Münster Oberseminar

Differential Geometry

Monday, June 7, 2021

d_p Convergence and ε-Regularity Theorems

for Entropy and Scalar Curvature Lower Bounds

joint w/ Man-Chun Lee and Aaron Naber

Theme:

(M^n, g) Riemannian manifold.

Put some constraints on the curvature $\rightsquigarrow$ What structure does this impose on (M, g)?

To study:

Take sequence (M_i, g_i) satisfying constraint. look at limit

$$M_i \longrightarrow X$$

In what sense?

What structure does X have?

$\underline{\text{Ex}}$ Spaces with Ricci lower bounds

(M_i, g_i) sat. $Ric_i \geq -\lambda g_i$

$(M_i, d_i) \xrightarrow{GH} (X, d)$ Cheeger - Colding.

ε-Regularity Thm (Colding '97, Cheeger-Colding '97)

Fix $\varepsilon > 0$. There exists $\delta = \delta(\varepsilon, n) > 0$ such that if (M, g) Riemannian manifold, $x \in M$ w/

$$\mathrm{Ric}_g \geq -\delta g \qquad \mathrm{vol}_g(B(x,1)) \geq (1-\delta)\,\omega_n$$

Then $d_{GH}(B(x,1), B(0^n,1)) \leq \varepsilon.$

In fact, GH can be improved to biHölder homeomorphic by iterating on all scales.

To summarize:

Almost Euclidean Ricci lower bound + Noncollapsing – Almost Euclidean volume

$\Downarrow$

Geometrically close to Euclidean

Our goal today:

ε regularity assuming only scalar curvature lower bounds.

Almost Euclidean <u>scalar</u> lower bound

$R_g \geq -\delta$

+

"Noncollapsing" - Almost Euclidean Perelman entropy

$\Downarrow$

"Geometrically close to Euclidean"

$\leftarrow$ Not in any metric space sense!

In "d_p sense."

Background — Perelman Entropy $\mu(g,\tau)$. $\tau > 0$

Optimal constant in family of log-Sobolev inequalities.

Def

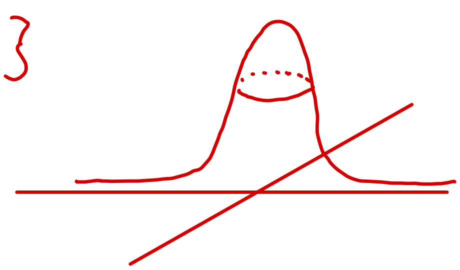
(a) $$\mu(g,1) = \inf\left\{ \widetilde{W}(u) \; : \; \int_M u^2 \, dvol_g = 1 \right\}$$

$$\widetilde{W}(u) = \int_M 4|\nabla u|^2 + R_g u^2 - u^2 \log u^2 \, dvol_g - n + \log(4\pi)^{-\frac{n}{2}}$$

cf: Log Sobolev on Euclidean space $\iff$ $\mu(g_{euc}, 1) = 0$.

$$\int u^2 \log u^2 \leq \int 4|\nabla u|^2 \, dx - n + \log(4\pi)^{-n/2} \qquad \text{if } \int u^2 = 1.$$

with $=$ $\iff$ $u^2 = (4\pi)^{-n/2} \exp\{-|x-x_0|^2/4\}$



(b) $$\mu(g,\tau) = \mu(\tau^{-1} g, 1)$$

Some key perspective notes

(M, g) complete with bounded curvature.

- $$\mu(g, \tau) \leq 0 \quad \text{for all} \quad \tau > 0$$

  $$\text{with equality} \iff (M, g) = (\mathbb{R}^n, g_{euc})$$

- $$\mu(g, \tau) \geq -\delta \ \text{ for } \tau \in (0, 1], \qquad R_g \geq -\delta$$

  $$\Rightarrow \quad \mathrm{vol}_g(B_g(x, r)) \geq (1 - \varepsilon) r^n \omega_n$$

  $$\forall \ r \in (0, 1].$$

We will assume

$$\boxed{\mu(g, \tau) \geq -\delta \quad \text{for} \quad \tau \in (0, 1]}$$

Now let's turn to understanding

"Geometrically close to Euclidean"

