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Scalar and mean curvature comparison via the Dirac operator

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Classical diameter estimates in positive curvature

Let (M, g) be a complete Riemannian n-manifold.

$$\sec \geqslant 1 \Rightarrow \mathrm{Ric} \geqslant (n-1) \Rightarrow \mathrm{scal} \geqslant n(n-1)$$

Theorem (Myers 1941; Bonnet 1855, Synge 1926 for sec $\geqslant 1$)

If $\mathrm{Ric}_g \geqslant (n-1)$, *then* $\mathrm{diam}(M, g) \leqslant \pi$.

Theorem (Cheng 1975)

If $\mathrm{Ric}_g \geqslant (n-1)$ *and* $\mathrm{diam}(M, g) = \pi$, *then* (M, g) *is isometric to the standard sphere.*

What about $\mathrm{scal} \geqslant n(n-1)$?

- No general diameter bound ($n \geqslant 3$), e.g.: $S^{n-1} \times \mathbb{R}$, $g = \sqrt{\frac{n-2}{n}} g_{S^{n-1}} + dx^2$
- Can you also find a complete metric of $\mathrm{scal} \geqslant n(n-1)$ on $T^{n-1} \times \mathbb{R}$?
  - **NO.** (Gromov–Lawson ’80s)

Let’s try anyway ... consider a warped product metric $g = \varphi^2 g_{\text{flat}} + dx^2$ on $T^{n-1} \times \mathbb{R}$

$$\rightsquigarrow \mathrm{scal}_g = -2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2$$

We can achieve $\mathrm{scal}_g = n(n-1)$ for a time ...

$$\varphi \colon \left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \to (0, \infty), \quad \varphi(x) = \cos\left(\frac{nx}{2}\right)^{\frac{2}{n}} \quad \to 0 \text{ at } \pm\frac{\pi}{n}$$

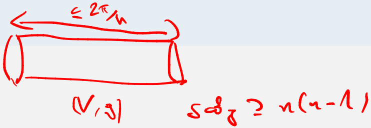
where $h = \varphi'/\varphi$ satisfies

$$h(x) = -\tan\left(\frac{nx}{2}\right). \quad \to \pm\infty \text{ at } \mp\frac{\pi}{n}$$

Conjecture (Gromov's band width conjecture 2018)

Let M be a closed connected manifold of dimension $n - 1 \neq 4$ such that M does not admit a metric of positive scalar curvature. Let g be a Riemannian metric on $V = M \times [-1, 1]$ of $\mathrm{scal}_g \geqslant n(n-1)$. Then

$$\mathrm{width}(V, g) = \mathrm{dist}_g(\partial_- V, \partial_+ V) \leqslant \frac{2\pi}{n}, \qquad \text{where } \partial_\pm V = M \times \{\pm 1\}.$$



Approaches:

- For the torus and related manifolds via minimal hypersurface ideas (Gromov)
- Via the spinor Dirac operator (Cecchini, Z.)
- Via "μ-bubbles" (Gromov, Räde)

Area-extremality of spheres & The long neck problem

Theorem (Llarull 1998)

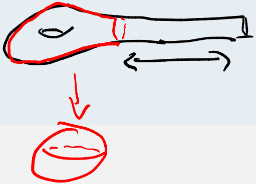
Let (M, g) be ***closed*** *connected Riemannian* ***spin*** *manifold of dimension $n \geqslant 2$ and $\mathrm{scal}_g \geqslant n(n-1)$. Then any smooth area non-increasing map $f\colon M \to S^n$ of non-zero degree is an* ***isometry****.*

Conjecture (Gromov's long neck conjecture)

Let (M, g) be a connected compact oriented Riemannian manifold of dimension $n \geqslant 2$ and $\mathrm{scal}_g \geqslant n(n-1)$ and $f\colon M \to S^n$ be a smooth area non-increasing map that is locally constant in a neighborhood of $\partial M \neq \emptyset$. If

$$\mathrm{dist}_g(\partial M, \mathrm{supp}(df)) \geqslant \frac{\pi}{n},$$

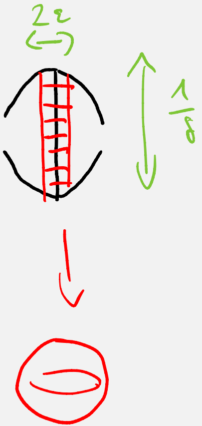
then f has vanishing mapping degree.



Optimality of the long neck estimate

$$\frac{1}{\delta} T^{n-1} \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \xrightarrow{\Phi \times \Psi} S^{n-1} \times S^1 \xrightarrow[\text{degree } 1]{} S^n$$

$\sim \frac{\delta}{\varepsilon}$ area contracting



- $\Phi : T^{n-1} \longrightarrow S^{n-1}$ degree 1
- $\Psi : \left(-\frac{\pi}{n}, \frac{\pi}{n}\right) \longrightarrow S^1$ degree 1
  constant outside $[-\varepsilon, \varepsilon]$.

