A LIE ALGEBRAIC APPROACH TO RICCI FLOW INVARIANT CURVATURE CONDITIONS AND HARNACK INEQUALITIES

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ABSTRACT. We consider a subset S of the complex Lie algebra $\mathfrak{so}(n,\mathbb{C})$ and the cone $C(S)$ of curvature operators which are nonnegative on S. We show that $C(S)$ defines a Ricci flow invariant curvature condition if S is invariant under $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$. The analogue for Kähler curvature operators holds as well. Although the proof is very simple and short it recovers all previously known invariant nonnegativity conditions. As an application we reprove that a compact Kähler manifold with positive orthogonal bisectional curvature evolves to a manifold with positive bisectional curvature and is thus biholomorphic to $\mathbb{CP}^n$. Moreover, the methods can also be applied to prove Harnack inequalities.

We consider a Lie algebra $\mathfrak{g}$ endowed with a scalar product $\langle \cdot, \cdot \rangle$ which is invariant under the adjoint representation of the Lie algebra. The reader should think of $\mathfrak{g}$ either as the space $\mathfrak{so}(n)$ of skew adjoint endomorphism of $\mathbb{R}^n$ with the scalar product $\langle A, B\rangle = -\frac{1}{2}\operatorname{tr}(AB)$ or of the Lie subalgebra $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$ corresponding to the unitary group $\mathsf{U}(n) \subset \mathrm{SO}(2n)$ endowed with the induced scalar product.

We consider the space of selfadjoint endomorphisms of $S^2(\mathfrak{g})$. Every selfadjoint endomorphism $R \in S^2(\mathfrak{g})$ is determined by the corresponding bilinear form $(x, y) \mapsto \langle Rx, y\rangle$. The extension of this form to a complex bilinear form

$$R\colon \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \times \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$$

will be denoted with the same letter R. Notice that for any $x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ the number $R(x, \bar{x})$ is real, where $x \mapsto \bar{x}$ is complex conjugation.

Also recall that the space of algebraic curvature operators $S^2_B(\mathfrak{so}(n))$ is a linear subspace of $S^2(\mathfrak{so}(n))$. Similarly the space of algebraic Kähler curvature operators $S^2_K(\mathfrak{u}(n))$ is a linear subspace of $S^2(\mathfrak{u}(n))$. The subspaces are also invariant under the Ricci flow ODE on $S^2(\mathfrak{g})$

$$R' = R^2 + R^{\#}$$

where $\langle R^{\#}x, y\rangle = -\frac{1}{2}\operatorname{tr}(\mathrm{ad}_x R\, \mathrm{ad}_y R)$ for $x, y \in \mathfrak{g}$. We have the following basic result

Theorem 1. *Let S be a subset of the complex Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and let $\mathsf{G}_{\mathbb{C}}$ denote a Lie group with Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. If S is invariant under the adjoint representation of $\mathsf{G}_{\mathbb{C}}$, then for $h \in \mathbb{R}$ the set*

$$C(S, h) := \{R \in S^2(\mathfrak{g}) \mid R(v, \bar{v}) \geq h \text{ for all } v \in S\}$$

is invariant under the ODE $R' = R^2 + R^{\#}$.

In many cases S is scaling invariant and then $h = 0$ is the only meaningful choice. For $h = 0$ the set $C(S) := C(S, 0)$ is a cone and the curvature condition $C(S)$ can be thought of as a nonnegativity condition. We recall that for a $O(n)$-invariant subset $C \subset S^2(\mathfrak{so}(n))$ we say that a manifold satisfies C if the curvature operator at each point is in $C \cap S^2_B(\mathfrak{so}(n))$. Although the proof of the theorem is just a few lines long its statement recovers via Hamilton's maximum principle [1986] the invariance of all previously known invariant nonnegativity conditions:

Remark 2. In all of the following examples we assume $h = 0$.

a) In the case of $S = \mathfrak{so}(n, \mathbb{C})$ the theorem recovers the invariance of the cone $C(S)$ of nonnegative operators – a result due to Hamilton.

b) In the case of $\mathfrak{g} = \mathfrak{so}(n, \mathbb{R})$ and $S = \{X \in \mathfrak{so}(n, \mathbb{C}) \mid \mathrm{tr}(X^2) = 0\}$ the theorem recovers the invariance of the cone $C(S)$ of 2 nonnegative operators. This result is also due to Hamilton.

c) The invariance of nonnegative isotropic curvature, which was shown independently by Nguyen [2007,2010] and Brendle and Schoen [2009], can be seen by setting $\mathfrak{g} = \mathfrak{so}(n)$ and

$$S := \{X \in \mathfrak{so}(n, \mathbb{C}) \mid \mathrm{rank}(X) = 2, X^2 = 0\}.$$

The equation $X^2 = 0$ is equivalent to saying that each vector v in the image of X is isotropic, i.e., the imaginary part and the real part are perpendicular and have the same norm. It is easy to see that $C(S) \cap S^2_B(\mathfrak{so}(n))$ is indeed the space of curvature operators with nonnegative isotropic curvature.

d) The invariance of the condition that the manifold crossed with $\mathbb{R}$ has nonnegative isotropic curvature due to Brendle and Schoen corresponds to

$$S := \{X \in \mathfrak{so}(n, \mathbb{C}) \mid \mathrm{rank}(X) = 2, X^3 = 0\}.$$

e) The invariance of the condition that the manifold crossed with $\mathbb{R}^2$ has nonnegative isotropic curvature due to Brendle and Schoen corresponds to

$$S := \{X \in \mathfrak{so}(n, \mathbb{C}) \mid \mathrm{rank}(X) = 2\}.$$

It was then observed by Ni and Wolfson [2008] that M satisfies $C(S)$ if and only if M has nonnegative complex curvature. Ni and Wolfson also gave a simpler proof that positive complex curvature is invariant under the Ricci flow. For the author this simplification was one indication that proofs should be simpler in the complex setting.

This invariance was the key new result in the proof of the differentiable quarter pinched sphere theorem of Brendle and Schoen [2009]. The convergence of the metric under Ricci flow toward constant curvature then followed from [Böhm and Wilking, 2008], see also subsection 4.1.

f) The invariance of nonnegative bisectional curvature due to Mok [1988] can be recovered from the theorem as well. The Lie algebra $\mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$ can be naturally identified with the algebra of complex $n \times n$ matrices $\mathfrak{gl}(n, \mathbb{C})$.

If we put

$$S = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \mathrm{rank}(X) = 1\},$$

one can check by straightforward computation that $C(S) \cap S^2_K(\mathfrak{u}(n))$ is given by the cone of Kähler curvature operators with nonnegative bisectional curvature. We would like to emphasize that Mok's proof of the invariance used a second variation argument for the first time in this context. The proof of the invariance of nonnegative isotropic curvature by Nguyen [2007,2010] and Brendle and Schoen [2009] also relied on second variation. The same is true for the proof of Theorem 1.

g) The theorem also shows the invariance of orthogonal bisectional curvature, if we put

$$S := \{X \in \mathfrak{gl}(n,\mathbb{C}) \mid \operatorname{rank}(X) = 1, X^2 = 0\}.$$

The invariance was announced by Hamilton and H.D. Cao in the early 90s and a proof was given by Gu and Zhang [2010].

The theorem can also be generalized to obtain Harnack inequalities: Let $(M, g(t))$ be a solution to the Ricci flow. We endow the Lie algebra $\mathfrak{g}(p,t)$ of the isometry group of $\operatorname{Iso}(T_pM, g(t))$ with a scalar product $((A,v),(B,w)) = -\frac{1}{2}\operatorname{tr}(AB) + g(t)(v,w)$ for skew adjoint endomorphisms A, B of $(T_pM, g(t))$ and $v, w \in T_pM$. The Harnack operator Hm can be viewed as a self adjoint endomorphism of $\mathfrak{g}(p,t)$. As a consequence of Hamilton's work [1993] the Harnack operator satisfies (cf. section 2) with respect to moving frames an evolution equation of the form

$$\mathrm{Hm}' = \Delta\,\mathrm{Hm} + 2(\mathrm{Hm}\,\mathrm{pr}\,\mathrm{Hm} + \mathrm{Hm}^{\#}) + \tfrac{2}{t}\,\mathrm{Hm}$$

Here $\mathrm{pr}\colon \mathfrak{g}(p,t) \to \mathfrak{so}(T_pM)$ denotes the orthogonal projection and

$$\langle \mathrm{Hm}^{\#}(x), y\rangle = -\tfrac{1}{2}\operatorname{tr}(\mathrm{ad}_{\cdot}^{tr}\, x\,\mathrm{Hm}\,\mathrm{ad}_{\cdot}^{tr}\, y\,\mathrm{Hm}),$$

where $\langle \mathrm{ad}_z^{tr}\, x, w\rangle = \langle x, [z,w]\rangle$ and $\mathrm{ad}_{\cdot}^{tr}\, x$ is the map $z \mapsto \mathrm{ad}_z^{tr}\, x$, which is easily seen to be skew adjoint with respect to $\langle\cdot,\cdot\rangle$.

Theorem 3. *Let $\mathfrak{g}$ be the Lie algebra of $Iso(\mathbb{R}^n)$ endowed with the scalar product from above. Let S be a subset of $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. We consider $\mathfrak{g}$ endowed with coadjoint representation $g \mapsto \mathrm{Ad}_{g^{-1}}^{tr}$, $\langle \mathrm{Ad}_a^{tr}\, v, w\rangle = \langle v, \mathrm{Ad}_a\, w\rangle$. We suppose that S is invariant under the natural extension of the coadjoint representation to a represention of $\mathrm{SO}(n,\mathbb{C}) \rtimes \mathbb{C}^n$. Then the cone*

$$C(S) = \big\{\mathrm{Hm} \in S^2(\mathfrak{g}) \mid \mathrm{Hm}(x,\bar{x}) \geq 0 \text{ for all } x \in S\big\}$$

defines a Ricci flow invariant condition.

It is not hard to see that the ODE $\mathrm{Hm}' = \mathrm{Hm}\,\mathrm{pr}\,\mathrm{Hm} + \mathrm{Hm}^{\#}$ is equivariant with respect to the action of $\operatorname{Iso}(\mathbb{R}^n)$ on $S^2(\mathfrak{g})$ given by $g \star \mathrm{Hm} := \mathrm{Ad}_g\,\mathrm{Hm}\,\mathrm{Ad}_g^{tr}$.

The theorem recovers Brendle's recent generalization of Hamilton's Harnack inequality by putting

$$S = \{(A,v) \mid A \in \mathfrak{so}(n,\mathbb{C}), \operatorname{rank}(A) = 2, v \in A(\mathbb{C}^n)\}$$

As is shown in [Brendle, 2009] this still implies the usual trace Harnack inequality.

A Kähler manifold M is said to have positive orthogonal bisectional curvature if $K(v,w) + K(v,iw) > 0$ holds for all unit vectors $v, w \in T_pM$ with $\mathbb{C}\cdot v \perp \mathbb{C}\cdot w$, where $K(v,w)$ denotes the sectional curvature of the plane spanned by v and w. A Kähler surface has nonnegative orthogonal bisectional curvature if and only if it has nonnegative isotropic curvature. Thus orthogonal bisectional curvature is independent of the traceless Ricci part if $n = 2$. Furthermore, M has nonnegative bisectional curvature if and only if $M \times \mathbb{C}$ has nonnegative orthogonal bisectional curvature. We will give a somewhat simpler proof of the following theorem.