Look at limits of sequences (M_i, g_i) satisfying

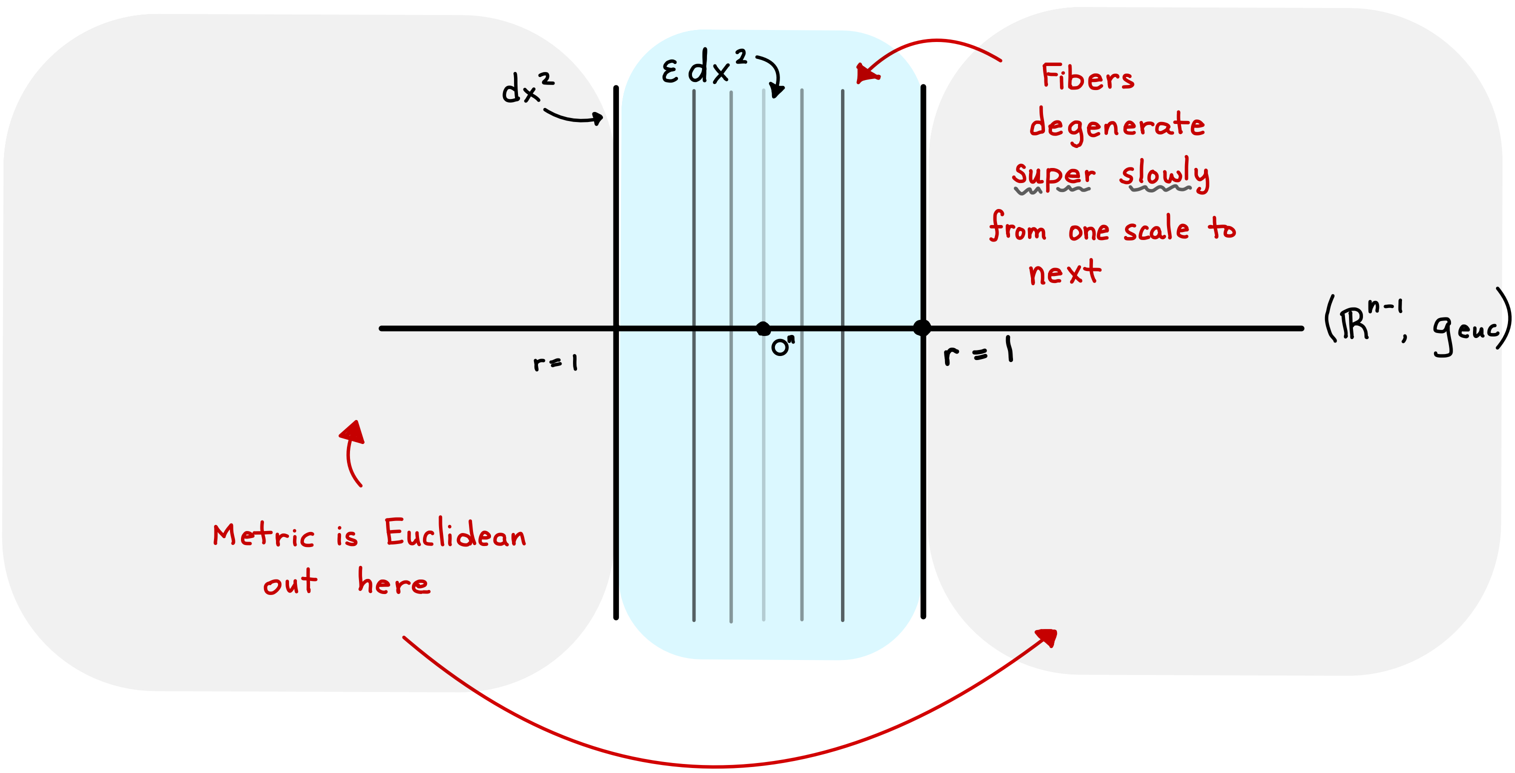
$$R_{g_i} \geq -1/i, \qquad \mu(g_i, \tau) \geq -1/i \quad \forall \ \tau \in (0, 1]$$

In what sense do they converge to $(\mathbb{R}^n, g_{euc})$?

