, by Theorem 3 the indices of closed geodesics in B^8 are odd.

2.4. Proof of Step 4. In the sequel we always talk about homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients. Recall that the first seven Betti numbers of the free loop space of $\Lambda\mathbb{S}^8$ are given by $b_0 = 1,\ b_1 = \cdots = b_6 = 0,\ b_7 = 1$.

Choose a number $c > \pi^2/2$ such that E has no critical values in $(\pi^2/2, c]$. Let $\Lambda^{<c}B$ denote the loops of energy $< c$. Since E is a Morse Bott function, the sublevel $\Lambda^{<c}B$ is homotopically equivalent to a complex that is obtained as follows: Consider all critical manifolds $C_1, \dots, C_h$ of positive energy $< c$, where the numbering is chosen such that manifolds with lower energy have lower numbers and where for each critical value each connected component is listed separately. For each critical manifold C_i there exists a vectorbundle $E_i \to C_i$ of rank equal to the index of C_i such that the Hessian of the energy function E is negative definite on the vector space $E_{i|q} \subset T_q\Lambda B^8$ for all $q \in C_i$. We choose a norm on E_i and denote by $E_i^{\leq 1}$ all vectors with norm ≤ 1. The complex can be obtained from $B^8 \cong \mathbb{S}^8$ by attaching successively $E_1^{\leq 1}, \dots, E_h^{\leq 1}$.

Notice that C_h corresponds necessarily to the energy $\pi^2/2$. For this energy there is, according to Step 1, only one critical manifold, and C_h is diffeomorphic to the unit tangent bundle T^1B of B^8. Moreover the index of C_h is 7, see Lemma 2.2.

Clearly, closed geodesics in B^8 of length $> \pi$ have index ≥ 14; in fact, according to Lemma 2.2, such a geodesic has 7 conjugate points on the open interval $(0, \pi)$ and π is a conjugate point with multiplicity 7. Therefore the first seven Betti numbers of $\Lambda^{<c}B$ are equal to the first seven Betti numbers of ΛB.

Suppose there is an exceptional normal geodesic $c\colon \mathbb{R} \to B$ with period π/k for $k > 1$. The index of $c_{|[0,\pi/k]}$ is at most $\mathrm{index}(c_{|[0,\pi]}) = 7$. Let $p \leq 7$

be the minimal (odd) number for which an exceptional geodesic of index p exists, and consider the maximal number $q < h$ for which $E_q \to C_q$ is a vectorbundle of rank p. Finally, we let X_l denote the complex that is obtained by attaching the first l bundles,

$$X_l := \big(\dots \big(\mathbb{S}^8 \cup_{f_1} E_1^{\leq 1} \big) \cup_{f_2} E_2^{\leq 1} \big) \cdots \cup_{f_l} E_l^{\leq 1} \big).$$

The relative homology group of the pair (X_l, X_{l-1}) is equal to the relative homology group of $(E_l^{\leq 1}, \partial E_l^{\leq 1})$, which by the Thom isomorphism is equal to the shifted homology of C_l,

$$(3) \qquad H_*(X_l, X_{l-1}) = H_{*-i}(C_l), \quad \text{where } i = \mathrm{index}(C_l) \geq p.$$

Consequently, the first $p-1$ Betti numbers of X_{q-1} are given by $b_0 = 1$, $b_1 = \cdots = b_{p-1} = 0$. Furthermore, for $l = q$ equation (3) implies that the p-th Betti number of X_q does not vanishes. Since the relative homology groups of the pair (X_i, X_{i-1}) have only entries in dimensions $\geq p+2$ for $q < i < h$, it follows that the p-th Betti number of X_{h-1} is positive.

If $p \leq 5$, then the same argument shows $b_p(X_h) \neq 0$ and we are done. If $p = 7$, then we observe that the relative homology $H_*(X_h, X_{h-1}) \cong H_{*-7}(T^1 B)$ has an entry in dimension 7 but no entry in dimension 8. This proves $b_7(X_h) \geq 2$ and we are done as well.