Theorem 4. *A compact Kähler manifold of complex dimension $n > 1$ with positive orthogonal bisectional curvature evolves under the Ricci flow to a manifold with positive bisectional curvature.*

The theorem is not new. Chen [2007] shows that for a compact solution to the Kähler Ricci flow which has positive first Chern class and positive orthogonal bisectional curvature throughout space time, the bisectional curvature becomes positive. Then Gu and Zhang [2010] show that indeed the first Chern class is positive and they also give a proof of the invariance of positive orthogonal bisectional curvature.

We decided to give a proof which is independent of [Chen, 2007] and Gu and Zhang [2010]. However, in our proof as well as in [Chen, 2007] a key ingredient is a result of Perelman, written up by Sesum and Tian [2006], ensuring that for a compact Kähler manifold with positive first Chern class all non flat blow up limits are compact.

Although we do not need it we should mention that Chen, Sun and Tian [2009] gave a new proof of the statement that a Kähler manifold with positive bisectional curvature evolves under the normalized Kähler Ricci flow to the Fubini study metric on $\mathbb{CP}^n$. The new proof does not need directly the solution of the Frankel conjecture due to Mori [1979] and Siu and Yau [1980].

We will explain in an appendix why Brendle and Schoen's strong maximum principle [2008] carries over to our more general setting. Therefore a Kähler metric on a compact manifold with nonnegative orthogonal bisectional curvature evolves under the Ricci flow to one with positive orthogonal bisectional curvature unless the holonomy group is not equal to $\mathsf{U}(T_pM)$. Combining with Berger's classification of holonomy groups [1955] and the solution of the Frankel conjecture, one can show that a locally irreducible compact Kähler manifold of dimension $n > 1$ with nonnegative orthogonal bisectional curvature is either biholomorphic to $\mathbb{CP}^n$ or locally isometric to a hermitian symmetric space. This recovers a rigidity theorem of Gu and Zhang [2010] which in turn generalized a result of Mok [1988].

The paper is organized as follows. Section 1 contains the proof of Theorem 1. Although the essential part of the argument in the proof of Theorem 3 is completely analogous, we do need a little extra preparation, which is done in section 2. Here we provide a maximum principle for Harnack operators. One of the main points is to explain that the invariance group needed for the maximum principle is naturally isomorphic to $\mathrm{Iso}(T_pM)$. This in fact is a simple consequence of the work of Chow and Chu [1995] as well as Brendle [2009]. Although we do not need it we have added a subsection showing that in quite a few cases the maximum principle for Harnack operators can not possibly yield any meaningful outcome. Using section 2 the proof of Theorem 3 is reduced to an ODE problem, which is solved, completely analogously to section 1 in section 3. In section 4 we show that often Theorem 1 can be used to show that a nonnegativity condition pinches towards a stronger nonnegativity condition. We show for example that nonnegative orthogonal bisectional curvature pinches toward nonnegative bisectional curvature. Section 5 is devoted to the proof of Theorem 4. An appendix is devoted to the strong maximum principle.

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1. Proof of Theorem 1.

Let $S \subset \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be invariant under $\mathrm{Ad}_{\mathsf{G}_{\mathbb{C}}}$ and put as in the theorem

$$C(S,h) := \{R \in S^2(\mathfrak{g}) \mid R(v,\bar{v}) \geq h \text{ for all } v \in S\}.$$

Since $C(S,h)$ does not change if we replace S by its closure we may assume that S is closed. As we will see below it suffices to show

Claim. If $R \in C(S,h)$ and $v \in S$ with $R(v,\bar{v}) = h$, then $R^2(v,\bar{v}) + R^{\#}(v,\bar{v}) \geq 0$.

Clearly, $R^2(v,\bar{v}) \geq 0$. We plan to establish the inequality by showing that the second summand is nonnegative as well:

$$2R^{\#}(v,\bar{v}) = -\operatorname{tr}(\mathrm{ad}_v R \,\mathrm{ad}_{\bar{v}} R) \geq 0.$$

Using that S is invariant under $\mathrm{Ad}_{\mathsf{G}_{\mathbb{C}}}$ we deduce that

$$h \leq R(\mathrm{Ad}_{\exp(tx)} v, \mathrm{Ad}_{\exp(t\bar{x})} \bar{v})$$

for all $x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and for all t and with equality at $t = 0$. Recall that $\mathrm{Ad}_{\exp(tx)} = \exp(t\,\mathrm{ad}_x)$. Thus differentiating twice with respect to t and evaluating at 0 gives

$$0 \leq 2R(\mathrm{ad}_x v, \mathrm{ad}_{\bar{x}} \bar{v}) + R(\mathrm{ad}_x \mathrm{ad}_x v, \bar{v}) + R(v, \mathrm{ad}_{\bar{x}} \mathrm{ad}_{\bar{x}} \bar{v})$$

If we replace x by ix, it is easy to see that the first summand in the above inequality remains unchanged while the other two summands change their sign. Therefore

$$(1) \qquad 0 \quad \leq \quad R(\mathrm{ad}_x v, \mathrm{ad}_{\bar{x}} \bar{v}) = R(\mathrm{ad}_v x, \mathrm{ad}_{\bar{v}} \bar{x}) \text{ for all } x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}.$$

In other words the hermitian operator $-\mathrm{ad}_{\bar{v}} R \,\mathrm{ad}_v$ and its conjugate $-\mathrm{ad}_v R \,\mathrm{ad}_{\bar{v}}$ on the unitary vectorspace $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ are nonnegative. Recall that we plan to show $\operatorname{tr}(-\mathrm{ad}_v R \,\mathrm{ad}_{\bar{v}} R) \geq 0$. It is an elementray well known lemma that the scalar product of two nonnegative hermitian matrices is nonnegative. By a slight extension of this lemma it suffices to show that the operator R induces a nonnegative sesquilinear form on the image of the first operator $-\mathrm{ad}_v R \,\mathrm{ad}_{\bar{v}}$. Clearly the image is contained in the image of ad_v and by (1) R is indeed nonnegative on it which completes the proof of the claim.

If $h = 0$, then we may assume that S is scaling invariant and the invariance of $C(S)$ follows immediately from the claim. In general we have to be a bit more cautious since we do not know that the infimum of $\{R(\bar{v}, v) \mid v \in S\}$ is attained.

In order to see that the above claim is sufficient in the general case we consider a solution $R(t)$ to the ODE $R' = X(R) = R^2 + R^{\#} + \varepsilon I$ for some $\varepsilon > 0$. We plan to show that if $R(0) \in C(S,h)$ then $R(t) \in C(S, h - \varepsilon t)$ for $t > 0$. By taking the limit $\varepsilon \to 0$ we get the desired result.

Suppose, on the contrary, that $R(t_i) \notin C(S, h - \varepsilon t_i)$ for some positive $t_i \to 0$. Thus there are $v_i \in S$ with $R(t_i)(v_i, \bar{v}_i) < h - \varepsilon t_i$. If v_i stays bounded we can assume that $v_i \to v \in S$ with $R(0)(v,\bar{v}) = h$. From the above claim $R'(0)(v,\bar{v}) \geq 0$. Thus there is a neighborhood U of v and $\delta > 0$ with $R'(t)(u,\bar{u}) \geq -\varepsilon/2$ for $u \in U$ and all $t \in [0,\delta]$. Clearly this gives a contradiction.

Thus we may assume $\|v_i\| \to \infty$. After passing to a subsequence, $\frac{v_i}{\|v_i\|} \to w$ with $R(w,\bar{w}) \leq 0$ and

$$w \in \partial_\infty S := \{Y \in \mathfrak{g}_{\mathbb{C}} \mid \text{ there exists } \lambda_i \in \mathbb{R} \text{ and } v_i \in S \text{ with } \lambda_i \to 0 \text{ and } \lambda_i v_i \to Y\}$$

We call $\partial_\infty S$ the boundary of S at infinity. Clearly $\partial_\infty S$ is scaling invariant and invariant under $\mathrm{Ad}_{\mathsf{G}_{\mathbb{C}}}$ using $R \in C(S,h)$ it is elementary to check that $R \in C(\partial_\infty S, 0)$, cf. Lemma 4.1 below. In particular $R(w,\bar w) = 0$ and from the above claim $(R^2 + R^\#)(w,\bar w) \geq 0$. Therefore $R'(0)(w,\bar w) \geq \varepsilon$. This in turn shows that there is a neighborhood U of w and $\delta > 0$ such that $R'(t)(u,\bar u) > \varepsilon/2$ for all $u \in U$ and $t \in [0,\delta]$. For large i we have $v_i = \|v_i\| u_i$ for some $u_i \in U$ and therefore $R(t_i)(v_i,\bar v_i) \geq R(0)(v_i,\bar v_i) \geq h$ for all large i – a contradiction.

Remark 1.1. In the case of $\mathfrak{g} = \mathfrak{u}(n)$, $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}(n,\mathbb{C})$ one can generalize the theorem slightly. For $h_1, h_2 \in \mathbb{R}$ the set

$$C(S,h_1,h_2) = \{R \in S^2(\mathfrak{u}(n)) \mid R(v,\bar v) + h_2 \operatorname{tr}(v)\operatorname{tr}(\bar v) \geq h_1, \text{ for all } v \in S\}$$

is invariant under the ODE as well, provided that S is $\mathrm{Ad}_{\mathrm{GL}(n,\mathbb{C})}$-invariant.

2. Maximum principle for Harnack operators.

In this section we establish a maximum principle for Harnack operators which only needs the invariance under a group action of $\mathrm{Iso}(\mathbb{R}^n)$. This is in fact a simple consequence of the work of Chow and Chu [1995], see also [Chow and Knopf, 2002].