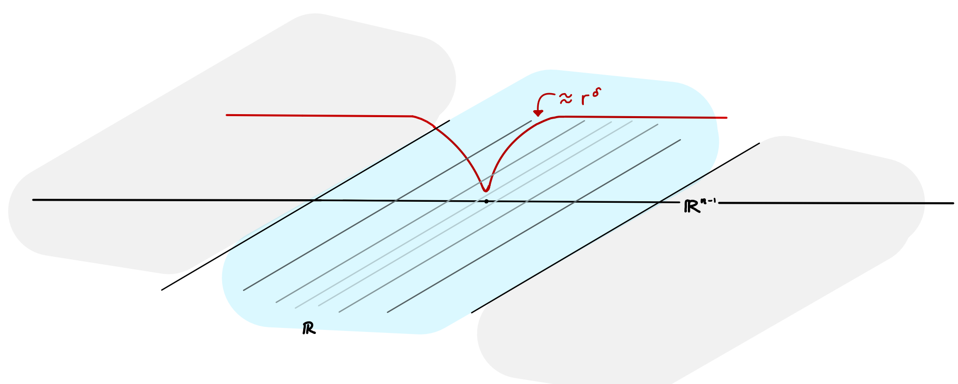
A key point:

Their metric space structures need not converge!

Ex 1: Basic Idea: Metric g_ε on $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$



Can make this change slow enough such that



- $R_{g_\varepsilon} \geq -c(\varepsilon) \to 0$,
- $\mu(g_\varepsilon, \tau) \geq -c(\varepsilon) \to 0$

  $\forall\ \tau \in (0,1]$

White lie: To get scalar lower bound, we need very flat cone metric on $\mathbb{R}^{n-1}$

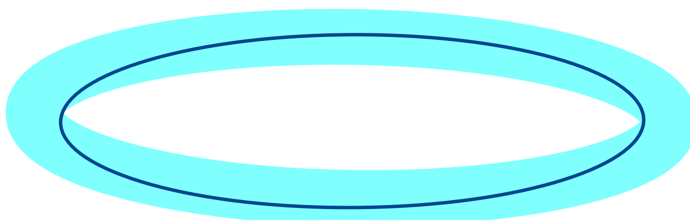
$n \geq 4$.

But In the pointed GH limit, central line collapses to a point.

and

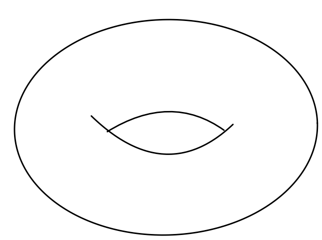
$$\mathrm{vol}_{g_\varepsilon}(B_{g_\varepsilon}(0,1)) \longrightarrow \infty$$

Note: After rescaling/chopping/$\mathbb{Z}$ action, we get

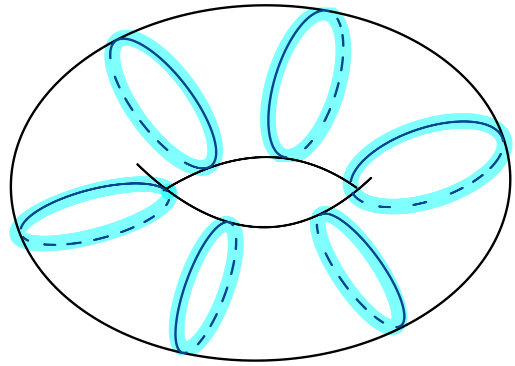


Ex 2:

① Take a flat torus (T^n, g_{flat})



② Obtain (T^n, g_i) by pasting increasingly dense collection of disjoint strips along parallel copies of S^1

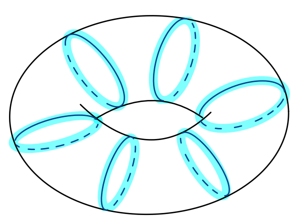


③ Can choose parameters such that

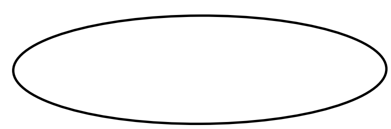
$$R_{g_i} \geq -1/i \qquad \mu(g_i, \tau) \geq -\delta_0 \qquad \forall\ \tau \in (0, 1]$$

But

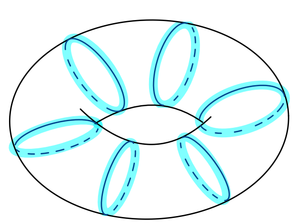
Gromov - Hausdorff limit is (T^{n-1}, g_{flat})