Normal radius estimates

A collar conjecture

Let M be a compact manifold of dimension $n \geqslant 2$ such that its double $\mathrm{d}\,M = M \cup_{\partial M} M^-$ does not admit a metric of positive scalar curvature. Then for any Riemannian metric g on M of $\mathrm{scal}_g \geqslant n(n-1)$,

$$\mathrm{rad}^{\circledcirc}(\partial M \subset M, g) \leqslant \frac{\pi}{n}.$$

mean curvature of ∂M w.r.t. interior unit normal.

- If $\mathrm{scal}_g > 0$ on M and $H_{(\partial M, g)} \geqslant 0$, then $\mathrm{d}\,M$ admits a metric of psc. (Gromov–Lawson 1980, Almeida 1985; see also Bär–Hanke 2020)
- **Again:** $T^{n-1} \times [-l, l]$ with $g = \varphi^2 g_{\mathrm{flat}} + \mathrm{d}x^2$, $\varphi(x) = \cos(nx/2)^{\frac{2}{n}}$, shows optimality.

Index theory of Dirac operators

(M, g) ... closed (even-dim.) Riemannian spin manifold.

$$\not{D}_g = \begin{pmatrix} 0 & \not{D}_g^- \\ \not{D}_g^+ & 0 \end{pmatrix}$$

$\rightsquigarrow$ **Spinor Dirac operator:** $\displaystyle \not{D}_g = \sum_{i=1}^{n} c(e^i)\nabla_{e_i} \colon C^\infty(M, \not{S}_M) \to C^\infty(M, \not{S}_M)$

$$\operatorname{index}(\not{D}_g) := \dim\ker(\not{D}_g^+) - \dim\ker(\not{D}_g^-) = \widehat{A}(M) \qquad \textbf{(Atiyah–Singer)}$$

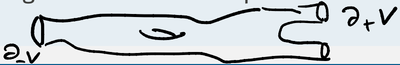
Schrödinger–Lichnerowicz formula: $\displaystyle \not{D}_g^2 = \underbrace{\nabla_g^*\nabla_g}_{\geq 0} + \frac{\mathrm{scal}_g}{4}$

- $\widehat{A}(M) \neq 0 \implies M$ does **not** admit a metric of psc.
- $\widehat{A}(M) \neq 0$ and $\mathrm{scal}_g \geq 0 \implies M$ admits a **parallel spinor** (and thus $\mathrm{Ric}_g = 0$).

Band estimates and rigidity via the Dirac operator

Definition

A **band** is a connected compact manifold V together with a decomposition $\partial V = \partial_- V \sqcup \partial_+ V$ such that $\partial_\pm V$ are unions of components.



Theorem (Cecchini–Z. ’21)

Let (V, g) *be a Riemannian spin band such that* $\widehat{A}(\partial_- V) \neq 0$ *and* $\mathrm{scal}_g \geqslant n(n-1)$.

1. *If* $H_g \geqslant -\tan(nl/2)$ *for some* $0 < l < \pi/n$, *then*

   $\to -\infty$ as $l \to \pi/n$

$$\mathrm{width}(V, g) = \mathrm{dist}_g(\partial_- V, \partial_+ V) \leqslant 2l.$$

2. *If equality is attained, then* V *is isometric to* $M \times [-l, l]$,

$$g = \cos(nx/2)^{2/n} g_M + dx^2,$$

   for some spin manifold (M, g_M) *that admits a parallel spinor.*

3. *In particular,* $\mathrm{width}(V, g) < 2\pi/n$.

Digression: Results via higher index theory

Precursor results (Z. '19, Cecchini '20, Z. '20)

We previously obtained the conclusion $\operatorname{width}(V, g) < 2\pi/n$ using higher index invariants, e.g. if the **Rosenberg index**

$$\alpha(M) \neq 0 \in KO_{n-1}(C^*\pi_1 M).$$

Corollary

Gromov's width conjecture for $V = M \times [-1, 1]$ holds for all manifolds M which are one of the following.

- *simply-connected of dimension $\geqslant 5$,*
- *spin and enlargeable,*
- *spin, aspherical and s.t. $\pi_1 M$ satisfies the strong Novikov conjecture.*

The long neck problem with mean curvature

Theorem (Cecchini–Z. ’21)

Let (M, g) be a compact connected n-dim. Riemannian spin manifold with boundary of $\mathrm{scal}_g \geqslant n(n-1)$, where $n \geqslant 2$ is even. Let $f\colon M \to S^n$ be a smooth area non-increasing map. Suppose that for some $0 < l < \pi/n$ the following estimates hold:

- $\mathrm{dist}_g(\partial M, \mathrm{supp}(df)) \geqslant l$.
- $H_g \geqslant -\tan(nl/2)$ *on* ∂M,

Then the mapping degree of f is zero.