3. The parity of the index of a closed geodesic

Throughout this section let M be an oriented n-dimensional Riemannian manifold, and let $c\colon [0,1] \to M$ be a non–constant closed geodesic, but see Remark 3.7.3 for the non–oriented case. We want to decide whether the Morse index of c in the free loop space ΛM of M is even or odd.

We will often identify the orthogonal complement $V \subset T_{c(0)}M$ of $\dot{c}(0)$ with $\mathbb{R}^{n-1}$ by using a linear isometry as identification map. Recall that the Poincare map P of c is the endomorphism of $V \times V$ given by

$$(v, w) \mapsto (J(1), J'(1)), \text{ where}$$

J is the Jacobifield along c with the initial conditions $J(0) = v$ and $J'(0) = w$. It is well known that P preserves the natural symplectic form

$$\omega((X_1, Y_1), (X_2, Y_2)) = \langle X_1, Y_2 \rangle - \langle X_2, Y_1 \rangle$$

of $V \times V$. The connected Lie group of all symplectic endomorphisms of $V \times V \cong \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ is denoted by $\mathsf{Sp}(n-1, \omega)$.

Proposition 3.1. *The parity of the index of c in the free loop space only depends on the Poincare map $P \in \mathsf{Sp}(n-1, \omega)$.*

Proof. By a formula of Morse [see Ballmann, Thorbergsson and Ziller, 1982] the index $\mathrm{index}_{\Lambda M}(c)$ of the closed geodesic c in the free loop space is given by

$$\mathrm{index}_{\Lambda M}(c) = \mathrm{index}_{c(0)}(c) + \mathrm{index}(\tilde{H}) + \dim(\mathrm{Ker}(\tilde{H})) - \mathrm{null}(c),$$

where $\mathrm{index}_{c(0)}(c)$ denotes the index of c in the loop space of M at $c(0)$, $\mathrm{null}(c)$ is the dimension of the space of normal 1-periodic Jacobifields along c and where $\mathrm{index}(\tilde{H}) + \dim(\mathrm{Ker}(\tilde{H}))$ denotes the number of nonpositive eigenvalues (counted with multiplicity) of the symmetric 2-form $\tilde{H}$ that is defined on the space $E := (P - \mathrm{id})^{-1}(\{0\} \times V)$ by

$$\tilde{H}(u, v) = -\omega((P - \mathrm{id})u, v) \quad \text{for all } u, v \in E.$$

In particular the quantity $\mathrm{index}(\tilde{H}) + \dim(\mathrm{Ker}(\tilde{H}))$ only depends on P. Furthermore the nullity $\mathrm{null}(c)$ is equal to the dimension of the eigenspace of P corresponding to 1. The question is therefore whether *mod 2* the number $\mathrm{index}_{c(0)}(c)$ is determined by P. Recall that according to the index theorem $\mathrm{index}_{c(0)}(c)$ is equal to the number of conjugate points of c on $(0, 1)$ (counted with multiplicity).

We consider the linear map

$$\hat{J}(t)\colon V = (\dot{c}(0))^{\perp} \to (\dot{c}(t))^{\perp}, \quad v \mapsto J_v(t),$$

where $J_v(t)$ denotes the Jacobifield along c with initial conditions $J_v(0) = 0$ and $J_v'(0) = v$. As M is oriented, an orientation on V induces naturally an orientation on each $(\dot{c}(t))^{\perp}$. Therefore the determinant of $\hat{J}(t)$ is well defined, i.e., we can define $\det(\hat{J}(t))$ as the determinant of a matrix representing $\hat{J}(t)$ with respect to oriented orthonormal bases of V and $(\dot{c}(t))^{\perp}$. Finally, we let h_t denote the number of conjugate points (counted with multiplicity) on the interval $(0, t)$. It is an elementary fact that the following holds

$$\det(\hat{J}(t)) \begin{cases} > 0 \text{ only if } h_t \text{ is even} \\ < 0 \text{ only if } h_t \text{ is odd} \\ = 0 \text{ if t is a conjugate point} \end{cases}.$$

Evidently, we can compute $\det(\hat{J}(1))$ if we know the Poincare map. This finishes the argument unless $\det(\hat{J}(1)) = 0$. If $\det(\hat{J}(1)) = 0$, then we have to show that the limit

$$\lim_{t \uparrow 1} \mathrm{sgn}(\det(\hat{J}(t))$$

only depends on P, where $\mathrm{sgn}(\det(J(t))$ denotes the sign of $\det(J(t))$.