Let $(M, g(t))$ be a solution to the Ricci flow $t \in (0,T)$. We consider $N = M \times [0,T)$. We define a Riemannian metric $\langle\cdot,\cdot\rangle$ on N by

$$\langle v,w\rangle = g(t)(v,w),\ \ \langle v, \tfrac{\partial}{\partial t}\rangle = 0,\ \ \langle \tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}\rangle = 1$$

We identify the Lie algebra $\mathfrak{iso}(T_pM)$ of the isometry group of T_pM with $\Lambda^2 T_pM \oplus \{\frac{\partial}{\partial t}\wedge v \mid v \in T_pM\}$ and define the Harnack operator Hk as an selfadjoint endomorphism of $\mathfrak{iso}(T_pM)$ by

$$\begin{aligned}
\langle \mathrm{Hk}(X\wedge Y), W\wedge Z\rangle &= R_{g(t)}(X\wedge Y, W\wedge Z)\\
\langle \mathrm{Hk}(X\wedge Y), \tfrac{\partial}{\partial t}\wedge Z\rangle &= t(\nabla^{g(t)}_X \mathrm{Ric})(Y,Z) - t(\nabla^{g(t)}_Y \mathrm{Ric})(X,Z)\\
\langle \mathrm{Hk}(\tfrac{\partial}{\partial t}\wedge X), \tfrac{\partial}{\partial t}\wedge Y\rangle &= t^2(\Delta \mathrm{Ric})_{g(t)}(X,Y) - \tfrac{t^2}{2}\mathrm{Hess}_{g(t)}(\mathrm{scal})(X,Y)\\
&\quad +2t^2\sum_i \mathrm{Ric}_{g(t)}(e_i,e_i)\,\mathrm{Rm}^{g(t)}(e_i\wedge X, e_i\wedge Y)\\
&\quad -t^2\,\mathrm{Ric}_{g(t)}(\mathrm{Ric}^{g(t)} X, Y) + \tfrac{t}{2}\mathrm{Ric}(X,Y)
\end{aligned}$$

where $\mathrm{Ric}^{g(t)}$ resp. $\mathrm{Ric}_{g(t)}$ is the Ricci tensor of $(M,g(t))$ viewed as $(1,1)$ resp. $(2,0)$ tensor, e_i is an orthonormal basis of eigenvectors of $\mathrm{Ric}^{g(t)}$ and where $\mathrm{Hess}_{g(t)}(\mathrm{scal})$ is the Hessian of the scalar curvature of $(M,g(t))$. By putting

$$\mathrm{Hm}(X\wedge Y + \tfrac{\partial}{\partial t}\wedge Z, X\wedge Y + \tfrac{\partial}{\partial t}\wedge Z) = \mathrm{Hk}(X\wedge Y + \tfrac{1}{t}\tfrac{\partial}{\partial t}\wedge Z, X\wedge Y + \tfrac{1}{t}\tfrac{\partial}{\partial t}\wedge Z)$$

we get back to the usual definition of the Harnack operator.

Let $\mathfrak{g}$ be the Lie algebra of $\mathrm{Iso}(\mathbb{R}^n)$ endowed with the natural scalar product from the introduction. Consider on $\mathfrak{g}$ the coadjoint representation

$$\mathrm{Iso}(M,g) \to \mathrm{GL}(\mathfrak{g}),\ \ g \mapsto \mathrm{Ad}^{tr}_{g^{-1}}$$

Let $S^2(\mathfrak{g})$ denote the vectorspace of selfadjoint endomorphisms of $\mathfrak{g}$ endowed the representation of $\mathrm{Iso}(\mathbb{R}^n)$ given by $g\star R = \mathrm{Ad}_g R\,\mathrm{Ad}_g^{tr}$ for $R \in S^2(\mathfrak{g})$ and $g \in \mathrm{Iso}(\mathbb{R}^n)$. Although it is not important for us, we mention that by Brendle [2009], the Harnack operator is always contained in a linear subspace $S^2_B(\mathfrak{g})$ of operators satisfying the first Bianchi identity.

Recall that a family of sets $C(t) \subset V$ $(t \in (a,b))$ in a vectorspace V is called invariant under a ODE $v' = f(v)$ if for any solution $v(t)$ $(t \in [t_0,s])$ with $v(t_0) \in C(t_0)$ we have $v(t) \in C(t)$ for $t \geq t_0$. In this section we want to prove

Theorem 2.1. *Suppose $C(t) \subset S^2_B(\mathfrak{g})$ is a family of closed convex sets which is invariant under the above representation of* $\mathrm{Iso}(\mathbb{R}^n)$. *We assume that C is invariant under the ODE*

$$\mathrm{Hk}' = 2(\mathrm{Hk}\,\mathrm{pr}\,\mathrm{Hk} + \mathrm{Hk}^{\#})$$

where $\mathrm{pr}\colon \mathfrak{iso}(\mathbb{R}^n) \to \mathfrak{so}(\mathbb{R}^n)$ *is the orthogonal projection,* $\mathrm{Hk}\,\mathrm{pr}\,\mathrm{Hk}$ *is the composition of the three endomorphisms and*

$$\langle \mathrm{Hk}^{\#} A, B\rangle = -\tfrac{1}{2}\,\mathrm{tr}\big(\mathrm{ad}^{tr}_{\cdot} A\,\mathrm{Hk}\,\mathrm{ad}^{tr}_{\cdot} B\,\mathrm{Hk}\big).$$

Then $C(t)$ defines a Ricci flow invariant condition, that is, if $(M, g(t))$ $(t \in (0,T))$ is a compact solution to the Ricci flow and $\mathrm{Hk}(p,0) \in C(0)$ *for all $p \in M$ then* $\mathrm{Hk}(p,t) \in C(t)$ *for all t.*

Of course maximum principles are well established in the literature including some for Harnack operators and generalization to open manifolds and the case of $t \to 0$ have been established. More important for us is that we only need C to be invariant under the above representation of the relatively small group $\mathrm{Iso}(\mathbb{R}^n)$.

We consider the connection

$$\nabla_X Y = \nabla^{g(t)}_X Y,\ \nabla_{\frac{\partial}{\partial t}}\frac{\partial}{\partial t} = -\tfrac{t}{2}\mathrm{grad}^{g(t)}(\mathrm{scal}) \tag{2}$$

$$\nabla_{\frac{\partial}{\partial t}} Y = -\mathrm{Ric}^{g(t)} Y + \tfrac{d}{dt} Y_{(p,t)},\ \nabla_Y \frac{\partial}{\partial t} = -t\,\mathrm{Ric}^{g(t)} Y - \tfrac{1}{2} Y$$

for vectorfields X, Y in $N = M \times [0,T)$ tangential to M. Notice that ∇ is neither torsion free nor Riemannian with respect to a background metric. However the distribution $TM \times [0,T)$ is parallel with respect to ∇ and ∇ respects the metric induced on this distribution by the background metric $\langle\cdot,\cdot\rangle$. The affine space $\frac{\partial}{\partial t}_{|(t,p)} + T_pM$ is also invariant under the parallel transport with respect to ∇.

The holonomy group of ∇ is thus in a natural fashion isomorphic to a subgroup of $\mathrm{Iso}(T_pM)$. In fact for a closed curve γ at (p,t) in N the parallel transport Par_γ is determined by the linear isometry $\mathrm{Par}_{\gamma|(T_pM, g(t))}$ and a translational part $\tau(\mathrm{Par}_\gamma) \in T_pM$ characterized by $\mathrm{Par}_\gamma(\frac{\partial}{\partial t}) = \frac{\partial}{\partial t} + \tau(\mathrm{Par}_\gamma)$. The map

$$\mathrm{Par}_\gamma \mapsto (\mathrm{Par}_{\gamma|(T_pM,g(t))}, \tau(\mathrm{Par}_\gamma)) \in \mathrm{O}(T_pM, g(t)) \rtimes T_pM$$

is a homomorphism.

We identify $\mathfrak{iso}(T_pM) = \Lambda^2 T_pN = \mathfrak{so}(T_pM) \oplus \{\frac{\partial}{\partial t}\wedge v \mid v \in T_pM\}$ where we view $\mathfrak{so}(T_pM) \cong \Lambda^2 T_pM$ as the vector space of skew adjoint endomorphism endowed with the scalar product $\langle X, Y\rangle = -\frac{1}{2}\mathrm{tr}(XY)$, the second summand $\frac{\partial}{\partial t}\wedge T_pM$ is orthogonal to $\mathfrak{so}(T_pM)$ and the scalar product is given by $\langle \frac{\partial}{\partial t}\wedge v, \frac{\partial}{\partial t}\wedge w\rangle = \langle v, w\rangle$. The Lie bracket is given by

$$[(X + \tfrac{\partial}{\partial t}\wedge v), Y + \tfrac{\partial}{\partial t}\wedge w)] = XY - YX + \tfrac{\partial}{\partial t}\wedge Xw - \tfrac{\partial}{\partial t}\wedge Yv.$$

Notice that the holonomy group of N with respect to T_pN acts naturally on $\mathfrak{iso}(T_pM)$. It is straightforward to check that this action corresponds to the coadjoint representation of $\mathrm{Iso}(T_pM)$ in $\mathfrak{iso}(T_pM)$ given by $g \mapsto Ad^{tr}_{g^{-1}}$.

For $A, B \in \mathfrak{iso}(T_pM)$ we define ad_A as usual $\mathrm{ad}_A B = [A,B]$ and let ad^{tr}_A denote the dual endomorphism and $\mathrm{ad}^{tr}_{\cdot} B$ the endomorphism $A \mapsto \mathrm{ad}^{tr}_A B$. It is straightforward to check that $\mathrm{ad}^{tr}_{\cdot} B$ is skew adjoint: $\langle \mathrm{ad}^{tr}_{\cdot} B(A), A\rangle = \langle B, [A,A]\rangle = 0$.

We extend the bilinear map $(A,B) \mapsto \mathrm{ad}^{tr}_A B$ to a complex bilinear map

$$\mathrm{ad}^{tr}\colon \mathfrak{iso}(T_pM)\otimes_{\mathbb{R}}\mathbb{C} \times \mathfrak{iso}(T_pM)\otimes_{\mathbb{R}}\mathbb{C} \to \mathfrak{iso}(T_pM)\otimes_{\mathbb{R}}\mathbb{C}.$$

Although we defined $\mathrm{Hk}(p,t)$ as a self adjoint endomorphism of $\mathfrak{iso}(T_pM, g(t))$, we view it as $(4,0)$-tensor in order to define $\Delta\,\mathrm{Hk}$: Choose a basis $b_1,\ldots,b_k$ of the Lie algebra $\mathfrak{iso}((T_pM,g(t)))$. For $v \in T_pM$ and small s we define $b_i(\exp(sv))$ as the parallel extension of b_i with respect to the connection ∇ on N defined by (2) along the geodesic $\exp(sv)$ in $(M, g(t))$. Then $\Delta\,\mathrm{Hk}(p,t)$ is the selfadjoint endomorphism of $\mathfrak{iso}(T_pM, g(t))$ characterized by

$$\big\langle \Delta\,\mathrm{Hk}(p,t) b_i, b_j\big\rangle = \sum_{k=1}^{n} \tfrac{d^2}{ds^2}_{|s=0} \big\langle \mathrm{Hk}(b_i(\exp(se_k))), b_j(\exp(se_k))\big\rangle.$$

where $e_1,\ldots,e_n$ is an orthonormal basis of $(T_pM, g(t))$.

Theorem 2.2. Hk *satisfies the tensor identity*

$$\nabla_{\frac{\partial}{\partial t}} \mathrm{Hk} = \Delta\,\mathrm{Hk} + 2(\mathrm{Hk}\,\mathrm{pr}\,\mathrm{Hk} + \mathrm{Hk}^{\#})$$

where ∇ *is the connection on* N *defined by* (2) *and* $(\Delta\,\mathrm{Hk})$ *is defined as above.*

The proof of Theorem 2.2 follows from Brendle [2009]. He derived a similar tensor identity for Hm using the following torsion free connection that is similar to the one introduced by Chow and Chu [1995].

$$D_X Y = \nabla^{g(t)}_X Y,\ D_{\frac{\partial}{\partial t}} \tfrac{\partial}{\partial t} = -\tfrac{1}{2}\mathrm{grad}^{g(t)}(\mathrm{scal}) - \tfrac{3}{2t}\tfrac{\partial}{\partial t}$$

$$D_{\frac{\partial}{\partial t}} Y = -\,\mathrm{Ric}^{g(t)} Y - \tfrac{Y}{2t} + \tfrac{d}{dt} Y_{(p,t)},$$

By Brendle the operator Hm satisfies the tensor identity

$$D_{\frac{\partial}{\partial t}} \mathrm{Hm} = \Delta\,\mathrm{Hm} + \tfrac{2}{t}\,\mathrm{Hm} + 2(\mathrm{Hm}\,\mathrm{pr}\,\mathrm{Hm} + \mathrm{Hm}^{\#}) \tag{3}$$

We should mention that Brendle has a different but equivalent definition of the algebraic expression $(\mathrm{Hm}\,\mathrm{pr}\,\mathrm{Hm} + \mathrm{Hm}^{\#})$. ¿From this equation Theorem 2.2 follows by a straightforward calculation.