- Gromov’s long neck conjecture holds for (even-dimensional) spin manifolds (compare Cecchini ’20).

Normal radius estimate for the boundary

Theorem (Cecchini–Z. ’21)

Let M *be a compact* n*-dim. spin manifold, where* n *even, with boundary* ∂M *such that its double* $\mathrm{d}W = W \cup_{\partial W} W^{-}$ *is compactly enlargeable. Let* g *be a Riemannian metric on* W *such that*

↳ e.g. T^n

- $\mathrm{scal}_g \geqslant n(n-1)$;
- $H_g \geqslant -\tan(nl/2)$ *for some* $0 < l < \pi/n$.

Then

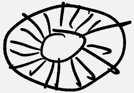
$$\mathrm{rad}^{\odot}(\partial W \subset W, g) \leqslant l.$$

In particular, $\mathrm{rad}^{\odot}(\partial W \subset W, g) < \pi/n$.

Scalar-mean extremality of annuli in space forms

Let (M_κ, g_κ) be the n-dimensional simply-connected space form of curvature $\kappa \in \mathbb{R}$. Consider the annulus around a base-point $p_0 \in M_\kappa$

$$A_{t_-,t_+} = \{p \in M_\kappa \mid t_- \leqslant d_{g_\kappa}(p, p_0) \leqslant t_+\},$$



where $0 < t_- < t_+ < t_\infty$ with $t_\infty = \pi/\sqrt{\kappa}$ if $\kappa > 0$ and $t_\infty = +\infty$ otherwise.

Theorem (Cecchini–Z. ’21)

Let $n \geqslant 3$ be odd and g be a Riemannian metric on A_{t_-,t_+} such that

- $g \geqslant g_\kappa$,
- $\mathrm{scal}_g \geqslant \mathrm{scal}_{g_\kappa} = \kappa n(n-1)$,
- $H_g \geqslant H_{g_\kappa} = \pm \mathrm{ct}_\kappa(t_\pm)$. on $\partial A_{t_-,t_+}$

Then $g = g_\kappa$.

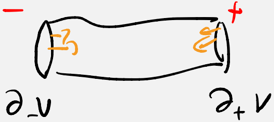
Proof ideas

$\hat{A}(\partial_- V) \neq 0$, $\mathrm{scal}_g \geq n(n-1)$, $H_g \geq -\tan\left(\frac{n\ell}{2}\right)$,
$\mathrm{width}(V,g) \geq 2\ell$

$$\Longrightarrow (V,g) \cong \left(M \times [-\ell,\ell],\ \varphi^2 g_M + dx^2\right)$$

- $\not{D}_V$, $\mathcal{D} = \begin{pmatrix} 0 & \not{D}_V \\ \not{D}_V & 0 \end{pmatrix}$, $\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

- $\chi(u|_{\partial V}) = \pm c(\nu)\sigma$



$$\mathrm{index}(\mathcal{D}_\chi) = \hat{A}(\partial_- V) \neq 0$$

(Bär–Ballmann)

$\chi(u|_{\partial V}) = u|_{\partial V}$

$x : V \to [-\ell, \ell]$ 1-Lipschitz function

s.th. $x|_{\partial_\pm V} = \pm \ell$.

$$B = \mathcal{D} + f(x)\sigma$$

$$B^2 = \mathcal{D}^2 + f(x)^2 + f'(x) \underbrace{c(dx)\sigma}_{\|\cdot\| \leq 1}$$

$u \in \mathrm{dom}(B_x)$
$u \in \ker(B_x)$

$$0 = \int_V |\mathcal{B}u|^2 \,\mathrm{vol}_V = \underline{\frac{n}{n-1}} \int_V \Big(\underbrace{|\mathcal{P}u|^2}_{\geq 0} + \underbrace{\frac{\mathrm{scal}_g}{4}}_{\geq \frac{n^2}{4}} |u|^2 \Big) \,\mathrm{vol}_V$$

$$+ \int_V \langle u, \underbrace{(f(x)^2 + f'(x)\, c(dx)\sigma)}_{\geq -\frac{n^2}{4}} u \rangle \,\mathrm{vol}_V$$

$$+ \int_{\partial V} \underbrace{(\tfrac{n}{2} H_g \pm f(x)}_{\geq 0} |u|^2 \,\mathrm{vol}_{\partial V}.$$

$f(x) = \frac{n}{2} \tan\left(\frac{nx}{2}\right)$

$f(x)^2 - f'(x) = -\frac{n^2}{4}$

$|dx| = 1$

where $\mathcal{P}_\xi u = \nabla_\xi u + \frac{1}{n} c(\xi^\flat) \mathcal{D} u$ (Penrose operator)

Literature

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