For that purpose we consider the kernel W of $\hat{J}(1)$ and its orthogonal complement $W^{\perp}$ in V. It is straightforward to check that the map

$$L\colon V = W \oplus W^{\perp} \to V, \quad (w \oplus v) \mapsto -J_w'(1) + J_v(1)$$

is an isomorphism. In particular $\det(L)$ is not zero and obviously $\det(L)$ only depends on the Poincare map P. Because of the equation

$$\lim_{t\uparrow 1} \operatorname{sgn}\left(\det(\hat{J}(t))\right) = \operatorname{sgn}(\det(L))$$

this completes the proof of the proposition. □

Proposition 3.2. *Let* $P\colon [0,1] \to \mathsf{Sp}(n-1,\omega),\ s \mapsto P_s$ *be a smooth curve, and let* $\mathrm{Sym}(k)$ *denote the real symmetric* $k \times k$ *matrices. Then there is a smooth map*

$$\mathrm{R}\colon [0,1]\times[0,1] \to \mathrm{Sym}(n-1),\ (s,t) \mapsto \mathrm{R}_s(t)$$

such that P_s *is the Poincare map of* R_s*, i.e., if* $J\colon [0,1] \to \mathbb{R}^{n-1}$ *is a solution of the Jacobi equation* $J''(t) + R_s(t)J(t) = 0$*, then* $P_s(J(0), J'(0)) = (J(1), J'(1))$*. Furthermore there is a* $\delta > 0$ *such that* $\mathrm{R}_s(t) = 0$ *for all* $t \in [0,\delta] \cup [1-\delta, 1]$*. In particular* R_s *has a smooth extension as a* 1*-periodic function* $\mathrm{R}_s\colon \mathbb{R} \to \mathrm{Sym}(n-1)$.

Although the conclusion of Proposition 3.2 is not surprising at all, its proof is rather long, see Subsect. 3.1. In fact one has to be careful with the statement: For example for all curvature tensors R_0 with Poincare map P_0, there is a curve P_s ($s \in [0,1]$) starting at P_0 for which a corresponding continuous lift R_s starting at R_0 does not exist. For special curvature operators R_0 a smooth lift might not even exist locally.

It is an immediate consequence of Proposition 3.2 that any endomorphism $P \in \mathsf{Sp}(n-1,\omega)$ occurs as Poincare map of a closed geodesic (with trivial holonomy).

Definition 3.3. *For an element in the symplectic group* $P \in \mathsf{Sp}(n-1,\omega)$ *we define the parity* $\mathrm{par}(P) \in \mathbb{Z}/2\mathbb{Z}$ *of* P *as follows: If* P *is the Poincare map of a closed geodesic* c *in an oriented* n*-manifold* M*, then we set*

$$\mathrm{par}(P) \equiv \mathrm{index}(c) \bmod 2,$$

where $\mathrm{index}(c)$ *denotes the index of* c *in the free loop space.*

The two propositions do not only ensure that the parity is well defined, but they also allow us to compute the invariant.

Corollary 3.4. *Let* $P\colon [0,1] \to \mathsf{Sp}(n-1,\omega),\ s \mapsto P_s$ *be a curve. If the dimension of the eigenspace corresponding to* 1 *of* P_s *is constant in* s*, then* $\mathrm{par}(P_s)$ *is constant in* s*, too.*

Proof. By Proposition 3.2 there is a continuous family of 1-periodic curvature operators $\mathrm{R}_s(t)$, $s \in [0,1]$, $t \in \mathbb{R}$ such that P_s is the Poincare map of $\mathrm{R}_{s|[0,1]}$. Consider the index form corresponding to R_s

$$I_s(X,Y) = \int_0^1 -\langle \mathrm{R}_s(t)X(t), Y(t)\rangle + \langle X'(t), Y'(t)\rangle\, dt,$$

for all piecewise smooth maps $X, Y\colon [0,1] \to \mathbb{R}^{n-1}$ with $X(0) = X(1)$, $Y(0) = Y(1)$. Evidently, we can view I_s as the index form of a closed geodesic in an oriented manifold. Therefore it is sufficient to show that the index i_s of I_s is constant in s.