The advantage of Theorem 2.2 over (3) is that the former is nonsingular at $t = 0$ and since the connection is fairly natural with respect to $\langle\cdot,\cdot\rangle$ it is easy to establish a dynamical version of the maximum principle. On the other hand D has similar properties to ∇ provided we endow N with the background metric

$$g(v,w) = \tfrac{1}{t} g(t)(v,w),\ \ g(v, \tfrac{\partial}{\partial t}) = 0,\ \ g(\tfrac{\partial}{\partial t}, \tfrac{\partial}{\partial t}) = \tfrac{1}{t^3}$$

The curvature tensor of D is given by $\frac{1}{t}\,\mathrm{Hm}$. Recently Cabezas-Rivas and Topping [2009] found a sequence g^k of metrics on N with the property that $g^k(v,w) = \frac{1}{t} g(t)(v,w)$ and $g^k(\frac{\partial}{\partial t}, w) = 0$ for $v, w \in T_pM$. The only constant $g^k(\frac{\partial}{\partial t}, \frac{\partial}{\partial t})$ depending on k diverges to infinity for $k \to \infty$. However the Levi Cevita connection of these metrics converge in the C^∞ topology to D. In particular the curvature tensor converges to $\frac{1}{t}\,\mathrm{Hm}$. Moreover (M, g^k) is a Ricci soliton up to order $\frac{1}{k}$. Cabezas-Rivas and Topping are then able to derive (3) from the evolution of a curvature tensor under the Ricci flow.

We now turn to the proof of Theorem 2.1. Since $C(t)$ is invariant under the representation of $\mathrm{Iso}(\mathbb{R}^n)$ we can identify it naturally with a subset of $S^2_B(\mathfrak{iso}(T_pM, g(s)))$ for all $(p,s) \in N$.

We choose an auxiliary smooth tensor field T such that $T(p,t)$ is a selfadjoint endomorphism of $\mathfrak{iso}(T_pM, g(t))$ representing an interior point of the closed convex set $C(t)$.

Lemma 2.3. *Any tensor* Hk *on* N *satisfying*

$$\nabla_{\frac{\partial}{\partial t}} \mathrm{Hk} = \Delta \mathrm{Hk} + 2(\mathrm{Hk} \operatorname{pr} \mathrm{Hk} + \mathrm{Hk}^{\#})$$

can be approximated by a sequence S_k *of tensors on* $M \times \left[\frac{1}{k}, T - \frac{1}{k}\right]$ *satisfying*

$$\nabla_{\frac{\partial}{\partial t}} S_k = \Delta S_k + 2(S_k \operatorname{pr} S_k + S_k^{\#}) + \varepsilon_k (T - S_k)$$

and $S(p, \frac{1}{k})$ *represents an interior point of* $C(\frac{1}{k})$ *and* $\varepsilon_k > 0$ *converges to* 0.

Clearly one can find an initial value $S(p, \frac{1}{k}) \in Int(C\delta_k)$ such that $S(p, \frac{1}{k}) - \mathrm{Hk}(p, \frac{1}{k})$ $(p \in M)$ converges to 0 in the C^{∞} topology. Moreover, $S(p, \frac{1}{k})$ is a solution if and only if $S_k - \mathrm{Hk}$ is a solution of an equation with the obvious modifications. Since one can prove similarly to Shi a priori estimates for the corresponding linearized equation, it follows that a solution of the initial value problem exists.

Proof of Theorem 2.1. Since $C(s) \subset S^2_B(\mathfrak{g})$ is invariant under $\mathrm{Iso}(\mathbb{R}^n)$ we can identify it naturally with a subset of $S^2_B(\mathfrak{iso}(T_pM, g(t))$ for all $(p, t) \in N$.

It suffices to prove that $S_k(p, t) \in C(t)$ $(t \in \left[\frac{1}{k}, T - \frac{1}{k}\right])$ for a sequence S_k as in the lemma. We assume, on the contrary, that for some minimal $t_0 > \frac{1}{k}$ we can find some $p \in M$ such that $S_k(p, t_0)$ is contained in the boundary of $C(t_0)$.

Because of the minimal choice of t_0 we know that $S_k(q, t) \in C(t)$ for all $t \leq t_0$ and $q \in M$. To get a contradiction we will show that $S_k(p, t_0 - h) \notin C(t_0 - h)$ for small positive h.

For small $h \geq 0$ and $q \in M$ we define $H(s) \in S^2_B(\mathfrak{iso}(T_q, g(t_0 - h)))$ as the solution of the ODE $H' = 2(H \operatorname{pr} H + H^{\#})$ with $H(0) = S_k(t_0 - h)$. Since the family $C(t)$ is invariant under the ODE we know that $P_k(q, t_0 - h) := H(h) \in C(t_0)$. By construction

$$\nabla_{\frac{\partial}{\partial t}} P_k(p, t_0) = \Delta P_k(p, t_0) + \varepsilon_k (T - P_k)(p, t_0)$$

Using that $P_k(q, t_0) = S_k(q, t_0) \in C(t_0)$ and that $C(t_0)$ is invariant under the representation of $\mathrm{Iso}(\mathbb{R}^n)$ it is immediate that $\Delta P_k(p, t_0) = T_{P_k(p, t_0)} C(t_0)$. Furthermore we know by construction that $\varepsilon_k (T - P_k)(p, t_0)$ is contained in the interior of the tangent cone $T_{P_k(p, t_0)} C(t_0)$. We deduce that $P_k(p, t_0 - h) \notin C(t_0)$ for small positive h – a contradiction. □

Remark 2.4.

a) If one carries out everything in this section in the special case that $(M, g(t))$ is a Kähler manifold, then the holonomy group of the connection ∇ is isomorphic to a subgroup of $\mathsf{U}(T_pM) \rtimes T_pM \subset \mathrm{SO}(T_pM) \rtimes T_pM$ and the image of the Harnack operator is contained in the Lie subalgebra $\mathfrak{g}'$ of this group. One can then formulate and prove an analogous statement for Harnack operators of Kähler manifolds.

b) Let Hk be a Harnack operator and $R = \mathrm{Hk}_{|so(n)}$. A simple computation shows that the trace Harnack inequality is equivalent to

$$\inf\{\mathrm{tr}\big(\mathrm{Ad}_v \mathrm{Hk} \mathrm{Ad}_v^{tr}\big) \mid v \in \mathbb{R}^n \subset \mathrm{Iso}(\mathbb{R}^n)\} - \mathrm{tr}(R) \geq 0.$$

If R has positive Ricci curvature, then there is a unique $v \in \mathbb{R}^n$ such that $\mathrm{Ad}_v \mathrm{Hk} \mathrm{Ad}_v^{tr}$ has minimal trace.

c) The reason for the somewhat complicated approach toward the maximum principle is that it is in general not true that a convex set $C(t)$ is contained in the interior of another slightly larger convex set C which is also invariant under the action of $\mathrm{Iso}(\mathbb{R}^n)$, cf. next subsection. This is of course related to the fact that the connection is not compatible with a metric on the space of curvature tensors.

2.1. **Some negative results on Harnack inequalities.** It is elementary to check that the subspaces

$$\begin{aligned} V &:= \big\{\mathrm{Hk} \in S^2_B(\mathfrak{g}) \mid \langle \mathrm{Hk}(v), w\rangle = 0 \text{ for all } v, w \in \mathfrak{so}(n)\big\} \\ W &:= \big\{\mathrm{Hk} \in S^2_B(\mathfrak{g}) \mid \mathfrak{so}(n) \subset \mathrm{kernel}(\mathrm{Hk})\big\} \end{aligned}$$

are invariant under the representation of $\mathrm{Iso}(\mathbb{R}^n)$. The space W can be characterized as the fixed point set of the normal subgroup $\mathbb{R}^n \subset \mathrm{Iso}(\mathbb{R}^n)$.

If the convex sets $C(t)$ in Theorem 2.1 have the form $C' + V$ for some subset $C' \subset S^2_B(\mathfrak{g})$, then the condition $C(t)$ can only provide restrictions for the curvature tensor $R = \mathrm{Hk}_{|\mathfrak{so}(n)\times\mathfrak{so}(n)}$.

Lemma 2.5. *Let $C \subset S^2_B(\mathfrak{g})$ be a closed convex set of maximal dimension which is invariant under* $\mathrm{Iso}(\mathbb{R}^n)$. *Suppose that C is not of the form $C' + V$. After possibly replacing C by $-C$ the following holds. For every* $\mathrm{Hk} \in C$ *the restriction* $\mathrm{Hk}_{|\mathfrak{so}(n)\times\mathfrak{so}(n)}$ *is a curvature operator with nonnegative Ricci curvature.*

Proof. Suppose, on the contrary, we can find $\mathrm{Hk}_i \in C$ and $v_i \in \mathbb{R}^n$ such that the Ricci curvature Ric_i of $\mathrm{Hk}_{i|\mathfrak{so}(n)\times\mathfrak{so}(n)}$ satisfies the following ($i = 1, 2$): $\mathrm{Ric}_1(v_1, v_1) > 0$ and $\mathrm{Ric}(v_2, v_2) < 0$. Put $\mathrm{Hk}_i(t) = \mathrm{Ad}_{tv_i} \mathrm{Hk}_i \mathrm{Ad}^{tr}_{tv_i}$. It is straightforward to check that the trace $\mathrm{tr}(\mathrm{Hk}_i(t))$ converges quadratically in t to $(-1)^{i+1}\infty$ as $t \to \infty$ ($i = 1, 2$). The element $X_i = \lim_{t\to\infty} \frac{\mathrm{Hk}_i(t)}{t^2}$ is an element in the cone at infinity

$$\partial_\infty C := \lim_{\lambda\to\infty} \tfrac{C}{\lambda}$$

of C. We know $\mathrm{tr}(X_1) > 0$, $\mathrm{tr}(X_2) < 0$ and it is straightforward to check $X_i \in W$. Moreover the operator X_i has $v_i \in \mathbb{R}^n$ in its kernel.

Clearly the cone $\partial_\infty C$ is invariant under the the representation of $\mathrm{Iso}(\mathbb{R}^n)$. The barry center of the $\mathrm{SO}(n)$-orbit of X_1 (resp. X_2) is a positive (resp. negative) multiple of the orthogonal projection of $\mathfrak{g}$ to $\mathbb{R}^n$.