$\xrightarrow{GH}$



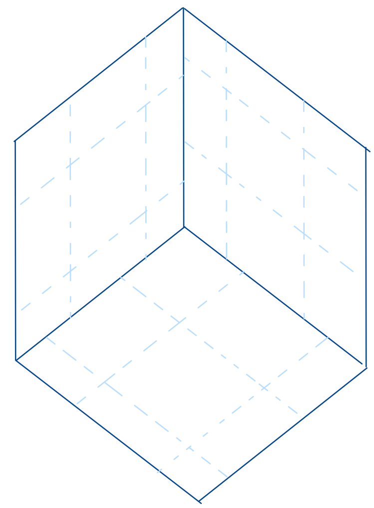
Intrinsic flat limit is zero current



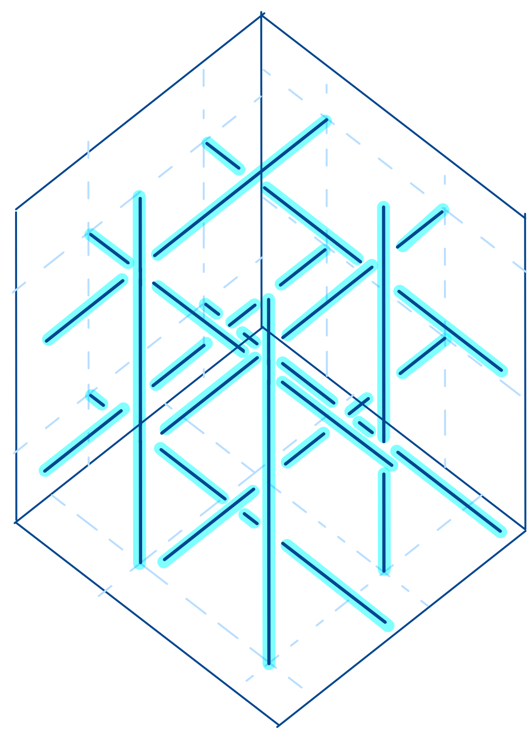
$\xrightarrow{IF}$ 0 - current.

Ex 3:

① Take a flat torus (T^n, g_{flat})



② Obtain (T^n, g_i) by pasting increasingly dense collection of disjoint strips along copies of S^1 parallel to coordinate directions

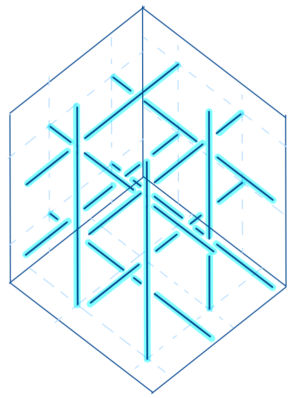


③ Can choose parameters such that

$$R_{g_i} \geq -1/i \qquad \mu(g_i, \tau) \geq -\delta_0 \qquad \forall\ \tau \in (0,1]$$

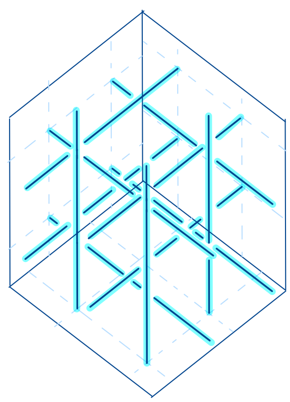
But

Gromov - Hausdorff limit is a point



$\xrightarrow{GH}$ $\bullet$

Intrinsic flat limit is zero current



$\xrightarrow{IF}$ 0 - current.

Upshot : under scalar curvature and entropy lower bounds,

<u>distance functions can</u>

<u>behave very poorly.</u>

$p \in \mathbb{R}, \quad p > n$

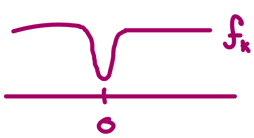
New notion: **d_p distance.**

$$d(x,y) = \sup\{ f(x) - f(y) \;:\; \|\nabla f\|_{L^\infty} \le 1 \}$$

Def: (M, g) Riemannian manifold, $x, y \in M$, $p > n$.