Notice that the nullity space

$$N_s := \{X \mid I_s(X, \cdot) = 0\}$$

consists precisely of the 1-periodic solutions $J\colon \mathbb{R} \to \mathbb{R}^{n-1}$ of the Jacobi equation $J'' + R_s J = 0$. Thus its dimension $n_s = \dim(N_s)$ is given by the dimension of the eigenspace of P_s corresponding to 1, and in particular n_s is constant. Consequently,

$$\begin{aligned} Z &:= \big\{s \in [0,1] \mid i_s = \max\{i_\tau \mid \tau \in [0,1]\}\big\} \\ &= \big\{s \in [0,1] \mid i_s + n_s = \max\{i_\tau + n_\tau \mid \tau \in [0,1]\}\big\}. \end{aligned}$$

The first equation implies that Z is open, and by second equation Z is closed in $[0, 1]$. Hence, $Z = [0, 1]$ and i_s is constant as well. □

Corollary 3.5. *The parity of the index of c only depends on the symplectic conjugacy class of the Poincare map* $P \in \mathsf{Sp}(n-1, \omega)$.

Proof. Let $P, S \in \mathsf{Sp}(n-1, \omega)$ and choose a curve $S(s)$ in the connected Lie group $\mathsf{Sp}(n-1, \omega)$ with $S(0) = \mathrm{id}$ and $S(1) = S$. The dimension of the eigenspace of $P_s = S(s)PS(s)^{-1}$ corresponding to 1 is constant in s, and accordingly $\mathrm{par}(P_s)$ is constant as well. □

Corollary 3.6. *We assume either that* 1 *is not an eigenvalue of P or that the algebraic multiplicity of the eigenvalue* 1 *of P is equal to the dimension of the corresponding eigenspace. Suppose that precisely k of the real eigenvalues of P lie in the open interval* $(0, 1)$ *(counted with algebraic multiplicity). Then the index of c is odd if and only if the number n + k is even,* $\mathrm{par}(P) \equiv n + k + 1 \bmod 2$.

Proof of Theorem 3. Notice that if M is a manifold all of whose geodesics are closed, then according to Wadsley [see Besse, 1978, Appendix A] there exists a common period. Hence, the Poincare map P of a closed geodesic in such a manifold is a root of the identity, i.e., $P^l = \mathrm{id}$ for some positive integer l. This implies $k = 0$ and hence Theorem 3. □

Proof of Corollary 3.6. Recall that $\mathsf{Sp}(n-1, \omega)$ is a real algebraic linear group. Let $\mathsf{A} \subset \mathsf{Sp}(n-1, \omega)$ be the real Zarisky closure of the cylic group generated by P. Clearly A is an abelian real algebraic linear group. Thus A is isomorphic to a direct product $\mathsf{R} \times \mathsf{U}$, where U is the unipotent radical of A and R is a reductive real algebraic group, [see Raghunathan, 1972, p. 11]. Consequently, there is a unipotent endomorphism $U \in \mathsf{A} \subset \mathsf{Sp}(n-1, \omega)$ and a semisimple endomorphism $S \in \mathsf{A} \subset \mathsf{Sp}(n-1, \omega)$ satisfying $P = SU = US$. Clearly, there is precisely one nilpotent element N in the Lie algebra of $\mathsf{Sp}(n-1,\omega)$ with $\mathrm{Exp}(N) = U$. Notice that S commutes with N and hence the eigenvalues of $P_s = S\,\mathrm{Exp}(sN)$ do not depend on s. Moreover our hypothesis implies that the eigenspace of P_s corresponding to 1 does not depend on s. Therefore $\mathrm{par}(P) = \mathrm{par}(P_0)$, by Corollary 3.4. In other words, we may assume that P is semisimple.