Under $\mathrm{SO}(n)$ the vectorspace W decomposes into a one dimensional trivial and an irreducible representation. Since $X_i \in W$ itself is not a multiple of the orthogonal projection we can also find a traceless operator $X \in \partial_\infty C \cap W$. Clearly this implies $W \subset \partial_\infty C$.

The quotient space V/W decomposes under $\mathrm{SO}(n)$ in two inequivalent irreducible nontrivial subrepresentations.

Using that C has maximal dimension we can find $\mathrm{Hk} \in C$ such that for some $v \in \mathbb{R}^n$, $\big(\mathrm{Ad}_v \mathrm{Hk} \mathrm{Ad}^{tr}_v - \mathrm{Hk}\big) \in V$ projects to an element in V/W which is not contained in a nontrivial invariant subspace.

For each t we choose $Y(t) \in W$ such that $L(t) := \mathrm{Ad}_t\, v \,\mathrm{Hk}\, \mathrm{Ad}_t\, v^{tr} - Y(t)$ has minimal norm. It then follows that the norm of L increases linearly and $L_\infty := \lim_{t\to\infty} \frac{L(t)}{t} \in V \cap \partial_\infty C$ corresponds in V/W to an element that does not lie in a nontrivial invariant subspace.

This in turn shows $V \subset \partial_\infty C$ and hence C is of the form $C' + V$.

□

Lemma 2.6. *Let $C \subset S^2_B(\mathfrak{g})$ be a convex subset of maximal dimension which is invariant under the representation of* $\mathrm{Iso}(\mathbb{R}^n)$*. Suppose there is an element* $\mathrm{Hk} \in C$ *such that* $R := \mathrm{Hk}_{|\mathfrak{so}(n)\times\mathfrak{so}(n)}$ *satisfies for some* $v \in \mathbb{R}^n$*:* $\mathrm{Ric}(v,v) = 0$ *and* $R(\cdot, v, v, \cdot) \neq 0$*. Then for any family of convex sets $C(t)$ which is invariant under the ODE, invariant under* $\mathrm{Iso}(\mathbb{R}^n)$ *with $C(0) = C$ we have $C(t) = C(t) + V$ for all $t > 0$.*

The lemma shows for example that one can not prove a Harnack inequality in the class of 3-manifolds with positive Ricci curvature evolving under the Ricci flow by means of the maximum principle of Theorem 2.1.

Proof. It is straightforward to check that $X := \lim_{t\to\infty} \frac{\mathrm{Ad}_{tv}\,\mathrm{Hk}\,\mathrm{Ad}^{tr}_{vt}}{t^2}$ is a traceless operator in W. Clearly $X \in \partial_\infty C$.

Since the traceless operators $W' \subset W$ form an irreducible subspace we deduce that $W' \subset \partial_\infty C$. Let $C(t)$ be as in the lemma.

Consider the element $-\mathrm{id} \in O(n)$ and let $\mathrm{Fix}(-\mathrm{id}) \subset S^2_B(\mathfrak{g})$ denote the fixed point set of $-\mathrm{id}$. Notice that $\mathrm{Fix}(-\mathrm{id})$ is still invariant under $\mathrm{SO}(n)$. It is easy to see that $\mathrm{Fix}(-id) \cap C$ has maximal dimension in $\mathrm{Fix}(-\mathrm{id})$. Moreover the set $\tilde{C}(t) = C(t) \cap \mathrm{Fix}(-\mathrm{id})$ is invariant under the ODE.

Notice that $\tilde{C}(0)$ contains a subset of the form $\lambda I + W'$ where I is the orthogonal projection of $\mathfrak{g}$ to $\mathfrak{so}(n)$ for some λ. If we evolve this set under the ODE we see that $W' \subset \partial_\infty C(t)$ for all t.

Thus $\tilde{C}(t) = \tilde{C}(t)' + W'$ where $\tilde{C}(t)' \subset W'^{\perp} \cap \mathrm{Fix}(-\mathrm{id})$ is convex.

We may assume that the norm of $\mathrm{Hk}_{|\mathfrak{so}(n)\times\mathfrak{so}(n)}$ is bounded by some a priori constant for all $\mathrm{Hk} \in C(t)$. This in turn implies that a sequence in $\tilde{C}(t)'$ tends to ∞ if and only if its trace is unbounded.

Using that $\tilde{C}(t)$ is invariant under the ODE and that $C'(t)$ has full dimension it is easy to see that for all positive t there are endomorphisms with arbitrary small as well as endomorphisms with arbitrary large trace in $\tilde{C}(t)$ and hence in $\tilde{C}(t)'$. Therefore $W \subset \partial_\infty \tilde{C}(t)$. This implies as before $V \subset \partial_\infty C(t)$ for all positive t as claimed.

□

3. Proof of Theorem 3.

We prove a slightly more general result which holds for any metric Lie algebra. Let $\mathfrak{g}$ be a Lie algebra endowed with a scalar product $\langle\cdot,\cdot\rangle$. Put $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ and let G and $\mathsf{G}_{\mathbb{C}}$ be associated groups. For a self adjoint endomorphism $R\colon \mathfrak{g} \to \mathfrak{g}$ we define $R^{\#}\colon \mathfrak{g} \to \mathfrak{g}$ by

$$\langle R^{\#} v, w\rangle = -\tfrac{1}{2} \operatorname{tr} \mathrm{ad}^{tr}_{\cdot} vR\, \mathrm{ad}^{tr}_{\cdot} wR \text{ for } v, w \in \mathfrak{g}.$$

Here $\mathrm{ad}^{tr}_{\cdot} v$ is the map $x \mapsto \mathrm{ad}^{tr}_x v$ which in turn is characterized by $\langle \mathrm{ad}^{tr}_x v, y\rangle = \langle v, [x,y]\rangle$. It is easy to see that $\mathrm{ad}^{tr}_{\cdot} v$ is skew adjoint with respect to the scalar product for each fixed $v \in \mathfrak{g}$. If G is compact, one can choose $\langle\cdot,\cdot\rangle$ to be Ad_{G} invariant and then $\mathrm{ad}^{tr}_{\cdot} v = \mathrm{ad}_v$ and we get back the earlier definition of $R^{\#}$.

Each $R \in S^2(\mathfrak{g})$ induces a complex symmetric bilinear form on $\mathfrak{g}_{\mathbb{C}}$ which we denote by $(x,y) \mapsto R(x,y)$. We extend the coadjoint representation $g \mapsto Ad^{tr}_{g^{-1}}$ with $\langle Ad^{tr}_{g^{-1}} x, y\rangle = \langle x, Ad_{g^{-1}} y\rangle$ to a representation of $\mathsf{G}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$.

Proposition 3.1. *Consider the vectorspace $S^2(\mathfrak{g})$ of selfadjoint endomorphisms of $\mathfrak{g}$ endowed with the ODE $R' = R^{\#}$ from above. Suppose $S \subset \mathfrak{g}_{\mathbb{C}}$ is invariant under the complexified coadjoint representation of $\mathsf{G}_{\mathbb{C}}$. Then for each $h \in \mathbb{R}$ the set*

$$C(S,h) = \{R \in S^2(\mathfrak{g}) \mid R(x,\bar{x}) \geq h \textit{ for all } s \in S\}$$

is invariant under the ODE $R' = R^{\#}$.

Combining the proposition with the maximum principle from the previous section and with the fact that the first summand in the Harnack ODE is a nonnegative operator Theorem 3 clearly follows.

Proof of the Proposition. As before we have to show.

Claim. If $R \in C(S,h)$ and $v \in S$ with $R(v,\bar{v}) = 0$, then $R^{\#}(v,\bar{v}) \geq 0$.

Up to some necessary changes in notation the proof is the same as the one in section 1. For convenience we repeat it here in the more general setting.

We extend the maps $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, $(x,v) \mapsto \mathrm{ad}_x^{tr} v$ to a complex bilinear map $\mathfrak{g}_{\mathbb{C}} \times \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ which we also denote by $(x,v) \mapsto \mathrm{ad}_x^{tr} v$.

Using that S is invariant under $\mathrm{Ad}^{tr}_{\mathsf{G}_{\mathbb{C}}}$ we deduce that for any $x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ we have for all $t \in \mathbb{R}$

$$h \leq R(\mathrm{Ad}^{tr}_{\exp(tx)} v, \mathrm{Ad}^{tr}_{\exp(t\bar{x})} \bar{v})$$

with equality at $t = 0$. Recall that $\mathrm{Ad}^{tr}_{\exp(tx)} = \exp(t\,\mathrm{ad}^{tr}_x)$. Thus differentiating twice with respect to t and evaluating at 0 gives

$$0 \leq 2R(\mathrm{ad}_x^{tr} v, \mathrm{ad}_{\bar{x}}^{tr} \bar{v}) + R(\mathrm{ad}_x^{tr}\mathrm{ad}_x^{tr} v, \bar{v}) + R(v, \mathrm{ad}_{\bar{x}}^{tr}\mathrm{ad}_{\bar{x}}^{tr} \bar{v})$$

If we now replace x by ix, then it is easy to see that the first summand in the above inequality remains unchanged while the other two summands change their sign.

Therefore

$$0 \leq R(\mathrm{ad}_x^{tr} v, \mathrm{ad}_{\bar{x}}^{tr} \bar{v}) \text{ for all } x \in \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}. \tag{4}$$

In other words, $-\mathrm{ad}^{tr}_{\cdot} \bar{v} R\, \mathrm{ad}^{tr}_{\cdot} v$ and its conjugate $-\mathrm{ad}^{tr}_{\cdot} v R\, \mathrm{ad}^{tr}_{\cdot} \bar{v}$ are nonnegative hermitian operators on the unitary vectorspace $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$.

In order to establish $\mathrm{tr}(-\mathrm{ad}^{tr}_{\cdot} v R\, \mathrm{ad}^{tr}_{\cdot} \bar{v} R) \geq 0$, it now suffices to show that R induces a nonnegative sesquilinear form on the image of the nonnegative operator $-\mathrm{ad}^{tr}_{\cdot} v R\, \mathrm{ad}^{tr}_{\cdot} \bar{v}$. Clearly the image is contained in the image of $\mathrm{ad}^{tr}_{\cdot} v$ and by (4) R is indeed nonnegative on it which completes the proof of the proposition. □

Remark 3.2. a) Using Remark 2.4 and the proposition in the case that $\mathfrak{g}$ is given by the Lie algebra of $\mathsf{U}(n) \rtimes \mathbb{C}^n$ one can derive Harnack inequalities for Kähler manifolds. The complexification of $\mathsf{U}(n) \rtimes \mathbb{C}^n$ is given by $\mathrm{GL}(n,\mathbb{C}) \rtimes (\mathbb{C}^n \oplus \mathbb{C}^n)$, where $\mathrm{GL}(n,\mathbb{C})$ acts in the standard way on the first summand and by $(A,v) \mapsto (\bar{A}^{tr})^{-1} v$ on the second $\mathbb{C}^n$-summand. Thus the set

$$S := \{(A,v,0) \in \mathfrak{gl}(n,\mathbb{C}) \times \mathbb{C}^n \times \mathbb{C}^n \mid \mathrm{rank}(A) = 1, v \in A(\mathbb{C}^n)\},$$

is invariant under the coadjoint representation. This gives a Harnack inequality for Kähler manifolds with positive bisectional curvature whose trace form is similar to Cao's [1992] Harnack inequality.

b) Let $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ be as in the proposition and let $G\colon \mathfrak{g} \to \mathfrak{g}$ denote a selfadjoint positive endomorphism. Put $g(v,w) = \langle\cdot, G\cdot\rangle$. The ODE $R' = R^{\#_g}$ corresponding to the metric Lie algebra $(\mathfrak{g}, g)$ is obtained by pulling back the corresponding ODE for the metric Lie algebra $(\mathfrak{g}, \langle\cdot,\cdot\rangle)$ under the linear map $S^2(\mathfrak{g}, g) \to S^2(\mathfrak{g}, \langle\cdot,\cdot\rangle), R \mapsto RG^{-1}$. Thus $R^{\#_g} = (RG^{-1})^{\#} \cdot G$.