(a) $$d_p(x,y) = \sup\left\{ |f(x) - f(y)| \;:\; \int_M |\nabla f|^p \, dvol_g \le 1 \right\}.$$

(b) $$B_p(x, r) = \{ y \in M \;:\; d_p(x,y) < r \}$$



$f_k \to 1$ a.e. $\qquad f_k \to 1$ in L^p.

$f_k \not\to 1$ in L^∞

Some key perspective notes

- $d_\infty(x,y) = d(x,y)$ and $\lim_{p \to \infty} d_p(x,y) = d(x,y)$
- d_p reflects behavior of $W^{1,p}$ Sobolev space

- On Euclidean space,

$$d_p(x,y) = S_{p,n} |x-y|^{1-n/p}.$$

- (M,g), (N,h) cpt Riemannian manifolds.

If $(M, d_{p,g})$, $(N, d_{p,h})$ isometric as metric spaces

then (M,g), (N,h) isometric. as Riem. mflds.

Thm (Lee - Naber - N. '20)

Fix $n \geq 2$, $p \geq n+1$, $\varepsilon > 0$. There exists $\delta = \delta(n,p,\varepsilon)$ such that if (M^n, g) complete with bounded curvature with

$$R_g \geq -\delta, \qquad \mu(g,\tau) \geq -\delta \quad \forall\, \tau \in (0,1]$$

then for any $x \in M$,

- $d_{GH}\Big((\mathcal{B}_{p,g}(x,1), d_{p,g}), (\mathcal{B}_{p,g_{euc}}(0^n,1), d_{p,euc}) \Big) \leq \varepsilon,$

- $$1-\varepsilon \leq \frac{\mathrm{vol}_g(\mathcal{B}_{p,g}(x,r))}{\mathrm{vol}_{g_{euc}}(\mathcal{B}_{p,g_{euc}}(0,r))} \leq 1+\varepsilon \quad \forall\, r \in (0,1).$$

Thm (Lee - Naber - N. '20)

Fix $n \geq 2$, $q < 1$, $\varepsilon > 0$. There exists $\delta = \delta(n, q, \varepsilon)$ such that <u>if</u> (M^n, g) closed with

$$R_g \geq -\delta, \qquad \mu(g, \tau) \geq -\delta \quad \forall \tau \in (0,1]$$

<u>then</u>

$$\boxed{\int_M |R_g|^q \, dvol_g \leq \varepsilon}$$

Schoen Yau: If (T^n, g) has $R_g \geq 0$, then $g = g_{flat}$.

Thm (Lee - Naber - N. '20)

Fix $n \geq 2$, $p \geq n+1$, $V > 0$. There exists $\delta = \delta(n, p)$ s.t. if (T^n, g_i) satisfy

$$R_{g_i} \geq -1/i, \qquad \mu(g_i, \tau) \geq -\delta \quad \forall \tau \in (0,1], \qquad Vol_{g_i}(M_i) \leq V.$$

Then up to subsequence,

$$\boxed{(M_i, g_i) \longrightarrow (T^n, g_{flat}) \quad \text{in } d_p \text{ sense.}}$$

Limit spaces are NOT metric spaces, they are

Rectifiable Riemannian spaces (X, g)

(X, m) topological space with a Borel measure, equipped with

- Atlas of charts with biLipschitz transition maps covering X up to set of measure zero.

- Possibly singular metric g defined in charts.

This is enough structure to define $W^{1,p}$.

Thm (Lee - Naber - N. '20)

Fix $n \geq 2$, $p \geq n+1$. There exists $\delta = \delta(n,p)$ such that <u>if</u> $\{(M_i, g_i, x_i)\}$ complete with bounded curvature with

$$R_{g_i} \geq -\delta, \qquad \mu(g_i, \tau) \geq -\delta \quad \forall\, \tau \in (0,1]$$

<u>then</u> up to subsequence,

$$(M_i, g_i, x_i) \longrightarrow (X, g, x) \quad \text{in pointed } d_p \text{ sense}$$

Some subtlety to defining this

where (X, g, x) is a rectifiable Riemannian space that is

- $W^{1,p}$ - rectifiably complete

  ($W^{1,p}$ space is "big" and "well - behaved")

- d_p - complete

  (d_p is metric generating same topology as X)

- X is a smooth topological manifold.

THANK YOU!