Since A is a reductive real algebraic abelian group, it follows that A is isomorphic to a direct product $\mathsf{K} \times \mathsf{S}$, where K is the maximal compact subgroup of A and S is the subgroup of matrices that have only positive real eigenvalues, see for example [Wilking, 2000, Lemma 4.4]. Consequently, there are semisimple endomorphisms $A, B \in \mathsf{A} \subset \mathsf{Sp}(n-1,\omega)$ such that $P = AB = BA$, A has only eigenvalues of absolute value one and B has only real positive eigenvalues. After passing to a symplectic conjugate endomorphism we may assume that A is contained in the maximal compact subgroup $\mathsf{U}(n-1)$ of $\mathsf{Sp}(n-1,\omega)$. Evidently, there is a continuous curve $A(s) \in \mathsf{U}(n-1)$ ($s \in [0,1]$) such that $A(0) = A$, $A(s)B = BA(s)$, the multiplicity of the eigenvalue 1 of $A(s)$ does not depend on s, and $A(1)$ has only the eigenvalues 1 and -1. Notice that the multiplicity of the eigenvalue 1 of $P(s) = A(s)B$ does not depend on s. Therefore

$$\mathrm{par}(P) = \mathrm{par}(P(0)) = \mathrm{par}(P(1)).$$

In other words, we may assume that the semisimple endomorphism P has only real eigenvalues. But then P is conjugate to an endomorphism $\tilde{P} \in \mathsf{Sp}(n-1,\omega)$ which has eigenvectors $(e_i, 0), (0, e_i) \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}$ ($i = 1, \ldots, n-1$). Without loss of generality P itself has these eigenvectors.

By making iterated use of Proposition 3.2 we see that P can be realized as a Poincare map of a curvature tensor $\mathrm{R}(t)$ that is diagonal for all t. If we view these diagonal elements as curvature operators in dimension two, then the index corresponding to $\mathrm{R}(t)$ is equal to the sum of the indices corresponding to the diagonal elements. In other words, we have reduced the problem to the case $n = 2$. So it is sufficient to consider a semisimple endomorphism $P \in \mathrm{SL}(2,\mathbb{R}) = \mathsf{Sp}(1,\omega)$ with real eigenvalues $\lambda, \frac{1}{\lambda}$. We distinguish among three cases.

If $\lambda < 0$, then we can connect P with $-\mathrm{id}$ in $\mathrm{SL}(2,\mathbb{R})$ without passing a unipotent matrix. Therefore we may assume $P = -\mathrm{id}$. A model geodesic for this situation is obtained as follows: Take an open distance tube neighborhood U of a closed geodesic in the round sphere $\mathbb{S}^2$. On U there is a free orientation preserving involution given by rotation with the angle π. The quotient of this action is a noncomplete Riemannian manifold with a closed geodesic of length π whose Poincare map is $-\mathrm{id}$. Furthermore it is straightforward to check that the index of the closed geodesic in the free loop space is 1. Hence, $\mathrm{par}(P) \equiv 1$.

If P is equal to the identity, then we can take a closed geodesic c of length 2π in $\mathbb{S}^2$ as a model to see that $\mathrm{par}(\mathrm{id}) \equiv 1$.

It remains to consider an endomorphism $P \in \mathrm{SL}(2,\mathbb{R})$ with two positive real eigenvalues different from 1. Notice that the set of endomorphisms of $\mathrm{SL}(2,\mathbb{R})$ with positive real eigenvalues different from 1 is connected. Hence, it is sufficient to check the assertion for one particular P. But for that we can just take a closed geodesic in a compact oriented hyperbolic surface as a model to see that $\operatorname{par}(P) \equiv 0$. □

Finally, we want to investigate the parity of the index of a closed geodesic in the cases that are not covered by Corollary 3.6.

