Uniqueness among scalar-flat toric metrics on non-compact surfaces

Rosa Sena-Dias

IST, Lisbon

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- Manifolds with metrics are the objects of Riemannian geometry which is more concrete and better understood than differential geometry. Is it ok to replace differential geometry with Riemannian geometry?
- Not unless we have god-given metrics.
- Finding canonical metrics motivates a lot of what we do in Riemannian geometry and it fuels some of what I am going to say today.

Riemann surfaces

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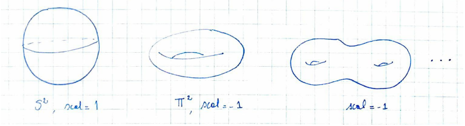

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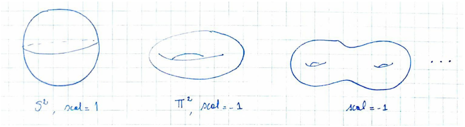

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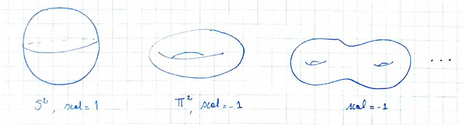

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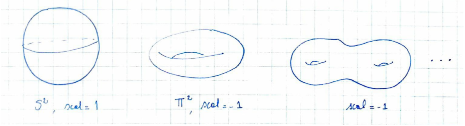

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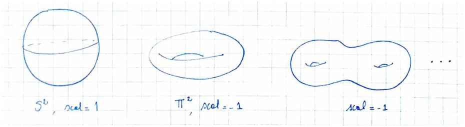

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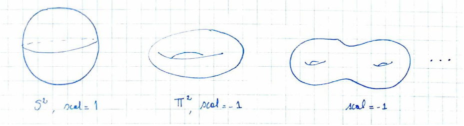

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- They are not always unique but are unique up to automorphism.

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- All Riemann surfaces and algebraic smooth varieties in $\mathbb{CP}^n$ are Kähler. So there are plenty of examples.

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- The set of such Kähler forms is called the Kähler class.
- Metrics corresponding to a Kähler class are parametrised by a function. It is much easier to find a god-given function than a god-given symmetric positive 2-tensor.

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- YTD conjecture: A Kähler manifold admits a cscK metric $\Longleftrightarrow$ it is K-polystable.

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- Even if one is interested only in the compact case, it is useful to understand the non-compact case. Let me explain why.

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- Using either method, even if there isn't convergence we can sometimes get information. Particularly if we can control how divergence occurs. This was observed by Uhlenbeck who discovered bubbling.

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Theorem (Joyce)

Let Γ be a discrete subgroup of $SU(m)$ acting freely with isolated fixed points on $\mathbb{C}^m$. Then there is a unique Krf metric on the minimal resolution of $\mathbb{C}^m/\Gamma$ which is ALE.

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such that

$$\left| \nabla^k \left(\pi_* g - g_{eucl} \right) \right|_{g_{eucl}} \leq R^{-m-k}.$$

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The above definition can be modified for spaces that are asymptotic fibrations over $\mathbb{R}^2$ or over $\mathbb{R}$. This way one gets the definitions for ALG or ALH.

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- To sum up: in the non-compact case the uniqueness question seems harder than existence.
- In particular all the results we have on uniqueness, require some restriction on behaviour at infinity.

Toric sf surfaces

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- There is also a uniqueness result given asymptotic behaviour.

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Definition

A symplectic manifold (X^{2m}, ω) *is toric if it admits an effective, Hamiltonian action of* $\mathbb{T}^m$.

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Let (X^{2m},ω) be a compact toric manifold. Then $\mu(X)$ is a convex polytope and it determines (X^{2m},ω).

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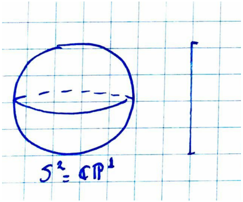
- We say that a non-compact toric surface is strictly non-compact if ν_1 and ν_d are independent.

Examples: 2d



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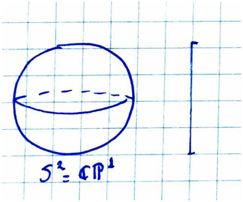

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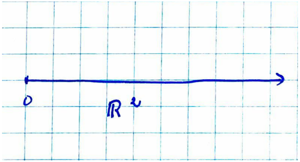
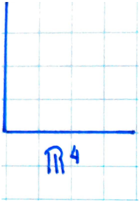


$$\mu(z) = |z|^2$$

Examples: 4d



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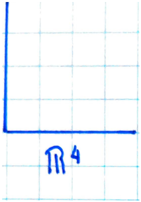

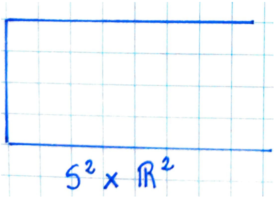

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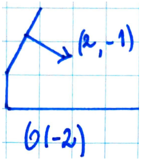
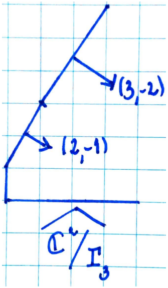
More examples: 4d





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The construction in [AS] really amounted to choosing the right boundary behaviour for ξ so that the $u - u_G$ is smooth and the resulting metrics are complete.

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The idea is is to use this fact together with our knowledge on harmonic functions to show uniqueness.

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- The upshot is $\xi_{ALE} - \xi$ is now axi-symmetric harmonic and smooth.

