OBERSEMINAR SUMMER 2018: AUTOMORPHISMS OF MANIFOLDS

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Talk 1 (Smoothing theory)**.** *Robin.* In this talk, an overview of smoothing theory should be given. The goal of the theory is to compare a suitably defined space CAT(M) of CAT = DIFF or CAT = PL structures on a topological manifold M with a space of sections of a suitable fibre bundle over M. The main result is [4, Theorem V.2.3] and Morlet's Theorem [4, Theorem V.3.4] as a corollary. Obviously, there won't be much time to go into the proofs of Theorem V.2.3 [4] (as it involves most of the material preceeding it in [4]) and the focus should be on statements. Related useful reading: [6], [9].

Talk 2 (An outline of surgery theory I: the structure set and normal invariants)**.** *Rudolf.* Spivak normal fibration of a Poincare complex [7, §3.2], normal maps [7, §3.3]. In particular, Theorem 3.45 and 3.48 of loc. cit. should be discussed. These give an interpretation of the homotopy groups of G/CAT (CAT = DIFF or PL) in terms of the set of normal invariants of the spheres. Then introduce the structure set [7, §5.1] and relate it to the groups of homotopy spheres as in [7, §6.1]. Related useful reading: [5].

Talk 3 (An outline of surgery theory II: The L-groups)**.** *Grigori.* This talk should continue the outline of surgery theory that was begun in the previous talk. Introduce the L-groups and the surgery exact sequence [7, §4 and 5]. To keep the exposition simple, concentrate on the case $\pi_1 = 1$. In that case, the odd-dimensional surgery groups vanish (even if they are complicated to define) and we suggest that you should not enter the definition of the odd L-groups at all. Using the computation $L_{4k}(\mathbb{Z}) = \mathbb{Z}$ and $L_{4k+2}(\mathbb{Z}) = \mathbb{Z}/2$, one can extract information about the homotopy groups of G/O, G/PL and related spaces, see [7, §6.6]. Related useful reading: [5].

Central to the next talk (and to the development of the whole theory) is the surgical classification of homotopy tori [4, Theorem V.5.4] which is proven in [4, Essay V, appendix B].

Talk 4 (The torus trick and computation of the homotopy groups of TOP/O, TOP/PL)**.** *Johannes.* In this talk, the strands of the preceding talks are tied together. It should be proven that $\pi_*(G/\mathrm{TOP}) \cong L_*(\mathbb{Z})$ (this implies that the inclusion $BO \to B\mathrm{TOP}$ is a rational equivalence). This is based on an instance of the famous "torus trick". Reference: [4, Essay V.5]. (Note: it is much easier to prove that $\pi_*(G/\mathrm{PL}) \cong L_*(\mathbb{Z})$ for $* \geq 5$, modulo the affirmed PL Poincaré conjecture in dimensions ≥ 5; indeed this could/should be mentioned in talk 3. Therefore this talk emphasizes the differences between CAT = TOP and CAT = PL. Technically they are big, computationally they turn out to be small.) Related useful reading: [6], [9].

Talk 5 (Algebraic K-theory and Waldhausen categories; generalities)**.** *Markus.* The standard (modern) foundational paper for this theme is [10]. A selection should be made from Part 1 "Abstract K-Theory", but we also need categories of retractive spaces, which is the first section of Part 2. (It may be interesting to make a comparison with [8], which was the standard foundational paper before [10].)

Talk 6 (Duality in Waldhausen categories and their K-theory spectra)**.** *Arthur.* Title almost self-explanatory; [15] is the reference. Duality refers to things like duality in categories of vector spaces, but also Spanier-Whitehead duality in stable homotopy theory. Duality notions in Waldhausen categories tend to determine involutions on the corresponding K-theory spectra.