Remark 3.7. 1. We call an element $P \in \mathsf{Sp}(n-1, \omega)$ (symplectically) decomposable if there is a decomposition $V \times V = W_1 \oplus W_2$ by P-invariant subspaces such that the restriction of ω to $W_i \neq \{0\}$ is non–degenerate, $i = 1, 2$. In that case $\operatorname{par}(P) = \operatorname{par}(P_{|W_1}) + \operatorname{par}(P_{|W_2})$, which is a consequence of the proof of Corollary 3.6.

2. Notice that Corollary 3.6 allows to compute the parity of an indecomposable element $P \in \mathsf{Sp}(n-1, \omega)$ unless P is unipotent. An indecomposable unipotent element $P \in \mathsf{Sp}(n-1, \omega)$ can be represented with respect to a symplectic basis by a matrix of the following form

$$\left(\begin{array}{cccc|cccc} 1 &&&&&&& \\ 1 & 1 &&&&&& \\ & \ddots & \ddots &&&&& \\ && 1 & 1 &&&& \\ \hline &&&& 1 & -1 && \\ &&&&& \ddots & \ddots & \\ &&&&&& 1 & -1 \\ &&& \sigma &&&& 1 \end{array}\right),$$

where all four blocks are $(n-1) \times (n-1)$ matrices, missing entries indicate a zero and where $\sigma \in \{-1, 0, 1\}$ [compare Ballmann, Thorbergsson and Ziller, 1982]; actually the case $\sigma = 0$ can only occur if $n-1$ is odd. Using the proof of Proposition 3.1, it is not hard to verify that

$$\operatorname{par}(P) \equiv \begin{cases} 1 & \text{if } \sigma \geq 0 \\ 0 & \text{if } \sigma < 0. \end{cases}$$

3. In all statements of this section the assumption that M is oriented can be replaced by the hypothesis that the holonomy transport along c is orientation preserving. Moreover it is easy see what happens in the case of an orientation reversing holonomy transport. Then the index of c in the free loop space is even if and only if $\operatorname{par}(P) \equiv 1 \bmod 2$.

3.1. Proof of Proposition 3.2. Set $h := \dim(\mathsf{Sp}(n-1, \omega)) = (2n-1)(n-1)$. We let

$$\operatorname{Map}_0\big([0, \lambda], \operatorname{Sym}(n-1)\big)$$

denote the vectorspace of smooth mappings $[0, \lambda] \to \operatorname{Sym}(n-1)$ vanishing in a neighborhood of the set $\{0, \lambda\}$ in $[0, \lambda]$. For $\mathrm{R}_1, \mathrm{R}_2 \in \operatorname{Map}_0\big([0, \lambda],$

$\mathrm{Sym}(n-1)\big)$ we can define an element $\mathrm{R}_1 \star \mathrm{R}_2$ in $\mathrm{Map}_0\big([0, 2\lambda], \mathrm{Sym}(n-1)\big)$ by using the map R_1 on the first half and R_2 on the second half of the interval $[0, 2\lambda]$. Clearly, the Poincare map of $\mathrm{R}_1 \star \mathrm{R}_2$ is the composition $P_2 \circ P_1$ of the Poincare maps of R_2 and R_1.

Lemma 3.8. *There are maps*

$$S_0, S_1, \dots, S_h \in \mathrm{Map}_0\big([0, 1], \mathrm{Sym}(n-1)\big)$$

such that if $P(s_1, \dots, s_h)$ *denotes the Poincare map of the curvature tensor*

$$S_0(t) + s_1 S_1(t) + \cdots + s_h S_h(t), \qquad t \in [0, 1],$$

then $P\colon (-1, 1)^h \to \mathsf{Sp}(n-1, \omega)$ *is a diffeomorphism onto a neighborhood* U *of* $\mathrm{id} \in \mathsf{Sp}(n-1, \omega)$.

Before proving the lemma we use it to complete the proof of Proposition 3.2. In the proof of the lemma we will first establish the special case $n = 2$. Finally, we use the special case $n = 2$ of Proposition 3.2, to prove the lemma in full generality.