4. Some pinching results

Theorem 1 gives a large family of invariant nonnegativity conditions. We will see in this section that it can also be used to show that some nonnegativity conditions pinch toward stronger nonnegativity conditions.

Lemma 4.1. *Consider, a* $\mathrm{Ad}_{\mathsf{G}_{\mathbb{C}}}$*-invariant subset* $S \subset \mathfrak{g}_{\mathbb{C}}$. *Then*

$$\partial_\infty S := \{Y \in \mathfrak{g}_{\mathbb{C}} \mid \text{ there exists } \lambda_i \in \mathbb{R} \text{ and } v_i \in S \text{ with } \lambda_i \to 0 \text{ and } \lambda_i v_i \to Y\}$$

is a scaling invariant $\mathrm{Ad}_{\mathsf{G}_{\mathbb{C}}}$*-invariant set, which we call, by slight abuse of notation, the boundary of* S *at infinity.*

a) *For any* $h \in \mathbb{R}$ *the set* $C(S, h)$ *is contained in* $C(\partial_\infty S, 0)$.
b) *The union* $\bigcup_{h<0} C(S, h)$ *contains all interior points of* $C(\partial_\infty S, 0)$.

Proof. a). Consider $R \notin C(\partial_\infty S, 0)$. Then there is a $Y \in \partial_\infty S$ with $R(Y, \bar{Y}) = a < 0$. Choose a sequence $v_i \in S$ and $\lambda_i \to 0$ with $\lambda_i v_i \to Y$. Then $R(v_i, \bar{v}_i) = -\frac{1}{\lambda_i^2} R(\lambda_i v_i, \lambda_i v_i) \to -\infty$. Thus $R \notin C(S, h)$ for all h.

b). If R is in the interior of $C(\partial_\infty S, 0)$, then we can find a scaling invariant open neighborhood U of $\partial_\infty S \setminus \{0\}$ such that $R(v, \bar{v}) > 0$ for all $v \in U$. It is straightforward to check that the set $S' = \{v \in S \mid v \notin U\}$ is bounded. Thus if we put $h := \inf_{v \in S'} R(v, \bar{v}) > -\infty$ we deduce $R \in C(S, \min\{0, h\})$. □

Applications.

a) Nonnegative orthogonal bisectional curvature pinches toward nonnegative bisectional curvature. Let $(M, g(0))$ be a compact Kähler manifold with nonnegative orthogonal bisectional curvature:

Using a strong maximum principle it is not hard to see that under the Ricci flow either the orthogonal bisectional curvature turns positive immediately or the manifold is covered by a product or a symmetric space. A Hermitian symmetric space has nonnegative bisectional curvature and if M is covered by a product one can argue for each factor separately. Thus we may assume that $(M, g(t_0))$ has positive orthogonal bisectional curvature and without loss of generality $t_0 = 0$.

We put $S = \{X \in \mathfrak{gl}(n, \mathbb{C}) \mid \mathrm{tr}(X) = \mathrm{rank}(X) = 1\}$. It is easy to see that $\partial_\infty S$ is given by the space of nilpotent matrices of rank ≤ 1. Thus $C(\partial_\infty S, 0)$ corresponds to the space of curvature operators with nonnegative orthogonal bisectional curvature. ¿From the above Lemma we deduce that $(M, g(t_0))$ satisfies the curvature condition $C(S, h)$ for some $h << 0$, that is $C(S, h)$ contains the compact set of curvature operators given by evaluating the curvature operator of $(M, g(0))$ at all base points.

By Theorem 1 we deduce that $(M, g(t))$ satisfies $C(S, h)$ for all t. This in turn implies that the bisectional curvature of $(M, g(t))$ stays bounded below by a fixed constant. Since the scalar curvature blows up at a singularity this shows that $(M, g(t))$ pinches toward nonnegative bisectional curvature.

b) Let $L \in [0, \infty]$ and put

$$S(L) := \{X + zI \in \mathfrak{gl}(n, \mathbb{C}) \mid z \in \mathbb{C}, |z| < L, \mathrm{rank}(X) = \mathrm{tr}(X) = 1\}$$

where I denotes the identity matrix. It is straightforward to check that $\partial_\infty S(L)$ is still given by the nilpotent rank 1 matrices, if $L < \infty$. Similarly to a) this in turn shows that nonnegative orthogonal bisectional curvature pinches toward the curvature condition $C(S(\infty), 0)$. In the case of $n = 2$, $C(S(\infty), 0)$ consists of the nonnegative Kähler curvature operators.

c) Suppose n is even and put $S = \{X \in \mathfrak{so}(n, \mathbb{C}) \mid X^2 = -\operatorname{id}\}$. Then $\partial_\infty S = \{X \in \mathfrak{so}(n, \mathbb{C}) \mid X^2 = 0\}$. As always $C(\partial_\infty S, 0)$ pinches toward $C(S, 0)$. If $n = 4, 6$, then $C(\partial_\infty S, 0)$ coincides with nonnegative isotropic curvature. In all even dimensions n the manifold $\mathbb{S}^{n-1} \times \mathbb{R}$ satisfies $C(S, 0)$ strictly.

d) Put $S = \{X \in \mathfrak{so}(n, \mathbb{C}) \mid \text{ the eigenvalues of } X \text{ have absolute value} \le 1\}$ Clearly $C(S, 0)$ corresponds to the cone of nonnegative curvature operators. Moreover $\partial_\infty S = \{X \in \mathfrak{so}(n, \mathbb{C}) \mid X^n = 0\}$. As before the curvature condition $C(\partial_\infty S, 0)$ pinches toward $C(S, 0)$.

4.1. **Manifolds satisfying PIC1.** Let (M, g) be a compact manifold such that $\mathbb{R} \times M$ has positive isotropic curvature (PIC1). Consider the subset $S \subset \mathfrak{so}(n, \mathbb{C})$ of rank 2 matrices with eigenvalues ± 1. It is easy to see that $\partial_\infty S$ consists of all nilpotent matrices X in $\mathfrak{so}(n, \mathbb{C})$ of rank ≤ 2. Moreover, such a matrix satisfies $X^3 = 0$, and if $X \ne 0$, then $\operatorname{rank}(X) = 2$. Thus (M, g) satisfies $C(\partial_\infty S, 0)$ strictly, see Remark 2 d). By Lemma 4.1 (M, g) satisfies $C(S, h)$ for some $h < 0$. By replacing h by $h - 1$ we may assume that (M, g) satisfies $C(S, h)$ strictly. We consider the linear map

$$l_s \colon S^2_B(\mathfrak{so}(n)) \to S^2_B(\mathfrak{so}(n)), R \mapsto R + 2s \operatorname{Ric} \wedge \operatorname{id} + (n-1)(n-2)s^2 R_I,$$

where R_I is the orthogonal projection of R to multiples of the identity.

Since the operators in $C(S, h) \subset C(\partial_\infty S, 0)$ have nonnegative Ricci curvature, we can apply [Böhm and Wilking 2008, Proposition 3.2], to see that $l_s(C(S, h))$ defines a Ricci flow invariant curvature condition for positive $s \le \frac{\sqrt{2n(n-2)+4}-2}{n(n-2)}$.

Clearly $(M, g(0))$ satisfies $l_s(C(S, h))$ for small $s > 0$. It is straightforward to check that the set $D = l_s(C(S, h)) \setminus l_{s/2}(C(S, 0))$ is bounded. Choose $L > 0$ such that all operators in D have trace $< L$. We also assume that the scalar curvature of (M, g) is bounded by L. Put $K = \{R \in l_{s/2}(C(S, 0)) \mid \operatorname{tr}(R) = L\}$. Notice that K is a compact subset of the interior of $C(S, 0)$. We now choose a convex, $\mathrm{O}(n)$–invariant, ODE invariant set $F \subset l_{s/2}(C(S, 0))$ with $K \subset F$ and $\lim_{\lambda\to\infty} \frac{1}{\lambda} F = \mathbb{R}_+ I$. The existence of this set follows immediately from Böhm and Wilking [2008, proof of Theorem 3.1 combined with Theorem 4.1], see [Theorem 6.1, Wilking 2007] – here we used that $C(S, 0)$ is a Ricci flow invariant curvature condition in between nonnegative curvature operator and nonnegative sectional curvature. We put

$$\hat{F} := \Big(\{R \in F \mid \operatorname{tr}(R) \ge L\} \cap l_s(C(S, h))\Big) \cup \{R \in l_s(C(S, h)) \mid \operatorname{tr}(R) \le L\}$$

It is easy to see that $\hat{F}$ is convex and $O(n)$-invariant and ODE-invariant. Clearly $\lim_{\lambda\to\infty} \frac{1}{\lambda} \hat{F} = \lim_{\lambda\to\infty} \frac{1}{\lambda} F = \mathbb{R}_+ I$. Moreover by construction (M, g) satisfies $\hat{F}$. By Theorem 5.1 in [Böhm and Wilking, 2008] which is a slight extension of an earlier convergence result of Hamilton [1986], g evolves under the normalized Ricci flow to a constant curvature limit metric on M.

This recovers the main theorem of [Brendle, 2008], which in turn generalized the main result of [Brendle and Schoen, 2009].

Part of the proof of Theorem 4 in the next section is analogous to the above arguments. However, there are a few additional twists which come from the fact that the ODE in the Kähler case behaves differently.

5. Kähler manifolds with positive orthogonal bisectional curvature

This section is devoted to the proof of Theorem 4.

We first want to explain the equivalence of the two definitions we gave in the introduction. We defined the cone of nonnegative orthogonal bisectional curvature as $C(S)$ where $S \subset \mathfrak{gl}(n, \mathbb{C}) \cong \mathfrak{u}(n) \otimes_{\mathbb{R}} \mathbb{C}$ is the space of nilpotent rank 1 matrices. We claim $R \in C(S)$ if and only if $K(v, w) + K(v, iw) \geq 0$ for all unit vectors v, w satisfying $\mathbb{C}v \perp \mathbb{C}w$. Since a complex rank one $n \times n$-matrix is via an element in $\mathsf{U}(n)$ conjugate to a matrix which is zero away from the upper 2×2 block it is clear that it suffices to explain the equivalence in the case of $n = 2$. Using the natural embedding $\mathfrak{u}(2) \subset \mathfrak{so}(4)$ the nilpotent rank 1 matrices in $\mathfrak{gl}(2, \mathbb{C})$ correspond to totally isotropic rank 2 matrices in $\mathfrak{so}(4, \mathbb{C})$. A totally isotropic rank 2 matrix in $\mathfrak{so}(4, \mathbb{C})$ is contained in an ideal $\mathfrak{su}(2, \mathbb{C}) \subset \mathfrak{so}(4, \mathbb{C})$. It follows easily (for n=2) that $C(S) \cap S^2_K(\mathfrak{u}(2))$ is given by the cone of those Kähler curvature operators in $S^2_K(\mathfrak{u}(2))$ with nonnegative isotropic curvature.