Talk 7 (Some Stiefel-Whitney theory in the style of James)**.** *Oliver ?* The source for this is probably [14], which is a little obsolete. James showed that the standard map

$$\mathbb{R}P^{n-1} \longrightarrow \mathrm{O}(n)$$

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(inclusion of the reflections) admits something like a stable splitting (left inverse), a map from $\mathrm{O}(n)$ to $\Sigma^\infty(\mathbb{R}P_+^{n-1})$. This was very important in Adams' work on vector fields on spheres. Briefly, [14] generalizes this construction (or generalizes a construction which James could have used to make his stable splitting), so that it can be used for example with $\mathrm{TOP}(n)$ instead of $\mathrm{O}(n)$, or with $\mathrm{G}(n)$ instead of $\mathrm{O}(n)$.

Talk 8 (General Stiefel-Whitney theory)**.** *Leon.* Sources: [12] which is too long and too detailed, and a chapter from [13] which MW can make available; this last one might be the best source. Theme, roughly: If a space X admits a filtration by subspaces $X(V)$ indexed by the finite-dimensional linear subspaces of $\mathbb{R}^\infty$, then there is an interesting theory of Stiefel-Whitney classes on X with coefficients in somewhat unexpected spectra. More precisely, X should be viewed as a functor from finite dimensional real vector spaces V with inner product to spaces. The standard examples are $X(V) = B\mathrm{O}(V)$, $X(V) = B\mathrm{TOP}(V)$, $X(V) = B\mathrm{G}(V)$, and these are also the most important for us. The coefficient spectra are made out of the spaces

$$\mathrm{hofiber}[X(\mathbb{R}^n) \to X(\mathbb{R}^{n+1})].$$

In the case where $X(V) = B\mathrm{O}(V)$, this gives a sphere spectrum, but in the case where $X(V) = B\mathrm{TOP}(V)$, it gives a spectrum which is homotopy equivalent to Waldhausen's $\mathbf{A}(*)$, a form of algebraic K-theory. This fact is important for us and the talk could end by stating it, at least.

Talk 9 (Algebraic K-theory and h-cobordism spaces)**.** *Thomas ?* Main topic: for a compact smooth manifold M, the space of smooth h-cobordisms on $M \times D^k$ (for $k \to \infty$) can be identified with the homotopy fiber of a map of infinite loop spaces $\Omega^\infty\Sigma^\infty M_+ \longrightarrow \Omega^\infty\mathbf{A}(M)$, where $\mathbf{A}(M)$ is the algebraic K-theory spectrum associated with the Waldhausen category of retractive spaces on M (subject to finiteness conditions). The main source for this is [11]. Take a look at [1], too; this introduces another point of view (index-theoretic) to explain why we have an interesting map from the space of smooth h-cobordisms to the homotopy fiber of $\Omega^\infty\Sigma^\infty M_+ \to \Omega^\infty\mathbf{A}(M)$. But [11] is still the place to look for a proof that the map is a homotopy equivalence.
In the case $M = *$, this result about $\mathbf{A}(*)$ and the h-cobordism space can easily be translated (using smoothing theory, talk 1) into a statement saying that $\mathbf{A}(*)$ is homotopy equivalent to a spectrum with n-th term $\mathrm{TOP}(n+1)/\mathrm{TOP}(n)$, analogous to the sphere spectrum which has n-th term $\mathrm{O}(n+1)/\mathrm{O}(n) = S^n$. This was mentioned in (the outline of) talk 8. Note that we ought to have such a statement *with involutions*; for example, $\mathbf{A}(*)$ has the involution given by Spanier-Whitehead 0-duality. But it is agreed that this would go rather far (into [16] and [13]), too far.