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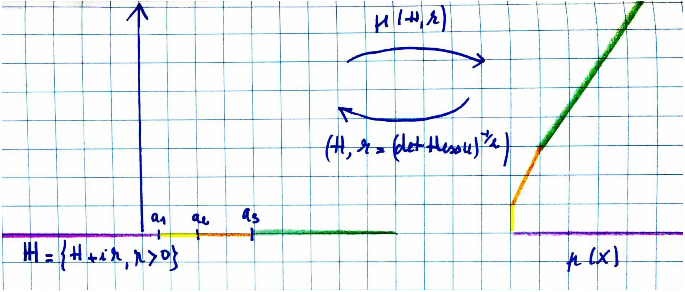

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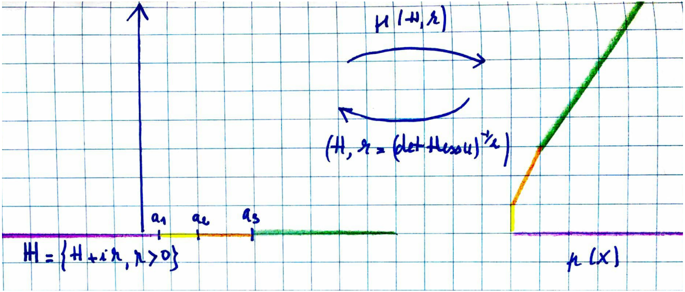

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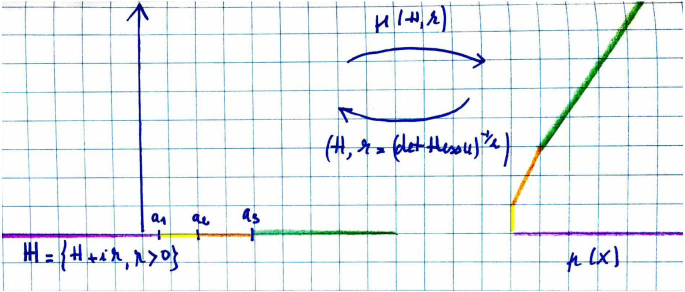

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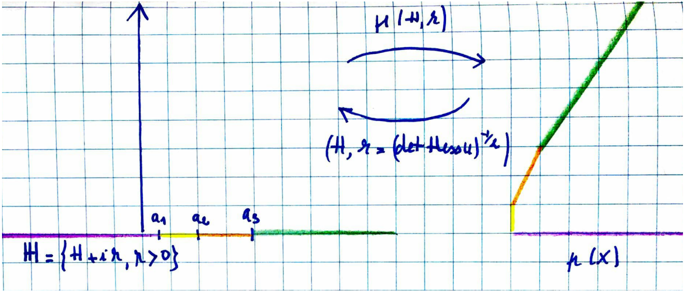

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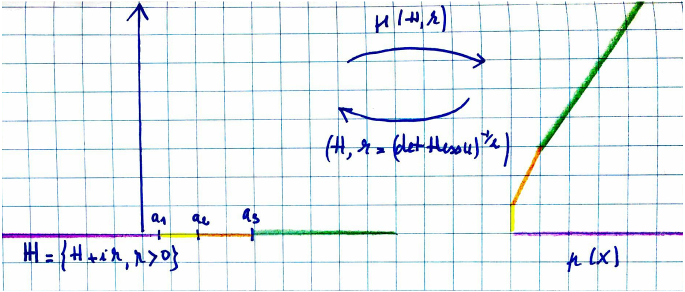

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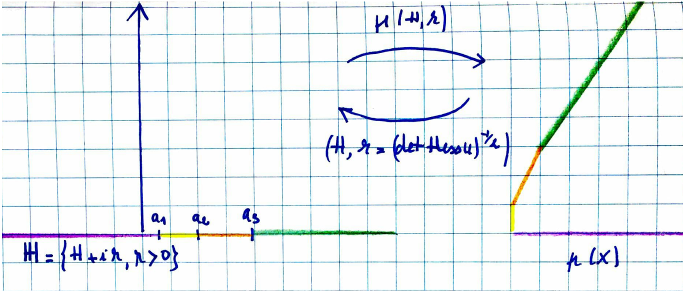

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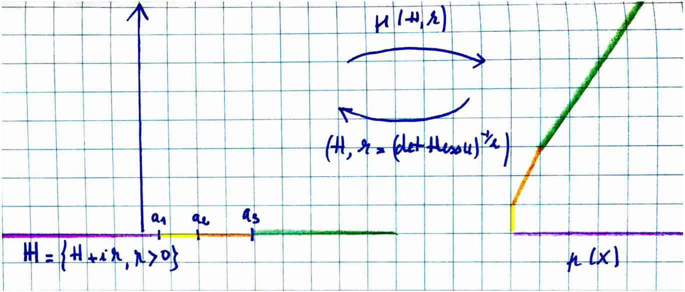

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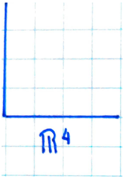

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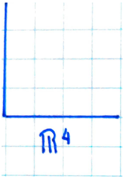

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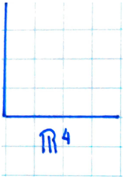

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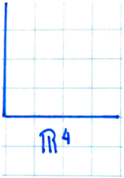

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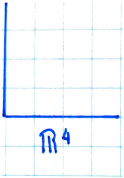

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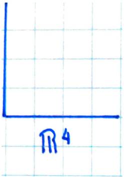

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- which in turn can be calculated in (H, r) coordinates and is proportional to $a_i - a_{i-1}$.

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- It follows from the smoothness of $\xi_{ALE} - \xi$ that $\mu_{ALE} - \mu$ vanish to second order on $\partial\mathbb{H}$.

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- This lemma is easy to prove but crucial.

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- Because v_1 and v_d are independent this then implies that f is constant. Here are the 2 parameters.

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- Higher dimensions is harder because we no longer have (H, r) coordinates. Perhaps the ALE case is doable?

Thank you!