Evidently, for some large integer k there are smooth curves $c_1, \dots, c_k$: $[0, 1] \to U$ such that $P_s = c_1(s) \circ \cdots \circ c_k(s)$. Combining the above observations it is easy to find a smooth family of maps

$$\tilde{\mathrm{R}}_s \in \mathrm{Map}_0\big([0, k], \mathrm{Sym}(n-1)\big)$$

such that the Poincare map of $\tilde{\mathrm{R}}_{s|[0,k]}$ is P_s, $s \in [0, 1]$. Notice that in general the Poincare maps of $k^2\tilde{\mathrm{R}}_s(kt)$ is not the same as the Poincare map of $\tilde{\mathrm{R}}_s$. However, the above statement implies that for some positive integer l we can find maps

$$\bar{\mathrm{R}}_1, \bar{\mathrm{R}}_2 \in \mathrm{Map}_0\big([0, l], \mathrm{Sym}(n-1)\big)$$

whose Poincare maps are

$$\bar{P} = \Big(\tfrac{1}{\sqrt{k+2}}\,\mathrm{id}\Big) \times \sqrt{k+2}\,\mathrm{id} \quad \text{and} \quad \bar{P}^{-1}.$$

The maps $l^2\bar{\mathrm{R}}_1(lt)$ and $l^2\bar{\mathrm{R}}_2(lt)$, $t \in [0, 1]$, have the same Poincare maps, as $\bar{\mathrm{R}}_1(lt)$ and $\bar{\mathrm{R}}_2(lt)$. Hence, we may assume $l = 1$. Then

$$\hat{\mathrm{R}}_s = \bar{\mathrm{R}}_1 \star \tilde{\mathrm{R}} \star \bar{\mathrm{R}}_2 \in \mathrm{Map}_0\big([0, k+2], \mathrm{Sym}(n-1)\big)$$

has the Poincare map $\bar{P}^{-1} P_s \bar{P}$. Thus the family

$$\mathrm{R}_s(t) := (k+2)^2\hat{\mathrm{R}}_s((k+2)\cdot t), \quad t, s \in [0, 1]$$

is a solution of the problem.

Proof of Lemma 3.8. We first consider the special case $n = 2$. Put

$$T_0(t) := 4\pi^2, \qquad T_1(t) := 1 + 2\cos(4\pi t),$$
$$T_2(t) := 4\pi^2(1 - 2\cos(4\pi t)), \quad T_3(t) := 8\pi \sin(4\pi t), \; t \in [0, 1].$$

The Poincare map $\hat{P}(s_1, s_2, s_3) \in \mathsf{Sp}(1) = \mathrm{SL}(2, \mathbb{R})$ of

$$T_0(t) + s_1 T_1(t) + s_2 T_2(t) + s_3 T_3(t)$$

fulfills $\hat{P}(0) = \mathrm{id}$ and

$$\frac{\partial}{\partial s_1}\hat{P}(0) = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial s_2}\hat{P}(0) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \frac{\partial}{\partial s_3}\hat{P}(0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where the right hand sides represent elements in the Lie algebra $\mathfrak{sl}(2, \mathbb{R})$ of $\mathrm{SL}(2, \mathbb{R})$. By the inverse function theorem, there is a positive number δ such that

$$\hat{P} \colon (-2\delta, 2\delta)^3 \to \mathrm{SL}(2, \mathbb{R})$$

is a diffeomorphism onto a neighborhood V of $\mathrm{id} \in \mathrm{SL}(2, \mathbb{R})$. Next we choose for each integer $i > 2$ a smooth function $\psi_i \colon [0, 1] \to [0, 1]$ satisfying $\psi_i(t) = 1$ for $t \in \left[\frac{1}{i}, \frac{i-1}{i}\right]$ and vanishing in a neighborhood of $\{0, 1\}$ in $[0, 1]$. Let $\hat{P}_i(s_1, s_2, s_3)$ denote the Poincare map of