Let $R \in S^2_K(\mathfrak{u}(2))$ with $n = 2$ and $v, w \in \mathbb{C}^2 \cong \mathbb{R}^4$ with $\mathbb{C}v \perp \mathbb{C}w$ then R induces an endomorphism of $\mathfrak{so}(4)$ such that $v \wedge w + iw \wedge iv$ and $v \wedge iw + iv \wedge w$ are in the kernel of R. It is now easy to see that $K(v, iw) + K(v, w) = K(iv, w) + K(iv, iw)$ is a positive multiple of the isotropic curvature $R(v \wedge w - iw \wedge iv, v \wedge w - iw \wedge iv) + R(v \wedge iw - iv \wedge w, v \wedge iw - iv \wedge w)$. Thus nonnegativity of $K(v, iw) + K(v, w)$ for all possible choices v and w is equivalent to $R \in C(S)$.

The Ricci flow on Kähler manifolds is particularly well behaved if the first Chern class is a multiple of the Kähler class. In our situation this will follow from

Lemma 5.1. *Let M be a compact Kähler manifold with nonnegative orthogonal bisectional curvature. Then the Bochner operator on two forms is nonnegative. In particular any harmonic two form is parallel. If (M, g) has positive orthogonal bisectional curvature, then $H^2(M, \mathbb{R}) \cong \mathbb{R}$.*

Proof. For a Riemannian manifold the Bochner operator on two forms is given by $\mathcal{R} := \mathrm{Ric} \wedge \mathrm{id} - R$ where R is the curvature operator Ric is the Ricci curvature and $\mathrm{Ric} \wedge \mathrm{id}(e_i \wedge e_j) = \frac{1}{2}\big(\mathrm{Ric}(e_i) \wedge e_j + e_i \wedge \mathrm{Ric}(e_j)\big)$. Compare for example [Ni and Wilking, 2009]. If R is the curvature operator of a Kähler manifold, then it is easy to see that $\mathrm{Ric} \wedge \mathrm{id}$ leaves the Lie algebra $\mathfrak{u}(n) \subset \mathfrak{so}(2n)$ invariant. Since the orthogonal complement of $\mathfrak{u}(n)^\perp$ is contained in kernel of R, we can show $\mathcal{R}_{|\mathfrak{u}(n)^\perp} \geq 0$ by establishing

Claim 1. For a Kähler manifold with nonnegative orthogonal bisectional curvature the following holds: If $v_1, v_2 \in T_pM$ are unit vectors with $\mathbb{C}v_1 \perp \mathbb{C}v_2$, then $\mathrm{Ric}(v_1, v_1) + \mathrm{Ric}(v_2, v_2) \geq 0$.

We extend v_1, v_2 to a complex orthonormal basis $v_1, \ldots, v_n$ of T_pM. Then

$$\begin{aligned}\mathrm{Ric}(v_1, v_1) + \mathrm{Ric}(v_2, v_2) \quad = \quad & K(v_1, iv_1) + 2K(v_1, v_2) + 2K(v_1, iv_2) + K(v_2, iv_2) + \\ & + \sum_{j=3}^{n} K(v_1, v_j) + K(v_2, v_j) + K(v_1, iv_j) + K(v_2, iv_j)\end{aligned}$$

Since R has nonnegative orthogonal bisectional curvature we know $K(v_j, v_k) + K(v_j, iv_k) \geq 0$ for $k \neq l$. In other words it suffices to show that the first four summands add up to a nonnegative number. This in turn is equivalent to establishing the claim for complex surfaces. But if $n = 2$, then $\mathrm{Ric}(v_1, v_1) + \mathrm{Ric}(v_2, v_2) = \frac{1}{2}\,\mathrm{scal} \geq 0$.

We can finish the proof of the first part of Lemma by establishing

Claim 2. $\mathcal{R}_{|\mathfrak{u}(n)} \geq 0$.

Suppose $\omega \in \mathfrak{u}(n)$ is an eigenvector of $\mathcal{R}$. We can find an orthonormal basis $v_1, \cdots, v_n$ with $\mathbb{C}v_j \perp \mathbb{C}v_k$ for $j \neq k$ and real numbers λ_i such that ω is given by $\omega = \sum_{j=1}^n \lambda_j v_j \wedge iv_j$. One now checks by straightforward computation

$$2\langle \mathcal{R}\omega, \omega\rangle = \sum_{j\neq k}^{n} (\lambda_j^2 + \lambda_k^2)\big(K(v_j, v_k) + K(v_j, iv_k)\big) - 2\sum_{j\neq k} \lambda_j \cdot \lambda_k R(v_j, iv_j, v_k, iv_k)$$

Since $v_j \wedge v_k + iv_k \wedge iv_j$ and $v_j \wedge iv_k + iv_j \wedge v_k$ are in the kernel of R it follows from the first Bianchi identity that $R(v_j, iv_j, v_k, iv_k) = K(v_j, iv_k) + K(v_j, v_k)$. Hence

$$2\langle \mathcal{R}\omega, \omega\rangle = \sum_{j\neq k}^{n} (\lambda_j - \lambda_k)^2\big(K(v_j, v_k) + K(v_j, iv_k)\big)$$

which is nonnegative as each summand is nonnegative.

This shows that $\mathcal{R}$ is nonnegative. Therefore any harmonic two form is parallel. If the orthogonal bisectional curvature is positive, it is easy to deduce that the kernel of $\mathcal{R}$ is given by multiples of the Kähler form and thus any harmonic two form is a multiple of the Kähler form. □

As before we consider the Lie algebra $\mathfrak{u}(n)$ of skew hermitian $n \times n$ matrices endowed with the scalar product $\langle u, v\rangle = -\operatorname{tr} u \cdot v$. The vectorspace of Kähler curvature operators $S^2_K(\mathfrak{u}(n))$ can be naturally seen as a subspace of the space $S^2(\mathfrak{u}(n))$ of selfadjoint endomorphism of $\mathfrak{u}(n)$.

Given two hermitian endomorphisms $A, B\colon \mathbb{C}^n \to \mathbb{C}^n$ we let $A \star B\colon \mathfrak{u}(n) \to \mathfrak{u}(n)$ denote the self adjoint endomorphism of $\mathfrak{u}(n)$ defined by

$$2\langle A \star Bu, v\rangle = -\operatorname{tr} AuBv - \operatorname{tr}(AvBu) - \operatorname{tr}(Au)\operatorname{tr}(Bv) - \operatorname{tr}(Av)\operatorname{tr}(Bu).$$

A straightforward computation shows that $A \star B$ is a Kähler curvature operator. We put $E = \mathrm{id} \star \mathrm{id}$. Then E corresponds to the curvature operator of $\mathbb{CP}^n$ scaled such that the sectional curvature lies in the interval [1/2,2]. Thus E has the eigenvalue 1 with multiplicity $n^2 - 1$ and the eigenvalue $n + 1$ with multiplicity 1. The operators of the form $A \star \mathrm{id}$ are precisely given by the orthogonal complement of the Ricci flat operators $\langle W\rangle$ in $S^2_K(\mathfrak{u}(n))$.

For $R \in S^2_K(\mathfrak{u}(n))$ we let $\mathrm{Ric}(R)$ denote its Ricci curvature which we can view as a hermitian $n \times n$ matrix. We define a linear map

$$l_s\colon S^2_K(\mathfrak{u}(n)) \to S^2_K(\mathfrak{u}(n)),\ R \mapsto R + 2s\,\mathrm{Ric}(R) \star \mathrm{id} + s^2\,\mathrm{scal}(R)E. \tag{5}$$

Similarly, to [Böhm and Wilking, 2008] we are interested in how the Ricci flow ODE changes if we pull it back under l_s. It is not hard but tedious to derive a formula similar to the one in [Böhm and Wilking, 2008]. However, for our purposes here the following simple formula will be sufficient.

Lemma 5.2. *For $R \in S^2_K(\mathfrak{u}(n))$ put $D(s)(R) = l_s^{-1}(l_s(R)^2 + l_s(R)^\#) - R^2 - R^\#$. Then*

$$\tfrac{d}{ds}_{|s=0} D(s)(R) = D'(0)(R) = 2\operatorname{Ric}(R) \star \operatorname{Ric}(R)$$

Proof. We let $\langle W \rangle \subset S^2_K(\mathfrak{u}(n))$ denote the kernel of $R \mapsto \operatorname{Ric}(R)$. For $R \in S^2_K(\mathfrak{u}(n))$ the orthogonal projection R_W of R to $\langle W \rangle$ is called the (Kähler)-Weyl part of R. As in the real case $\langle W \rangle$ is an irreducible module and analogously to [Böhm and Wilking, 2008] one can show that $D(t)(R)$ is independent of R_W. We let

$$B(s)(R_1, R_2) = \tfrac{1}{4}\big(D(s)(R_1 + R_2) - D(s)(R_1 - R_2)\big)$$

denote the corresponding bilinear form. Since any Ricci tensor is the sum of commuting rank one tensors it suffices to prove the corresponding statement for B in the special case that $\operatorname{Ric}(R_1)$ and $\operatorname{Ric}(R_2)$ are commuting rank 1 matrices. Clearly we may assume that $\operatorname{Ric}(R_1)$ have 1 as an eigenvalue. Using the polarization

$$B(s)(R_1, R_2) = \tfrac{1}{2}\big(D(s)(R_1 + R_2) - D(t)(R_1) - D(s)(R_2)).$$

we deduce that it suffices to prove the original statement for the following two special cases: The statement holds for one curvature operator for which $\operatorname{Ric}(R)$ has rank 1 and the statement holds for one curvature operator for which $\operatorname{Ric}(R)$ has rank 2 with 2 equal nonzero eigenvalues. In particular, it suffices to check that the statement holds in the case that R is the curvature operator R_k of $\mathbb{CP}^k \times \mathbb{C}^{n-k}$, $k = 1, \dots, n$.

It is straightforward to check that $D'(0)(R)$ and $2\operatorname{Ric}\star\operatorname{Ric}$ have the same trace. Clearly this proves the formula for R_k in the case $k = n$.