Talk 10. *Michael W.* According to talks 7, 8, 9 and the stability theorem of Igusa [3], the space $B\mathrm{TOP}(n)$ is well approximated by the *homotopical vanishing locus* of a characteristic class or cocycle on $B\mathrm{TOP}$ with twisted coefficients in a spectrum of the form $(S^{n+1} \wedge \mathbf{A}(*))_{h\mathbb{Z}/2}$. Here the subscript $h\mathbb{Z}/2$ is for a homotopy orbit (alias Borel) construction. In this talk, the calculations available (or not available) should be wrapped up so that we can at least get an idea of the *rational* homotopy type of $B\mathrm{TOP}(n)$ in a certain range. Going rational leads to many simplifications ... and looking back one could certainly say that it was excessive to pay so much attention to the prime 2, if the rational homotopy type of $B\mathrm{TOP}(n)$ was all we wanted. In any case this will also allow us to understand the rational homotopy type of $\Omega^{n+1}(\mathrm{TOP}(n)/\mathrm{O}(n))$ in a certain range (roughly, the $n/3$-skeleton). By talk 1, this is homotopy equivalent to the space of smooth automorphisms of D^n relative to the boundary S^{n-1}. In this way, we can understand or recover an old theorem due to Farrell and Hsiang [2] which describes the rational homotopy groups of these automorphism spaces (in the above-mentioned range).

References

[1] W. Dwyer, M. Weiss, and B. Williams. An index theorem for the algebraic K–theory Euler class. *Acta Math.*, 190:1–104, 2003.

[2] F. T. Farrell and W. C. Hsiang. On the rational homotopy groups of the diffeomorphism groups of discs, spheres and aspherical manifolds. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 1*, Proc. Sympos. Pure Math., XXXII, pages 325–337. Amer. Math. Soc., Providence, R.I., 1978.

[3] Kiyoshi Igusa. The stability theorem for smooth pseudoisotopies. *K-Theory*, 2(1-2):vi+355, 1988.

[4] Robion C. Kirby and Laurence C. Siebenmann. *Foundational essays on topological manifolds, smoothings, and triangulations*. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1977. With notes by John Milnor and Michael Atiyah, Annals of Mathematics Studies, No. 88.

[5] Matthias Kreck and Wolfgang Lück. *The Novikov conjecture*, volume 33 of *Oberwolfach Seminars*. Birkhäuser Verlag, Basel, 2005. Geometry and algebra.

[6] S. Kupers. Diffeomorphisms of manifolds. Lecture notes, available at http://math.harvard.edu/ kupers/teaching/272x/index.html.

[7] Wolfgang Lück. A basic introduction to surgery theory. In *Topology of high-dimensional manifolds, No. 1, 2 (Trieste, 2001)*, volume 9 of *ICTP Lect. Notes*, pages 1–224. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2002.

[8] Daniel Quillen. Higher algebraic K-theory. I. pages 85–147. Lecture Notes in Math., Vol. 341, 1973.

[9] Y. B. Rudyak. Piecewise linear structures on topological manifolds. *ArXiv Mathematics e-prints*, May 2001.

[10] F. Waldhausen. Algebraic K–theory of spaces. In *Proceedings of 1983 Rutgers conference on algebraic and geometric topology*, volume 1126 of *Springer Lect. Notes in Math.*, pages 318–419, New Brunswick, NJ, 1985. Springer–Verlag.

[11] Friedhelm Waldhausen, Bjørn Jahren, and John Rognes. *Spaces of PL manifolds and categories of simple maps*, volume 186 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

[12] M. Weiss. Orthogonal calculus. *Trans. Amer. Math. Soc.*, 347:3743–3796, 1995. Erratum in vol 350.

[13] M. Weiss and B. Williams. Automorphisms of manifolds and algebraic K–theory: IV. in preparation.

[14] M. Weiss and B. Williams. Automorphisms of manifolds and algebraic K–theory: I. *K–theory*, 1:575–626, 1988.

[15] M. Weiss and B. Williams. Duality in Waldhausen categories. *Forum Math.*, 10:533–603, 1998.

[16] M. Weiss and B. Williams. Automorphisms of manifolds and algebraic K–theory: III. *Memoirs Amer. Math. Soc.*, 231:vi+110pp., 2014.