$$\psi_i(t)\big(T_0(t) + s_1 T_1(t) + s_2 T_2(t) + s_3 T_3(t)\big).$$

It is straightforward to check that the sequence $\hat{P}_i$ converges on compact sets uniformly to $\hat{P}$. Moreover the same holds for the differential of $\hat{P}_i$. Consequently, $\hat{P}_i \colon (-\delta, \delta)^3 \to \mathrm{SL}(2, \mathbb{R})$ is a diffeomorphism onto a neighborhood U of $\mathrm{id} \in \mathrm{SL}(2, \mathbb{R})$ for a sufficiently large number i. The statement of the lemma follows with $S_0(t) := \psi_i(t) T_0(t)$ and $S_j(t) := \delta \cdot \psi_i(t) T_j(t)$, $j = 1, 2, 3$.

For the general case we notice that each vector $v \in V$ gives rise to an embedding

$$\mathrm{SL}(2, \mathbb{R}) \hookrightarrow \mathsf{Sp}(n-1, \omega)$$

such that this copy of $\mathrm{SL}(2, \mathbb{R})$ acts trivially on the orthogonal complement of $\mathrm{span}_{\mathbb{R}}((v, 0), (0, v))$ in $V \times V$. It is an immediate consequence of the special case $n = 2$ of Proposition 3.2 that any element in this copy of $\mathrm{SL}(2, \mathbb{R})$ can be realized as the Poincare map of a curvature tensor $\mathrm{R} \in \mathrm{Map}_0([0, 1], \mathrm{Sym}(n-1))$. Since $\mathsf{Sp}(n-1, \omega)$ is generated by all these copies of $\mathrm{SL}(2, \mathbb{R})$, it follows that any element $P \in \mathsf{Sp}(n-1, \omega)$ can be realized as a Poincare map of a curvature tensor $\mathrm{R} \in \mathrm{Map}_0([0, k], \mathrm{Sym}(n-1))$, where k is a sufficiently large integer. Using the same trick as in the proof of Proposition 3.2 we see that we may choose $k = 1$ or more generally that we can replace k by any positive real number.

Applying the special case $n = 2$ of the lemma, we can find elements $T_0, T_1 \in \mathrm{Map}\big([0, \frac{1}{3h}], \mathrm{Sym}(n-1)\big)$ such that if $\bar{P}(s)$ denotes the Poincare map of $T_0 + sT_1$, then $\bar{P}(0) = \mathrm{id}$ and the tangent vector $\dot{\bar{P}}(0)$ in the Lie algebra $\mathfrak{sp}(n-1, \omega)$ does not vanish. Since $\mathsf{Sp}(n-1, \omega)$ is a simple Lie group, there are elements $P_1, \dots, P_h \in \mathsf{Sp}(n-1, \omega)$ such that the vectors $P_j \dot{\bar{P}}(0) P_j^{-1}$ $(j = 1, \dots, h)$ form a basis of $\mathfrak{sp}(n-1, \omega)$. As explained before, there are elements $U_j, V_j \in \mathrm{Map}\big([0, \frac{1}{3h}], \mathrm{Sym}(n-1)\big)$ whose Poincare maps are P_j^{-1} and P_j. Set

$$S_0 := U_1 \star T_0 \star V_1 \star \cdots \star U_h \star T_0 \star V_h,$$

$$S_j(t) = \begin{cases} 0 & \text{for } t \notin \left[\frac{3j+1}{3h}, \frac{3j+2}{3h}\right] \\ T_1\big(t - \frac{3j+1}{3h}\big) & \text{for } t \in \left[\frac{3j+1}{3h}, \frac{3j+2}{3h}\right] \end{cases},$$

$j = 1, \dots, h$ and define $P(s_1, \dots, s_h)$ as in the lemma. It is straightforward to check $P(0) = \mathrm{id}$ and $\frac{\partial}{\partial s_j} P(0) = P_j \dot{\bar{P}}(0) P_j^{-1}$. Thus P is a local diffeomorphism in a neighborhood of 0 and the lemma follows after scaling $S_1, \dots, S_h$ by a suitable small factor. □

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