Notice that $R_k^2 + R_k^\# = (k+1)R_k$. It is easy to see that $D'(0)(R_k)_{|\mathfrak{u}(k)}$ is independent of n. Using that we know the formula in the case $k = n$ we deduce $D'(0)(R_k)_{|\mathfrak{u}(k)} = 2\operatorname{Ric}\star\operatorname{Ric}_{|\mathfrak{u}(k)}$. Moreover it is easy to see that both operators contain the subalgebra $\mathfrak{u}(n-k)$ in their kernel. It remains to check that $D'(0)(R_k)$ restricted to $(\mathfrak{u}(n-k) \oplus \mathfrak{u}(k))^\perp$ vanishes. For symmetry reasons this restriction is given by a multiple of the identity. Combining this with the facts that $D'(0)(R_k)$ and $2\operatorname{Ric}\star\operatorname{Ric}$ coincide on $\mathfrak{u}(n-k) \oplus \mathfrak{u}(k)$ and have the same trace the lemma follows. □

Definition 5.3. We consider two subsets $C_1, C_2 \subset S^2_K(\mathfrak{u}(n))$ which are convex, closed and $\mathsf{U}(n)$-invariant. Recall that we say that a Kähler manifold (M, g) satisfies C_i if the curvature operator at each point is contained in C_i. We say that C_1 defines a Ricci flow invariant curvature condition under the constraint C_2 if the following holds: Any compact solution $(M, g(t))$ to the unnormalized Kähler Ricci flow ($t \in [0, T]$) satisfying C_2 at all times and satisfying C_1 at $t = 0$, satisfies C_1 at all times.

One can carry over [see Chow and Lu, 2004] the proof of Hamilton's maximum principle to show

Theorem 5.4. *Suppose for all $R \in C_1 \cap C_2$ we have $R^2 + R^\# \in T_R C_1$, then C_1 defines a Ricci flow invariant curvature condition under the constraint C_2.*

As consequence of this and the previous Lemma we obtain

Corollary 5.5. *Suppose $C \subset S^2_K(\mathfrak{u}(n))$ is a convex $\mathsf{U}(n)$-invariant set which is invariant under the ODE $R' = R^2 + R^\#$ and contains the space of nonnegative*

Kähler curvature operators. Let $p \in (0,1)$ and put

$$C_2(p) = \{R \in S^2_K(\mathfrak{u}(n)) \mid \mathrm{Ric}(R) \geq p\tfrac{\mathrm{scal}}{2n}\}$$

Then there is an $s_0 = s_0(p, C) > 0$ such that the set $l_s(C)$ defines a Ricci flow invariant curvature condition under the constraint $C_2(p)$ for all $s \in [0, s_0]$, where l_s is the linear map defined by (5).

Proof. Put $X(s)(R) = l_s^{-1}(l_s(R)^2 + l_s(R)^{\#})$. By the above Lemma

$$X(s) = R^2 + R^{\#} + 2s\,\mathrm{Ric}(R) \star \mathrm{Ric}(R) + O(s^2)$$

where $O(s^2)$ stands for an operator satisfying $\|O(s^2)\| \leq Cs^2\|\mathrm{Ric}(R)\|^2$ for some constant $C > 0$, $s \in [0,1]$.

We choose s_0 so small that $l_s^{-1}(C_2(p)) \subset C_2(p/2)$ for all $s \in [0, s_0]$. Thus it suffices to check that $X(s)(R) \in T_RC$ for all $R \in C \cap C_2(p/2)$ and for all small s. Using our estimate on $O(s^2)$ we can find s_0 such that the operator $2s\,\mathrm{Ric}(R) \star \mathrm{Ric}(R) + O(s^2)$ is positive for all $R \in C \cap C_2(p/2)$ and all $s \in (0, s_0]$. Since the positive operators are contained in C they are also contained in the tangent cone T_RC. By assumption $R^2 + R^{\#} \in T_RC$ and thus $X(s)(R) \in T_RC$ for all $R \in C \cap C_2(p/2)$.

□

Proof of Theorem 4. We consider a solution to the unnormalized Ricci flow $(M, g(t))$, $t \in [0, T)$. Since the scalar curvature is positive ($n \geq 2$), a finite time singularity T occurs. By Lemma 5.1 we have $H^2(M, \mathbb{R}) \cong \mathbb{R}$. Thus the first Chern class is a multiple of the Kähler class. Since a finite time singularity occurs this in turn implies it is positive.

Claim. For some $\varepsilon, p > 0$ we have that $(M, g(t))$ satisfies $C_2(p)$ (see Corollary 5.5 for a definition) for all $t \in [T - \varepsilon, T)$.

By Lemma 4.1 we can find some small h such that $(M, g(t))$ satisfies $C(S, h)$, where $S \subset \mathfrak{gl}(n, \mathbb{C})$ is the set of rank 1 matrices which have 1 as an eigenvalue.

We argue by contradiction and assume that we can find $p_i \to 0$ and $t_i \to T$ such that $(M, g(t_i))$ does not satisfy $C_2(p_i)$. We rescale the manifold to have maximal curvature one. By an argument of Perelman which was written up by Sesum and Tian [2006] $(M, \lambda_i g(t_i))$ subconverges to a compact limit manifold (M, g_∞).

(M, g_∞) is a Kähler manifold satisfying the curvature condition $\lim_{i\to\infty} \frac{1}{\lambda_i} C(S, h) = C(S, 0)$. Recall that $C(S, 0)$ is the cone of curvature operators with nonnegative bisectional curvature. Thus (M, g_∞) has nonnegative bisectional curvature and in particular nonnegative Ricci curvature. By compactness we can assume that g_∞ has a backward solution to the Ricci flow with nonnegative bisectional curvature. Since (M, g_∞) is diffeomorphic to M we know from Lemma 5.1 that its second homology is isomorphic to $\mathbb{R}$. Thus (M, g_∞) does not have any flat factors. Combining with the strong maximum principle we deduce that (M, g_∞) has positive Ricci curvature. But this contradicts our choice of $(M, g(t_i))$.

After replacing $g(0)$ by $g(T - \varepsilon)$ we may assume that $(M, g(t))$ satisfies $C_2(p)$ for all $t \in [0, T)$. Moreover we can assume that the curvature operator of $(M, g(0))$ at each point is contained in the interior of $C(S, h)$ – otherwise one can just replace h by $h - 1$.

This in turn shows that $(M, g(0))$ satisfies $l_s(C(S, h))$ for sufficiently small $s > 0$. By Corollary 5.5 $(M, g(t))$ satisfies $l_s(C(S, h))$ for all $t \in [0, T]$ and some $s > 0$.

It is elementary to check that there is an $\varepsilon > 0$ and $C > 0$ such that for all $R \in l_s(C(S,h))$ we have

$$R - (\varepsilon\,\mathrm{scal}(R) - C)E \in C(S,0)$$

Since for the unnormalized Ricci flow the scalar curvature of $(M, g(t))$ converges uniformly to ∞ for $t \to T$, we deduce that $(M, g(t))$ has positive bisectional curvature for some t.

□

Appendix: Strong maximum principle for the Ricci flow.

In this appendix we will sketch the argument for the following extension of Brendle and Schoen's maximum principle.

Theorem 5.6. *Let $S \subset \mathfrak{so}(n,\mathbb{C})$ be an $\mathrm{Ad}_{\mathrm{SO}(n,\mathbb{C})}$–invariant subset and consider a solution to the Ricci flow $(M, g(t))$, $t \in [0,T)$, satisfying $C(S)$ for all t. By choosing a linear isometry between $(T_pM, g(t))$ and $\mathbb{R}^n$ we obtain a subset $S(p,t) \subset \mathfrak{so}((T_pM, g(t))) \otimes_{\mathbb{R}} \mathbb{C}$ corresponding to S for each (p,t). Put*

$$N(p,t) = \{X \in S(p,t) \mid R_{g(t)}(X, \bar{X}) = 0\}$$

Then $N(p,t)$ is invariant under parallel transport for $t > 0$.

As usual with strong maximum principles we do not require that $(M, g(t))$ is compact or complete. The methods used to derive the above theorem from Theorem 1 (and its proof) are due to Brendle and Schoen [2008]. In fact, Proposition 8 in that paper is the special case of the above theorem where S is given by the totally isotropic rank 2 matrices. The analogue of the above theorem for Kähler manifolds holds as well with the same proof.

A delicate part of Brendle and Schoen's proof of the strong maximum principle for isotropic curvature is that it does not just use the invariance of positive isotropic curvature but also the proof of the invariance by means of first and second variation formulas. This is here true as well.

Proof. In the following we can assume that S is invariant under scaling with positive numbers. Moreover we may assume that for X, $Y \in S$ there is some $g \in \mathrm{SO}(n,\mathbb{C})$ and $\lambda > 0$ with $\mathrm{Ad}_g X = \lambda Y$. In fact otherwise we decompose S into subsets with this property and prove the theorem for each subset separately. Notice that these assumptions imply in particular that S is a submanifold with a transitive smooth action of $\mathrm{SO}(n,\mathbb{C}) \times \mathbb{R}_+$. Therefore $S(p,t)$ defines a bundle over $M \times (0,T)$ whose totalspace we denote by T. We consider on T the function

$$u\colon T \to \mathbb{R},\ u(v) = R_{g(t)}(v, \bar{v}).$$

We lift the vectorfield $\frac{\partial}{\partial t}$ horizontally to a vectorfield on T using the connection induced on T by the connection ∇ on $M \times (0,T)$ from section 2. We denote this horizontal lift again by $\frac{\partial}{\partial t}$. Then

$$\tfrac{\partial u}{\partial t}(v, \bar{v}) = \Delta_h u + 2(R^2(v,\bar{v}) + R^{\#}(v,\bar{v})),$$

where $\Delta_h u$ is the horizontal Laplacian which is defined as follows. Choose in a neighborhood of (p,t) vectorfields $X_1, \ldots, X_n$ tangential to M with $g(t)(X_i, X_j) = \delta_{ij}$ and put $Y_j = \nabla^{g(t)}_{X_i} X_i$. Let $\hat{X}_i$ and $\hat{Y}_i$ denote the horizontal lifts of X_i and Y_i to T. Then $\Delta_h u = \sum_{i=1}^n \hat{X}_i \hat{X}_i u - \hat{Y}_i u$.

From the proof of the invariance in section 1 we can derive

$$R^{\#}_{g(t)}(v,\bar v) \geq C\|R\| \inf\left\{ \tfrac{d^2}{dt^2} u(Ad_{\exp(tx)} v) \;\middle|\; x\in \mathfrak{so}((T_pM, g(t)),\mathbb{C}), \|x\|\leq 1\right\}$$

for all $v\in T_pM$ where $C=C(n)$ is a constant. We now introduce coordinates on some relative compact subset $U\subset T$, which have an extension to a neighborhood of $\bar U$. Then the corresponding function $\tilde u$ in local coordinates, which is defined on some open subset $V\subset \mathbb{R}^k$, satisfies

$$\sum_{i=1}^n \tilde X_i \tilde X_i \tilde u \leq -K \inf\left\{\mathrm{Hess}(\tilde u)(a,a) \mid a\in\mathbb{R}^n, \|a\|\leq 1\right\} + K\|\mathrm{grad}(\tilde u)\|$$

for some large constant K, where $\tilde X_i$ denote the corresponding vectorfields in coordinates. We can now apply Proposition 4 from [Brendle and Schoen, 2008], to see that the level set $\tilde u^{-1}(0)$ is invariant under the (local) flows of the vectorfields X_i. Translating this back we obtain that the level set $u^{-1}(0)$ in T is invariant under spacial parallel translation. □

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